Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com
Volume 7 Issue 3(2016), Pages 12-25.

AN IMPROVED VERSION OF PERTURBED COMPANION OF OSTROWSKI TYPE INEQUALITIES

[1]M. Z. SARIKAYA, [1]H. BUDAK, AND [2]A. QAYYUM

Abstract. The purpose of this paper is to establish an improved version of perturbed companion of Ostrowski type integral inequalities for functions whose first derivatives are either bounded or of bounded variation.

1. Introduction

In 1938, Ostrowski first announced his inequality result for different differentiable mappings. Ostrowski inequality has potential applications in Mathematical Sciences. In the past, many researchers have worked on Ostrowski type inequalities for functions (bounded, of bounded variation, etc.) see for example ([1]-[6], [8]-[11], [16],[17],[20]-[21]). Furthermore, several works were devoted to study of perturbed Ostrowski type inequalities for bounded functions and functions of bounded variation, please refer to ([7], [12]-[15],[19]). The structure of this paper is as follows: in Section 2 we present inequalities for mappings of bounded variation. In Section 3, we provide inequalities for functions whose derivatives are bounded. Finally, in Section 4 we extend inequalities for Lipschitzian mappings. Some previous results are recaptured as special cases.

Ostrowski proved a useful inequality, which gives an upper bound for the approximation of the integral average by the value of mapping at a certain point of the interval, which is given below:

Theorem 1. [18] *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

2000 *Mathematics Subject Classification.* 26D07; 26D10; 26D15.
Key words and phrases. Ostrowski inequality, function of bounded variation.

Submitted March 15 2016. Published June 21, 2016.
This paper is in final form and no version of it will be submitted for publication elsewhere.

2. Some Identities

In order to prove our inequalities, we need the following identity which although of interest in itself.

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous on $[a,b]$. Then for any $\lambda_i(x)$, $i = 1,2,...,5$ complex numbers, we have*

$$\begin{aligned} &\frac{1}{b-a}\left\{ \int_a^{\frac{a+x}{2}} (t-a)\left[f'(t)-\lambda_1(x)\right]dt + \int_{\frac{a+x}{2}}^{x} \left(t-\frac{3a+b}{4}\right)\left[f'(t)-\lambda_2(x)\right]dt \right. \\ (2.1)\ &+ \int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_3(x)\right]dt \\ &\left. + \int_{a+b-x}^{\frac{a+2b-x}{2}} \left(t-\frac{a+3b}{4}\right)\left[f'(t)-\lambda_4(x)\right]dt + \int_{\frac{a+2b-x}{2}}^{b} (t-b)\left[f'(t)-\lambda_5(x)\right]dt \right\} \\ =\ & \frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &+\frac{1}{8(b-a)}\left\{(x-a)^2\left[\lambda_5(x)-\lambda_1(x)\right]\right. \\ &\left. + \left[\left(x-\frac{a+b}{2}\right)^2 - 4\left(x-\frac{3a+b}{4}\right)^2\right](\lambda_2(x)-\lambda_4(x))\right\} \end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Using the integration by parts for Riemann-Stieltjes integral for each integral, we can easily obtain the desired result (2.1). □

Remark 1. *By substituting $x = a$ in (2.1), we get*

$$\frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_3\right]dt = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt$$

which was given by Dragomir in [13].

Corollary 1. *Choosing* $x=\frac{a+b}{2}$ *in (2.1), we get*

$$\frac{1}{b-a}\left\{\int_{a}^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(t-\frac{3a+b}{4}\right)\left[f'(t)-\lambda_2\right]dt\right.$$

$$\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(t-\frac{a+3b}{4}\right)\left[f'(t)-\lambda_4\right]dt+\int_{\frac{a+3b}{4}}^{b}(t-b)\left[f'(t)-\lambda_5\right]dt\right\}$$

$$=\frac{1}{4}\left[2f\left(\frac{a+b}{2}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt$$

$$+\frac{1}{32}(b-a)\left[\lambda_5+\lambda_4-\lambda_1-\lambda_2\right]$$

Corollary 2. *If we substitute* $x=\frac{3a+b}{4}$ *in (2.1), we get*

$$\frac{1}{b-a}\left\{\int_{a}^{\frac{7a+b}{8}}(t-a)\left[f'(t)-\lambda_1\right]dt+\int_{\frac{7a+b}{8}}^{\frac{3a+b}{4}}\left(t-\frac{3a+b}{4}\right)\left[f'(t)-\lambda_2\right]dt\right.$$

$$+\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_3\right]dt$$

$$\left.+\int_{\frac{a+3b}{4}}^{\frac{a+7b}{8}}\left(t-\frac{a+3b}{4}\right)\left[f'(t)-\lambda_4\right]dt+\int_{\frac{a+7b}{8}}^{b}(t-b)\left[f'(t)-\lambda_5\right]dt\right\}$$

$$=\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt$$

$$+\frac{b-a}{128}\left[\lambda_5-\lambda_1+\lambda_2-\lambda_4\right]$$

Now using above lemma, we will present inequalities via three different cases.

3. Inequalities for Mappings of bounded variation

In this section we give some companion of perturbed Ostrowski inequalities for function whose derivatives are of bounded variation.

Let $f:[a,b]\to\mathbb{C}$ be a differentiable function on I°(I° is the interior of I) and $[a,b]\subset I^{\circ}$.Then from we have for

$$\lambda_1(x)=f'(a)$$

$$\lambda_2(x)=\frac{f'\left(\frac{a+x}{2}\right)+f'(x)}{2}$$

$$\lambda_3(x)=\frac{f'(x)+f'(a+b-x)}{2}$$

$$\lambda_4(x) = \frac{f'(a+b-x) + f'\left(\frac{a+2b-x}{2}\right)}{2}$$

$$\lambda_5(x) = f'(b)$$

$$\begin{aligned}
(3.1)\quad & \frac{1}{b-a}\left\{ \int_a^{\frac{a+x}{2}} (t-a)\left[f'(t) - f'(a)\right] dt + \int_{\frac{a+x}{2}}^{x} \left(t - \frac{3a+b}{4}\right) \right. \\
& \times \left[f'(t) - \frac{f'\left(\frac{a+x}{2}\right) + f'(x)}{2}\right] dt \\
& + \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right)\left[f'(t) - \frac{f'(x) + f'(a+b-x)}{2}\right] dt \\
& + \int_{a+b-x}^{\frac{a+2b-x}{2}} \left(t - \frac{a+3b}{4}\right)\left[f'(t) - \frac{f'(a+b-x) + f'\left(\frac{a+2b-x}{2}\right)}{2}\right] dt \\
& \left. + \int_{\frac{a+2b-x}{2}}^{b} (t-b)\left[f'(t) - f'(b)\right] dt \right\} \\
= \; & \frac{1}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\
& + \frac{1}{8(b-a)}\left\{(x-a)^2\left[f'(b) - f'(a)\right]\right. \\
& + \frac{1}{2}\left[\left(x - \frac{a+b}{2}\right)^2 - 4\left(x - \frac{3a+b}{4}\right)^2\right] \\
& \left. \times \left(f'\left(\frac{a+x}{2}\right) + f'(x) - f'(a+b-x) - f'\left(\frac{a+2b-x}{2}\right)\right)\right\}
\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Theorem 2. *Let $f : [a,b] \to \mathbb{C}$ be a differentiable function on I° (I° is the interior of I) and $[a,b] \subset I^\circ$. If the second derivative f' is of bounded variation on $[a,b]$, then,*

$$\begin{aligned}
& \left| \frac{1}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right. \\
& + \frac{1}{8(b-a)}\left\{(x-a)^2\left[f'(b) - f'(a)\right]\right. \\
& + \frac{1}{2}\left[\left(x - \frac{a+b}{2}\right)^2 - 4\left(x - \frac{3a+b}{4}\right)^2\right] \\
& \left.\left. \times \left(f'\left(\frac{a+x}{2}\right) + f'(x) - f'(a+b-x) - f'\left(\frac{a+2b-x}{2}\right)\right)\right\} \right|
\end{aligned}$$

$$\leq \frac{1}{8(b-a)}\left\{(x-a)^2\bigvee_a^{\frac{a+x}{2}}(f')\right.$$
$$+sgn\left(\frac{3a+b}{4}-x\right)\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_{\frac{a+x}{2}}^{x}(f\,')$$
$$+4\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f\,')+\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f\,')$$
$$\left.+(x-a)^2\bigvee_{\frac{a+2b-x}{2}}^{b}(f\,')\right\}.$$

Proof. From (3.1),

$$\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.$$
$$+\frac{1}{8(b-a)}\left\{(x-a)^2\left[f\,'(b)-f\,'(a)\right]\right.$$
$$+\frac{1}{2}\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right]$$
$$\left.\left.\times\left(f\,'\left(\frac{a+x}{2}\right)+f\,'(x)-f\,'(a+b-x)-f\,'\left(\frac{a+2b-x}{2}\right)\right)\right\}\right|$$
$$\leq \frac{1}{b-a}\left\{\int_a^{\frac{a+x}{2}}|t-a|\,|f\,'(t)-f\,'(a)|\,dt\right.$$
$$+\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|\left|f\,'(t)-\frac{f\,'\left(\frac{a+x}{2}\right)+f\,'(x)}{2}\right|dt$$
$$+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f\,'(t)-\frac{f\,'(x)+f\,'(a+b-x)}{2}\right|dt$$
$$+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left|t-\frac{a+3b}{4}\right|\left|f\,'(t)-\frac{f\,'(a+b-x)+f\,'\left(\frac{a+2b-x}{2}\right)}{2}\right|dt$$
$$\left.+\int_{\frac{a+2b-x}{2}}^{b}|t-b|\,|f\,'(t)-f\,'(b)|\,dt\right\}.$$

Since $f\,'$ is of bounded variation on $[a,b]$, we get

$$|f\,'(t)-f\,'(a)|\leq\bigvee_a^t(f\,')$$

for $t \in \left[a, \frac{a+x}{2}\right]$

$$\left| f'(t) - \frac{f'\left(\frac{a+x}{2}\right) + f'(x)}{2} \right| \leq \frac{1}{2} \bigvee_{\frac{a+x}{2}}^{x} (f') < \bigvee_{\frac{a+x}{2}}^{x} (f')$$

for $t \in \left[\frac{a+x}{2}, x\right]$

$$\left| f'(t) - \frac{f'(x) + f'(a+b-x)}{2} \right| \leq \frac{1}{2} \bigvee_{x}^{a+b-x} (f')$$

for $t \in [x, a+b-x]$

$$\left| f'(t) - \frac{f'(a+b-x) + f'\left(\frac{a+2b-x}{2}\right)}{2} \right| \leq \frac{1}{2} \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f') < \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f')$$

for $t \in \left[a+b-x, \frac{a+2b-x}{2}\right]$

$$|f'(t) - f'(b)| \leq \bigvee_{t}^{b} (f')$$

for $t \in \left[\frac{a+2b-x}{2}, b\right]$.

Thus, we have

$$\begin{aligned}
&\left| \frac{1}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right. \\
&+\frac{1}{8(b-a)}\left\{(x-a)^2\left[f'(b) - f'(a)\right]\right. \\
&+\frac{1}{2}\left[\left(x - \frac{a+b}{2}\right)^2 - 4\left(x - \frac{3a+b}{4}\right)^2\right] \\
&\left.\left.\times\left(f'\left(\frac{a+x}{2}\right) + f'(x) - f'(a+b-x) - f'\left(\frac{a+2b-x}{2}\right)\right)\right\}\right| \\
\leq\ & \frac{1}{b-a}\left\{\int_a^{\frac{a+x}{2}} (t-a)\bigvee_a^t (f')\,dt + \int_{\frac{a+x}{2}}^{x} \left|t - \frac{3a+b}{4}\right| \bigvee_{\frac{a+x}{2}}^{x} (f')\,dt\right. \\
&+\frac{1}{2}\int_x^{a+b-x}\left|t - \frac{a+b}{2}\right| \bigvee_x^{a+b-x} (f')\,dt + \int_{a+b-x}^{\frac{a+2b-x}{2}} \left(\frac{a+3b}{4} - t\right) \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f')\,dt \\
&\left.+\int_{\frac{a+2b-x}{2}}^{b} (b-t)\bigvee_t^b (f')\,dt\right\}
\end{aligned}$$

$$\leq \frac{1}{b-a}\left\{ \bigvee_{a}^{\frac{a+x}{2}}(f') \int_{a}^{\frac{a+x}{2}}(t-a)\,dt + \bigvee_{\frac{a+x}{2}}^{x}(f') \int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right| dt \right.$$
$$+\frac{1}{2}\bigvee_{x}^{a+b-x}(f') \int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right| dt + \bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f') \int_{a+b-x}^{\frac{a+2b-x}{2}}\left(\frac{a+3b}{4}-t\right) dt$$
$$\left. + \bigvee_{\frac{a+2b-x}{2}}^{b}(f') \int_{\frac{a+2b-x}{2}}^{b}(b-t)\,dt \right\}$$
$$= \frac{1}{8(b-a)}\left\{ (x-a)^2 \bigvee_{a}^{\frac{a+x}{2}}(f') \right.$$
$$+sgn\left(x-\frac{3a+b}{4}\right)\frac{1}{2}\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_{\frac{a+x}{2}}^{x}(f')$$
$$+4\left(x-\frac{a+b}{2}\right)^2\bigvee_{x}^{a+b-x}(f')+\frac{1}{2}\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f')$$
$$\left. +(x-a)^2\bigvee_{\frac{a+2b-x}{2}}^{b}(f') \right\}.$$

Here,

$$\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right| dt = \frac{1}{8}\left(x-\frac{a+b}{2}\right)^2-\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2$$

for $x \in \left[a, \frac{3a+b}{4}\right]$ and

$$\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right| dt = \frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2-\frac{1}{8}\left(x-\frac{a+b}{2}\right)^2$$

for $x \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$.

This completes the proof. □

Corollary 3. *If we choose $x=a$ in Theorem 2, then we get the Liu's result* [17].

Corollary 4. *Under assumption of Theorem 2 with $x=\frac{a+b}{2}$, we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\&\left.+\frac{(b-a)}{32}\left[f'(b)-f'(a)+\frac{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)}{2}\right]\right|\\\leq\ &\frac{(b-a)}{32}\bigvee_a^b(f').\end{aligned}$$

4. Inequalities for Functions Whose Derivatives are Bounded

Now, we obtain some inequalities for bounded function using the identity (2.1). Recall the sets of complex-valued functions:

$$\begin{aligned}&\overline{U}_{[a,b]}(\gamma,\Gamma)\\:\ &=\left\{f:[a,b]\to\mathbb{C}|Re\left[(\Gamma-f(t))\left(\overline{f(t)}\right)-\overline{\gamma}\right]\geq 0\text{ for almost every }t\in[a,b]\right\}\end{aligned}$$

and

$$\overline{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}\left|\left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\ \text{ for a.e. }t\in[a,b]\right.\right\}.$$

Proposition 1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\overline{U}_{[a,b]}(\gamma,\Gamma)$ and $\overline{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty and closed sets and*

$$\overline{U}_{[a,b]}(\gamma,\Gamma)=\overline{\Delta}_{[a,b]}(\gamma,\Gamma).$$

Let $I_1=\left[a,\frac{a+x}{2}\right]$, $I_2=\left[\frac{a+x}{2},x\right]$ $I_3=[x,a+b-x]$ $I_4=\left[a+b-x,\frac{a+2b-x}{2}\right]$ and $I_5=\left[\frac{a+2b-x}{2},b\right]$.

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a differantiable function on (a,b). Suppose that $\gamma_i(x),\Gamma_i(x)\in\mathbb{C}$, $\gamma_i(x)\neq\Gamma_i(x)$, $i=1,2,3,4,5$ and*

$$f^{'}\in\cap_{i=1}^{5}\overline{U}_{I_i}(\gamma_i,\Gamma_i)$$

then we have the inequality

$$\begin{aligned}&\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\&+\frac{1}{16(b-a)}\left\{(x-a)^2\left[\gamma_5(x)+\Gamma_5(x)-\gamma_1(x)-\Gamma_1(x)\right]\right.\\&\left.\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right](\gamma_2(x)+\Gamma_2(x)-\gamma_4(x)-\Gamma_4(x))\right\}\right|\end{aligned}$$

$$\leq \frac{1}{8(b-a)}\left\{(x-a)^2|\Gamma_1(x)-\gamma_1(x)|\right.$$
$$+sgn\left(x-\frac{3a+b}{4}\right)\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]|\Gamma_2(x)-\gamma_2(x)|$$
$$+4\left(x-\frac{a+b}{2}\right)^2|\Gamma_3(x)-\gamma_3(x)|$$
$$+\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]|\Gamma_4(x)-\gamma_4(x)|$$
$$\left.+(x-a)^2|\Gamma_5(x)-\gamma_5(x)|\right\}.$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. Taking the modulus identity (2.1) for $\lambda_i(x)=\frac{\gamma_i(x)+\Gamma_i(x)}{2}$, $i=1,2,...,5$, since $f^{'}\in\cap_{i=1}^{5}\overline{U}_{I_i}(\gamma_i,\Gamma_i)$, we have

$$\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.$$
$$+\frac{1}{16(b-a)}\left\{(x-a)^2[\gamma_5(x)+\Gamma_5(x)-\gamma_1(x)-\Gamma_1(x)]\right.$$
$$\left.\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right](\gamma_2(x)+\Gamma_2(x)-\gamma_4(x)-\Gamma_4(x))\right\}\right|$$
$$\leq \frac{1}{b-a}\left\{\int_a^{\frac{a+x}{2}}|t-a|\left|f'(t)-\frac{\gamma_1(x)+\Gamma_1(x)}{2}\right|dt\right.$$
$$+\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|\left|f'(t)-\frac{\gamma_2(x)+\Gamma_2(x)}{2}\right|dt$$
$$+\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)-\frac{\gamma_3(x)+\Gamma_3(x)}{2}\right|dt$$
$$+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left|t-\frac{a+3b}{4}\right|\left|f'(t)-\frac{\gamma_4(x)+\Gamma_4(x)}{2}\right|dt$$
$$\left.+\int_{\frac{a+2b-x}{2}}^{b}|t-b|\left|f'(t)-\frac{\gamma_5(x)+\Gamma_5(x)}{2}\right|dt\right\}$$

$$
\begin{aligned}
&\leq \frac{1}{2(b-a)}\left\{\left|\Gamma_1(x)-\gamma_1(x)\right|\int_a^{\frac{a+x}{2}}(t-a)\,dt+\left|\Gamma_2(x)-\gamma_2(x)\right|\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|dt\right.\\
&\quad+\left|\Gamma_3(x)-\gamma_3(x)\right|\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt\\
&\quad\left.+\left|\Gamma_4(x)-\gamma_4(x)\right|\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(\frac{a+3b}{4}-t\right)dt+\left|\Gamma_5(x)-\gamma_5(x)\right|\int_{\frac{a+2b-x}{2}}^{b}(b-t)dt\right\}\\
&= \frac{1}{8(b-a)}\left\{(x-a)^2\left|\Gamma_1(x)-\gamma_1(x)\right|\right.\\
&\quad+sgn\left(x-\frac{3a+b}{4}\right)\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]\left|\Gamma_2(x)-\gamma_2(x)\right|\\
&\quad+4\left(x-\frac{a+b}{2}\right)^2\left|\Gamma_3(x)-\gamma_3(x)\right|\\
&\quad\left.\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]\left|\Gamma_4(x)-\gamma_4(x)\right|+(x-a)^2\left|\Gamma_5(x)-\gamma_5(x)\right|\right\}.
\end{aligned}
$$

This completes the proof. □

Remark 2. *If we choose $x=a$ in Theorem 3, then we get a new result, proved by Dragomir* [13].

Corollary 5. *Under assumption of Theorem 3 with $x=\frac{a+b}{2}$, we get the inequality*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\
&+\frac{1}{64}(b-a)\left[\gamma_5(x)+\Gamma_5(x)-\gamma_1(x)-\Gamma_1(x)\right.\\
&\left.\left.+\gamma_2(x)+\Gamma_2(x)-\gamma_4(x)+\Gamma_4(x)\right]\right|\\
&\leq \frac{b-a}{16}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|+\left|\Gamma_2(x)-\gamma_2(x)\right|\right].
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\
&\left.+\frac{1}{64}(b-a)\left[\gamma_1(x)+\Gamma_1(x)+\gamma_2(x)+\Gamma_2(x)\right]\right|\\
\leq\ &\frac{b-a}{16}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|+\left|\Gamma_2(x)-\gamma_2(x)\right|\right].
\end{aligned}
$$

5. Inequalities for Lipschitzian Mappings

In this section, we obtain some inequalities for function whose derivatives are Lipschitzian.

We say that the function $g:[a,b]\rightarrow\mathbb{C}$ is Lipschitzian with the constant $L>0$ if

$$|g(t)-g(s)|\leq L\,|t-s|$$

for any $t,s\in[a,b]$.

Theorem 4. *Let $f:[a,b]\rightarrow\mathbb{C}$ be a twice differantiable function on (a,b). If the derivative f' is a Lipschitzian mapping with the constant $L>0$, then we have the inequality*

$$\begin{aligned}(5.1)\quad&\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\&+\frac{1}{8(b-a)}\left\{(x-a)^2\left[f'(b)-f'(a)\right]\right.\\&\left.\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right]\left(f'\left(\frac{3a+b}{4}\right)-f'\left(\frac{a+3b}{4}\right)\right)\right\}\right|\\\leq\;&\frac{L}{12(b-a)}\left[(x-a)^3+8\left(x-\frac{3a+b}{4}\right)^3+7\left(x-\frac{a+b}{2}\right)^3\right]\end{aligned}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. If we take $\lambda_1=f'(a)$, $\lambda_2=f'\left(\frac{3a+b}{4}\right)$, $\lambda_3=f'\left(\frac{a+b}{2}\right)$, $\lambda_4=f'\left(\frac{a+3b}{4}\right)$ and $\lambda_5=f'(b)$ in equality (2.1), we have

$$\begin{aligned}(5.2)\quad&\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\&+\frac{1}{8(b-a)}\left\{(x-a)^2\left[f'(b)-f'(a)\right]\right.\\&\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right]\left(f'\left(\frac{3a+b}{4}\right)-f'\left(\frac{a+3b}{4}\right)\right)\right\}\\=\;&\frac{1}{b-a}\left\{\int_a^{\frac{a+x}{2}}(t-a)\left[f'(t)-f'(a)\right]dt\right.\\&+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)\left[f'(t)-f'\left(\frac{3a+b}{4}\right)\right]dt\\&+\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt\end{aligned}$$

$$+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)\left[f'(t)-f'\left(\frac{a+3b}{4}\right)\right]dt$$

$$\left.+\int_{\frac{a+2b-x}{2}}^{b}(t-b)\left[f'(t)-f'(b)\right]dt\right\}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Since f' is lipschitzian, taking the madulus in (5.2), we have

$$\begin{aligned}
&\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&+\frac{1}{8(b-a)}\left\{(x-a)^2\left[f'(b)-f'(a)\right]\right.\\
&\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right]\left(f'\left(\frac{3a+b}{4}\right)-f'\left(\frac{a+3b}{4}\right)\right)\right\}\\
\le\ &\frac{1}{b-a}\left\{\int_a^{\frac{a+x}{2}}|t-a|\,|f'(t)-f'(a)|\,dt+\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|\left|f'(t)-f'\left(\frac{3a+b}{4}\right)\right|dt\right.\\
&+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|dt\\
&\left.+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left|t-\frac{a+3b}{4}\right|\left|f'(t)-f'\left(\frac{a+3b}{4}\right)\right|dt+\int_{\frac{a+2b-x}{2}}^{b}|t-b|\,|f'(t)-f'(b)|\,dt\right\}\\
\le\ &\frac{L}{b-a}\left\{\int_a^{\frac{a+x}{2}}(t-a)^2\,dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^2dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2dt\right.\\
&\left.+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2dt+\int_{\frac{a+2b-x}{2}}^{b}(t-b)^2\,dt\right\}\\
=\ &\frac{L}{b-a}\left[\frac{(x-a)^3}{12}+\frac{2}{3}\left(x-\frac{3a+b}{4}\right)^3+\frac{7}{12}\left(x-\frac{a+b}{2}\right)^3\right]
\end{aligned}$$

which completes the proof. □

Remark 3. *If we choose $x=a$ in Theorem 3, then we get another inequality which was proved by Dragomir* [14].

Corollary 6. *Under assumption of Theorem 4 with* $x = \frac{a+b}{2}$, *we get the inequality*

$$\begin{aligned} & \left| \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right. \\ & \left. \frac{1}{32}(b-a)\left[f'(b) - f'(a) + f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right] \right| \\ \leq & \frac{(b-a)^2}{48} L. \end{aligned}$$

Corollary 7. *Under assumption of Theorem 4 with* $x = \frac{3a+b}{4}$, *we get the inequality*

$$\begin{aligned} & \left| \frac{1}{4}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right. \\ & \left. \frac{1}{128}(b-a)\left[f'(b) - f'(a) + f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right] \right| \\ \leq & \frac{(b-a)^2}{96} L. \end{aligned}$$

References

[1] M. W. Alomari, *A Generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation*, RGMIA Research Report Collection, 14(2011), Art. 87, 11 pp.

[2] H. Budak and M.Z. Sarıkaya, *On generalization of Dragomir's inequalities*, RGMIA Research Report Collection, 17(2014), Article 155, 10 pp.

[3] H. Budak and M.Z. Sarikaya, *A new generalization of Ostrowski type inequality for mappings of bounded variation*, RGMIA Research Report Collection, 18(2015), Article 47, 9 pp.

[4] H. Budak and M. Z. Sarikaya, *A new Ostrowski type inequality for functions whose first derivatives are of bounded variation*, Moroccan J. Pure Appl. Anal., 2(1)(2016), 1–11.

[5] H. Budak and M.Z. Sarikaya, *A companion of Ostrowski type inequalities for mappings of bounded variation and some applications*, RGMIA Research Report Collection, 19(2016), Article 24, 10 pp.

[6] H. Budak, M.Z. Sarikaya and A. Qayyum, *Improvement in companion of Ostrowski type inequalities for mappings whose first derivatives are of bounded variation and application*, RGMIA Research Report Collection, 19(2016), Article 25, 11 pp.

[7] H. Budak and M. Z. Sarikaya, *Some perturbed Ostrowski type inequality for functions whose first derivatives are of bounded variation*, RGMIA Research Report Collection, 19 (2016), Article 54, 13 pp.

[8] S. S. Dragomir and N.S. Barnett, *An Ostrowski type inequality for mappings whose second derivatives are bounded and applications*, RGMIA Research Report Collection, 1(2)(1998) .

[9] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bulletin of the Australian Mathematical Society, 60(1) (1999), 495-508.

[10] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Mathematical Inequalities & Applications, 4 (2001), no. 1, 59–66.

[11] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, International Journal of Nonlinear Analysis and Applications, 5 (2014) No. 1, 89-97 pp.

[12] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation,* Asian-European Journal of Mathematics, 8(4)(2015,),14 pages.

[13] S. S. Dragomir, *Perturbed Companions of Ostrowski's Inequality for Absolutely Continuous Functions (I)*, RGMIA Research Report Collection, 17(2014), Article 7, 15 pp.

[14] S. S. Dragomir, *Perturbed Companions of Ostrowski's Inequality for Absolutely Continuous Functions (II)*, GMIA Research Report Collection, 17(2014), Article 19, 11 pp.

[15] S. S. Dragomir, *Perturbed Companions of Ostrowski's Inequality for Functions of Bounded Variation,* RGMIA Research Report Collection, 17(2014), Article 1, 16 pp.

[16] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, (2012).

[17] Z. Liu, *Some Ostrowski type inequalities*, Mathematical and Computer Modelling 48 (2008) 949–960.

[18] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[19] M. Z. Sarikaya, H. Budak, T. Tunc, S. Erden and H. Yaldiz, *Perturbed companion of Ostrowski type inequality for twice differentiable functions*, RGMIA Research Report Collection, 19 (2016), Article 59, 13 pp.

[20] A. Qayyum, I. Faye and M. Shoaib, *A companion of Ostrowski Type Integral Inequality using a 5-step kernel With Some Applications*, Filomat, (In Press).

[21] A. Qayyum, M. Shoaib and I. Faye, Companion of Ostrowski-type inequality based on 5-step quadratic kernel and applications, Journal of Nonlinear Sciences and Applications, 9 (2016), 537-552.

[1]Department of Mathematics, Faculty of Science and Arts,, Düzce University, Düzce, Turkey.

E-mail address: sarikayamz@gmail.com

[1]Department of Mathematics, Faculty of Science and Arts,, Düzce University, Düzce, Turkey.

E-mail address: hsyn.budak@gmail.com

[2]Department of Mathematics, University of Hail, Kingdom of Saudi Arabia.

E-mail address: atherqayyum@gmail.com