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ON SOME INEQUALITIES FOR GAMMA AND BETA k-FUNCTIONS VIA SOME CLASSICAL INEQUALITIES

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Abstract. In this paper, We improve several results recently established by Rehman et al. [14] for the gamma and beta k-functions by using some classical inequalities like Chebychev, Hölder and Grüss inequalities.

1. Introduction

Diaz and Pariguan [1] have introduced the generalization of the classical gamma and beta functions in terms of a new parameter $k > 0$, called gamma and beta k-functions respectively. The gamma k-function is defined as follows

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k > 0, x \in \mathbb{C} \setminus kZ^-, \tag{1.1}$$

where

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \quad n \geq 1,$$

is called Pochhammer k-symbol, which is the generalization of usual Pochhammer symbol. The relation between gamma and beta k-functions is given by

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)},\ Re(x) > 0,\ Re(y) > 0$$

and integral representations of gamma and beta k-functions are respectively given below

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\,dt,\ Re(x) > 0 \tag{1.2}$$

and

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad Re(x) > 0, Re(y) > 0. \tag{1.3}$$

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Later, the researchers [4, 5, 7, 9, 11] have worked on the generalized gamma and beta k-functions and discussed the following properties of these functions for $k > 0$, $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\Gamma_k(x+k) = x\Gamma_k(x),$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)},$$

$$\Gamma_k(k) = 1,$$

$$\Gamma_k(x) = \alpha^{\frac{x}{k}} \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}\alpha} dt,$$

$$\Gamma_k(\alpha k) = k^{\alpha-1}\Gamma(\alpha),$$

$$\Gamma_k(nk) = k^{n-1}(n-1)!,$$

$$\beta_k(x+k, y) = \frac{x}{x+y}\beta_k(x, y), \tag{1.4}$$

$$\beta_k(x, y+k) = \frac{y}{x+y}\beta_k(x, y). \tag{1.5}$$

From relation (1.2), one can easily calculate the nth-derivative of gamma k-function as

$$\Gamma_k^{(n)}(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} (\ln t)^n dt, \ Re(x) > 0. \tag{1.6}$$

Kokologiannaki et al. [6] formed some important results of gamma, beta, zeta and hypergeometric k-functions. They also gave completely monotonicity properties and inequalities for functions involving the Γ_k functions and their logarithmic derivatives ψ_k. Krasniqi [8] proved some properties of special k-functions and also gave a limit for the gamma and beta k-functions. Suryanarayana and Rao [17] established some properties and inequalities for the generalized special k-functions. Zhang and Shi [18] proved two double inequalities for gamma and Riemann zeta k-functions and gave their applications. Secer et al. [16] established the q-extension of the Grüss type integral inequality related to the integrable functions involving the Riemann-Liouville fractional q-integral operators.

Mubeen et al. [12] discussed some properties and inequalities involving gamma, beta and Psi k-functions and also proved some extended results for the said functions. They also proved an inequality for beta k-function via logarithmic derivative of the gamma k-function. Rehman and Mubeen [15] presented some inequalities involving gamma and beta k-functions via some classical inequalities like the Chebychev inequality for synchronous (asynchronous) mappings, Grüss and the Ostrowski's inequality. Also, they proved the log-convexity of the gamma and beta k-functions by using Hölder's inequality and also they gave applications of beta k-function in probability distributions.

2. Inequalities via Chebychev's integral inequality

In this section, we prove some inequalities involving the derivative of gamma k-function by using Chebychev's integral inequality. The following result is the well known Chebychev's integral inequality for synchronous (asynchronous) mappings [13].

Lemma 2.1. *Let $f, g, h : I \subseteq \mathbb{R} \to \mathbb{R}$ be such that $h(x) \geq 0$ for all $x \in I$ and h, hfg, hf and hg are integrable on I. If f, g are synchronous (asynchronous) on I, that is, if it holds*

$$(f(x) - f(y))(g(x) - g(y)) \geq (\leq) 0 \text{ for all } x,\, y \in I. \tag{2.1}$$

Then, we have the following inequality

$$\int_I h(x)dx \int_I h(x)f(x)g(x)dx \geq (\leq) \int_I h(x)f(x)dx \int_I h(x)g(x)dx. \tag{2.2}$$

Theorem 2.2. *For $k > 0$, let s, u and v be real numbers with $s, u > 0$ and $u > v > -s$, if*

$$v(u - s - v) \geq (\leq) 0, \tag{2.3}$$

then for a non-negative integer n, we have

$$\Gamma_k^{(n)}(u)\Gamma_k^{(n)}(s) \geq (\leq) \Gamma_k^{(n)}(s + v)\Gamma_k^{(n)}(u - v). \tag{2.4}$$

Proof. Consider the mappings $f, g, h : [0, \infty) \to [0, \infty)$ defined as

$$f(t) = t^{u-s-v}, \quad g(t) = t^v, \quad h(t) = t^{s-1}e^{-\frac{t^k}{k}}(\ln t)^n.$$

If the condition given in (2.3) is true, then the functions f and g are synchronous (asynchronous) on $[0, \infty)$. So, by Chebychev's inequality for $I = [0, \infty)$, we have

$$\int_0^\infty t^{s-1}e^{-\frac{t^k}{k}}(\ln t)^n dt \int_0^\infty t^{u-v-s}t^v t^{s-1}e^{-\frac{t^k}{k}}(\ln t)^n dt$$
$$\geq (\leq) \int_0^\infty t^{u-v-s}t^{s-1}e^{-\frac{t^k}{k}}(\ln t)^n dt \int_0^\infty t^v t^{s-1}e^{-\frac{t^k}{k}}(\ln t)^n dt.$$

This implies that

$$\int_0^\infty t^{s-1}e^{-\frac{t^k}{k}}(\ln t)^n dt \int_0^\infty t^{u-1}e^{-\frac{t^k}{k}}(\ln t)^n dt$$
$$\geq (\leq) \int_0^\infty t^{u-v-1}e^{-\frac{t^k}{k}}(\ln t)^n dt \int_0^\infty t^{s+v-1}e^{-\frac{t^k}{k}}(\ln t)^n dt. \tag{2.5}$$

Hence from relation (1.6), we get the required inequality (2.4). □

Corollary 2.3. *For $k > 0$, let $u > 0$ and $r \in R$ be such that $|r| \leq u$, then for a non-negative integer n, we have*

$$\left[\Gamma_k^{(n)}(u)\right]^2 \leq \Gamma_k^{(n)}(u - r)\Gamma_k^{(n)}(u + r). \tag{2.6}$$

Proof. The inequality (2.6) can be obtained by using $s = u$ and $v = r$ in theorem 2.2. □

Remark. *For $s = 2$, $u = p + q$ and $v = q - 1$, the condition (2.3) becomes*

$$(p - 1)(q - 1) \geq (\leq) 0, \tag{2.7}$$

that is the positive real numbers p and q are said to be similarly (oppositely) unitary.

Theorem 2.4. *For $k>0$, if $p,q>0$ are similarly (oppositely) unitary, then*

$$\Gamma_k^{(n)}(p+q+k-1)\Gamma_k^{(n)}(k+1) \geq (\leq) \Gamma_k^{(n)}(p+k)\Gamma_k^{(n)}(q+k). \tag{2.8}$$

Proof. For $k>0$, consider the mappings $f,g,h:[0,\infty)\to[0,\infty)$ defined as

$$f(t)=t^{p-1}, \quad g(t)=t^{q-1}, \quad h(t)=t^k e^{-\frac{t^k}{k}}(\ln t)^n$$

If the condition given in (2.7) is true, then the functions f and g are synchronous (asynchronous) on $[0,\infty)$. So, by Chebychev's inequality for the functions f, g and h defined above, we have

$$\int_0^\infty t^k e^{-\frac{t^k}{k}}(\ln t)^n dt \int_0^\infty t^{p-1}t^{q-1}t^k e^{-\frac{t^k}{k}}(\ln t)^n dt$$
$$\geq (\leq) \int_0^\infty t^{p-1}t^k e^{-\frac{t^k}{k}}(\ln t)^n dt \int_0^\infty t^{q-1}t^k e^{-\frac{t^k}{k}}(\ln t)^n dt.$$

This implies taht

$$\int_0^\infty t^{k+1-1} e^{-\frac{t^k}{k}}(\ln t)^n dt \int_0^\infty t^{p+q+k-1-1} e^{-\frac{t^k}{k}}(\ln t)^n dt$$
$$\geq (\leq) \int_0^\infty t^{p+k-1} e^{-\frac{t^k}{k}}(\ln t)^n dt \int_0^\infty t^{q+k-1} e^{-\frac{t^k}{k}} dt. \tag{2.9}$$

Hence from relation (1.6), we get the required inequality (2.8). □

Theorem 2.5. *For $k>0$, if $p,q>0$ are similarly (oppositely) unitary, then*

$$\Gamma_k^{(n)}(k)\Gamma_k^{(n)}(p+q+k) \geq (\leq) \Gamma_k^{(n)}(p+k)\Gamma_k^{(n)}(q+k). \tag{2.10}$$

Proof. For $k>0$, consider the mappings $f,g,h:[0,\infty)\to[0,\infty)$ defined as

$$f(t)=t^p, \quad g(t)=t^q, \quad h(t)=t^{k-1}e^{-\frac{t^k}{k}}(\ln t)^n.$$

If the conditions of previous theorem hold, then the functions f and g are synchronous (asynchronous) on $[0,\infty)$. So, by following the steps of previous theorem for the functions f, g and h defined above, we have the required inequality (2.10). □

Theorem 2.6. *For $k>0$, a given real $s>0$ and a non-negative integer n, consider the mapping*

$$\xi_{s,n,k}(x) = \frac{\Gamma_k^{(n)}(x+s)}{\Gamma_k^{(n)}(s)}. \tag{2.11}$$

Show that the mapping $\xi_{s,n,k}(.)$ is supermultiplicative on $[0,\infty)$, in the sense

$$\xi_{s,n,k}(x+y) \geq \xi_{s,n,k}(x)\xi_{s,n,k}(y). \tag{2.12}$$

Proof. Put $u=x+y+s$ and $v=y$, then condition (2.3) becomes

$$xy \geq 0, \tag{2.13}$$

and the inequality (2.4) takes the form

$$\Gamma_k^{(n)}(s)\Gamma_k^{(n)}(x+y+s) \geq \Gamma_k^{(n)}(x+s)\Gamma_k^{(n)}(y+s), \tag{2.14}$$

which is equivalent to (2.12). □

3. An inequality via Hölder's inequality

In this section, we prove an inequality involving the derivative of gamma k-function, which shows the logarithmic convexity of $\Gamma_k^{(n)}$. The well known Hölder's inequality is defined as follows.
Let p and q be two positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, and let $f,g:I\subseteq R\to R$ satisfying

$$\int_I |f(t)|^p dt, \int_I |g(t)|^q dt \leq \infty, \tag{3.1}$$

then

$$\left|\int_I f(t)g(t)dt\right| \leq \left(\int_I |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_I |g(t)|^q dt\right)^{\frac{1}{q}}. \tag{3.2}$$

Theorem 3.1. *For $k>0$, let $a,b\geq 0$ with $a+b=1$. Also let $x,y>0$ be real numbers, then for a non-negative integer n, we have*

$$\Gamma_k^{(n)}(ax+by) \leq \left[\Gamma_k^{(n)}(x)\right]^a \left[\Gamma_k^{(n)}(y)\right]^b. \tag{3.3}$$

Proof. Use the followin mappings

$$f(t)=t^{a(x-1)}\left(e^{-\frac{t^k}{k}}(\ln t)^n\right)^a, \quad g(t)=t^{b(y-1)}\left(e^{-\frac{t^k}{k}}(\ln t)^n\right)^b, \quad t\in(0,\infty)$$

in inequality (3.2) (for $I=(0,\infty)$, $p=\frac{1}{a}$ and $q=\frac{1}{b}$), we get the required inequality (3.3). □

4. Inequalities via Grüss inequality.

In this section, we prove the Grüss type inequalities [2] involving beta and gamma k-functions. The Grüss inequality was given by Grüss in [3]. This inequality actually connects the integral of the product of two functions with the product of their integrals. The following interpolation of Grüss inequality is well known (see [10]).

Lemma 4.1. *Let f, g and h be integrable functions defined on $[a,b]$ and φ, ϕ, ψ and Ψ are given constants such that*

$$\varphi \leq f(t) \leq \phi, \quad \psi \leq g(t) \leq \Psi, \quad \forall t\in[a,b],$$

then

$$|D(f,g;h)| \leq [D(f,f;h)]^{\frac{1}{2}}[D(g,g;h)]^{\frac{1}{2}} \leq \frac{1}{4}(\phi-\varphi)(\Psi-\psi)\left[\int_a^b h(t)dt\right]^2, \tag{4.1}$$

where

$$D(f,g;h)=\int_a^b h(t)dt\int_a^b h(t)f(t)g(t)dt-\int_a^b h(t)f(t)dt\int_a^b h(t)g(t)dt. \tag{4.2}$$

Theorem 4.2. *For $k>0$, let $p,q,c,d>0$, then we have*

$$\begin{aligned}
&\beta_k(c,d)\beta_k(c+p,d+q)-\beta_k(c+p,d)\beta_k(c,d+q)\\
&\quad\leq \left[\beta_k(c,d)\beta_k(c+2p,d)-(\beta_k(c+p,d)^2)\right]^{\frac{1}{2}}\\
&\qquad\times\left[\beta_k(c,d)\beta_k(c,d+2q)-(\beta_k(c,d+q)^2)\right]^{\frac{1}{2}}\\
&\qquad\leq \frac{1}{4}\left[\beta_k(c,d)\right]^2.
\end{aligned} \tag{4.3}$$

Proof. Set the following mappings

$$f(t) = t^{\frac{p}{k}}, \quad g(t) = (1-t)^{\frac{q}{k}}, \quad h(t) = t^{\frac{c}{k}-1}(1-t)^{\frac{d}{k}-1},$$

$a = 0$, $b = 1$, $\varphi = \psi = 0$ and $\phi = \Psi = 1$ in lemma 4.1, we get the desired double inequality (4.3). □

Corollary 4.3. *For $c = d = k$, the inequality (4.3) of previous theorem reduces to*

$$\begin{aligned} &\beta_k(p+k, q+k) - \frac{k}{(p+k)(q+k)} \\ &\leq \frac{pq}{(p+k)(q+k)\sqrt{(2p+k)(2q+k)}} \\ &\leq \frac{1}{4k}. \end{aligned} \tag{4.4}$$

Proof. It can be proved by using

$$\beta_k(k,k) = \frac{1}{k}, \quad \beta_k(x,k) = \frac{1}{x}, \quad \beta_k(k,y) = \frac{1}{y},$$

and relations (1.4) and (1.5) in previous theorem. □

Theorem 4.4. *For $k > 0$, let $m, n, p, q, c, d > 0$. Then we have*

$$\begin{aligned} &\beta_k(c,d)\beta_k(c+m+p, d+n+q) - \beta_k(c+m, d+n)\beta_k(c+p, d+q) \\ &\leq \left[\beta_k(c,d)\beta_k(c+2m, d+2n) - (\beta_k(c+m, d+n))^2\right]^{\frac{1}{2}} \\ &\times \left[\beta_k(c,d)\beta_k(c+2p, d+2q) - (\beta_k(c+p, d+q))^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}\frac{m^{\frac{m}{k}} n^{\frac{n}{k}}}{(m+n)^{\frac{m+n}{k}}}\frac{p^{\frac{p}{k}} q^{\frac{q}{k}}}{(p+q)^{\frac{p+q}{k}}}\left[\beta_k(c,d)\right]^2. \end{aligned} \tag{4.5}$$

Proof. Set

$$f(t) = t^{\frac{m}{k}}(1-t)^{\frac{n}{k}}, \quad g(t) = t^{\frac{p}{k}}(1-t)^{\frac{q}{k}}, \quad h(t) = t^{\frac{c}{k}-1}(1-t)^{\frac{d}{k}-1},$$

$a = 0$ and $b = 1$ in relation (4.2), we have

$$D(f,g;h) = k^2\left[\beta_k(c,d)\beta_k(c+m+p, d+n+q) - \beta_k(c+m, d+n)\beta_k(c+p, d+q)\right],$$

$$D(f,f;h) = k^2\left[\beta_k(c,d)\beta_k(c+2m, d+2n) - (\beta_k(c+m, d+n))^2\right],$$

$$D(g,g;h) = k^2\left[\beta_k(c,d)\beta_k(c+2p, d+2q) - (\beta_k(c+p, d+q))^2\right].$$

Now for the application of Grüss inequality, we have to find ϕ and Ψ, the maximum of f and g respectively.
Let

$$I_{m,n}(t) = t^{\frac{m}{k}}(1-t)^{\frac{n}{k}},$$

and

$$I_{p,q}(t) = t^{\frac{p}{k}}(1-t)^{\frac{q}{k}}.$$

Then

$$\frac{dI_{m,n}(t)}{dt} = \frac{1}{k}t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}[m(1-t) - nt].$$

From here, we observe that the unique solution of $I'_{m,n}(t) = 0$ in $(0,1)$ is

$$t_0 = \frac{m}{m+n}.$$

Also since $I'_{m,n}(t) > 0$ on $(0, t_0)$ and $I'_{m,n}(t) < 0$ on $(t_0, 1)$, therefore t_0 is its maximum point in $(0, 1)$.
Consequently,

$$\varphi = \inf_{t\in[0,1]} I_{m,n}(t) = 0$$

and

$$\phi = \sup_{t\in[0,1]} I_{m,n}(t) = I_{m,n}(\frac{m}{m+n}) = \frac{m^{\frac{m}{k}} n^{\frac{n}{k}}}{(m+n)^{\frac{m+n}{k}}}.$$

By following the same steps for $I_{p,q}(t)$, we can find $\psi = 0$ and

$$\Psi = \frac{p^{\frac{p}{k}} q^{\frac{q}{k}}}{(p+q)^{\frac{p+q}{k}}}.$$

So by using all values in inequality (4.1), we get the required double inequality (4.5). □

Theorem 4.5. *For $k > 0$, let $p, q, r, u, v, w > 0$, then we have*

$$\left|\frac{\Gamma_k(p+q+r)\Gamma_k(c)}{(u+v+w)^{\frac{p+q+r}{k}} w^{\frac{r}{k}}} - \frac{\Gamma_k(p+r)\Gamma_k(q+r)}{(u+w)^{\frac{p+r}{k}}(v+w)^{\frac{q+r}{k}}}\right|$$
$$\leq \left[\frac{\Gamma_k(2p+r)\Gamma_k(r)}{(2u+w)^{\frac{2p+r}{k}} w^{\frac{r}{k}}} - \frac{(\Gamma_k(p+r))^2}{(u+w)^{\frac{2(p+r)}{k}}}\right]^{\frac{1}{2}}$$
$$\times \left[\frac{\Gamma_k(2q+r)\Gamma_k(r)}{(2v+w)^{\frac{2q+r}{k}} w^{\frac{r}{k}}} - \frac{(\Gamma_k(q+r))^2}{(u+w)^{\frac{2(q+r)}{k}}}\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left(\frac{p}{ue}\right)^{\frac{p}{k}}\left(\frac{q}{ve}\right)^{\frac{q}{k}}\frac{(\Gamma_k(r))^2}{w^{\frac{2r}{k}}}. \tag{4.6}$$

Proof. Consider the mapping

$$f_{p,u}(t) = t^p e^{-u\frac{t^k}{k}} \tag{4.7}$$

on $(0, \infty)$. Then

$$f'_{p,u}(t) = t^{p-1} e^{-u\frac{t^k}{k}}(p - ut^k). \tag{4.8}$$

It shows that $f_{p,u}$ is increasing on $(0, (\frac{p}{u})^{\frac{1}{k}})$ and decreasing on $((\frac{p}{u})^{\frac{1}{k}}, \infty)$. So maximal value of $f_{p,u}$ is

$$f_{p,u}\left(\left(\frac{p}{u}\right)^{\frac{1}{k}}\right) = \left(\frac{p}{ue}\right)^{\frac{p}{k}} \tag{4.9}$$

Similarly, we can find the maximal value of $f_{q,v}$. Now set

$$f(t) = f_{p,u}(t), \quad g(t) = f_{q,v}(t), \quad h(t) = f_{r-1,w}(t), \tag{4.10}$$

$a = 0$, $b \to \infty$, $\phi = (\frac{p}{ue})^{\frac{p}{k}}$ and $\Psi = (\frac{q}{ve})^{\frac{q}{k}}$ in inequality (4.1), we get the required double inequality (4.6). □

Corollary 4.6. *Using $u = v = w = k$ in inequality (4.6), we get*

$$\left|\frac{\Gamma_k(p+q+r)\Gamma_k(r)}{3^{\frac{p+q+r}{k}}k^{\frac{p+q+2r}{k}}} - \frac{\Gamma_k(p+r)\Gamma_k(q+r)}{(2k)^{\frac{p+q+2r}{k}}}\right|$$
$$\leq \left[\frac{\Gamma_k(2p+r)\Gamma_k(r)}{3^{\frac{2p+r}{k}}k^{\frac{r}{k}}} - \frac{(\Gamma_k(p+r))^2}{(2k)^{\frac{2(p+r)}{k}}}\right]^{\frac{1}{2}}$$
$$\times \left[\frac{\Gamma_k(2q+r)\Gamma_k(r)}{3^{\frac{2q+r}{k}}k^{\frac{2(q+r)}{k}}} - \frac{(\Gamma_k(q+r))^2}{(2k)^{\frac{2(q+r)}{k}}}\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left(\frac{p}{ue}\right)^{\frac{p}{k}}\left(\frac{q}{ve}\right)^{\frac{q}{k}}\frac{(\Gamma_k(r))^2}{k^{\frac{2r}{k}}}. \tag{4.11}$$

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