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NEW CONNECTION FORMULAE BETWEEN $(p,q)-$FIBONACCI POLYNOMIALS AND CERTAIN JACOBI POLYNOMIALS

WALEED ABD-ELHAMEED

Abstract. The main purpose of this article is to solve the connection problems between $(p,q)-$Fibonacci polynomials and the two polynomials, namely Chebyshev polynomials of third and fourth kinds which are considered as two nonsymmetric polynomials of the Jacobi polynomials. Moreover, the inversion connection formulae for the latter formulae are given. We show that all the connection coefficients are expressed in terms of hypergeometric functions of the type ${}_2F_1$ of certain arguments which can be summed in some cases. As special cases of the derived formulae, the connection formulae between Fibonacci, Pell, Fermat, second kind Chebyshev, second kind Dickson polynomials and Chebyshev polynomials of third and fourth kinds are deduced. Moreover, as applications of the introduced connection formulae, some new expressions for the celebrated numbers of Fibonacci, Pell, Mersenne and Fermat numbers and their derivatives sequences are presented. As another application of the derived connection formulae, some new definite integrals involving products of $(p,q)-$Fibonacci polynomials and Chebyshev polynomials of third and fourth kinds are given.

1. Introduction

Hypergeometric functions are crucial in mathematical analysis and its applications. These functions appear in many important problems related to special functions. There are old and recent studies concerned with one-variable hypergeometric functions. They appear in the work of Euler, Gauss, Riemann, and Kummer. Their integral representations were studied by Barnes and Mellin, and special properties of them by Schwarz and Goursat. Several special functions are represented by the Gaussian or ordinary hypergeometric function ${}_2F_1(a,b;c;z)$ as specific or limiting cases. For some studies on hypergeometric functions and their various transformations, see for example [4, 6]. Moreover, hypergeometric functions appear in linearization, duplication and connection problems. In fact, almost all linearization, duplication and connection coefficients are expressed in terms of several kinds of hypergeometric functions [1, 2, 7, 21, 25].

Many number and polynomial sequences can be generated by recurrence relations of order two. Among these sequences, Fibonacci, Lucas, Pell and Fermat.

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It is well-known that these sequences of polynomials and numbers have prominent roles in mathematics. In fact, they are of fundamental importance in combinatorics, number theory and numerical analysis, see for example [17, 18]. Recently, the studies concerning different generalizations of Fibonacci and Lucas polynomials and their related numbers have attracted a number of authors, see for example [26, 16, 24, 14, 15, 28, 27].

The Jacobi polynomials play distinguished roles in theoretical analysis as well as applied analysis. This class of polynomials contains six well-known subclasses. The Legendre, ultraspherical and Chebyshev polynomials of the first and second kinds are symmetric Jacobi polynomials, while the two special polynomials, namely Chebyshev polynomials of third and fourth kinds are nonsymmetric Jacobi polynomials. Several theoretical and practical studies concerning these polynomials have been done by several authors, see for example [11, 10, 12].

The connection coefficients play an important part in many problems in several disciplines such as applied mathematics and mathematical physics. Many authors are interested in finding closed formulae for the solutions of various connection problems. For some articles in this direction, one can be referred to Area et al. [5], Doha [8, 9], Doha and Ahmed [13], Maroni and Rocha [19], Sánchez-Ruiz [22], and Szwarc [23].

Given two sets of polynomials $\{A_i(x)\}_{i\geq 0}$ and $\{B_j(x)\}_{j\geq 0}$, the so-called connection problem between these polynomials is to find the coefficients $C_{i,j}$ in the expression

$$A_i(x)=\sum_{j=0}^{i} C_{i,j}\, B_j(x).$$

In fact, the connection coefficients $C_{i,j}$ are often expressed in terms of terminating hypergeometric series of various types. For example, the connection coefficients in case of Jacobi-Jacobi connection problem is expressed in terms of a terminating hypergeometric series of the type ${}_3F_2(1)$, see [9]. Recently, Abd-Elhameed et al. in [3] have established new connection formulae between Chebyshev polynomials of the first and second kinds and Fibonacci polynomials. These formulae involve terminating hypergeometric functions of the type ${}_2F_1$ of certain arguments.

Our goal in this article is to solve the connection problems between (p,q)-Fibonacci polynomials and the two orthogonal polynomials, namely Chebyshev polynomials of third and fourth kinds $V_n(x)$ and $W_n(x)$, then deduce several connection formulae between some special polynomials of $(p,q)-$ Fibonacci polynomials, and in in particular between Fibonacci, Pell, Fermat, second kind Chebyshev, second kind Dickson polynomials and the polynomials $V_n(x)$ and $W_n(x)$. The connection coefficients of the latter formulae involve terminating hypergeometric series ${}_2F_1$ of certain arguments. Based on the new derived connection formulae, we exhibit several identities expressing the celebrated numbers of Fibonacci, Pell, Mersenne and Fermat and also their derivatives sequences. Moreover, we give new expressions for some weighted definite integrals of products of $(p,q)-$Fibonacci polynomials and Chebyshev polynomials of third and fourth kinds. To the best of our knowledge, all the solutions of the connection problems between $(p,q)-$Fibonacci polynomials and Chebyshev polynomials of third and fourth kinds are new and also the presented applications for these formulae are traceless in literature. This gives us a motivation for establishing these formulae.

The paper is organized as follows. In Section 2, an overview on $(p,q)-$Fibonacci polynomials and Chebyshev polynomials of third and fourth kinds is presented. Section 3 is interested in stating and proving two important theorems in which Chebyshev polynomials of third and fourth kinds are linked with $(p,q)-$Fibonacci polynomials. Moreover, some special connection formulae are exhibited. The inversion formulae of those obtained in Section 3 are displayed in Section 4. Section 5 is interested in presenting three applications based on the results obtained in Sections 3 and 4. In the first application, we give some new expressions involving the celebrated Fibonacci, Pell, Mersenne and Fermat numbers, while in the second, some identities of the derivatives sequences of the latter numbers are given. Finally, we give some definite integrals involving certain products of Chebyshev polynomials of third and fourth kinds and $(p,q)-$Fibonacci polynomials.

2. Preliminaries and some Fundamental properties

In this section, we present some preliminaries and properties of $(p,q)-$Fibonacci polynomials. Also, some relevant properties of Chebyshev polynomials of third and fourth kinds are given.

2.1. **Some relevant properties of $(p,q)-$Fibonacci polynomials.** Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The $(p,q)-$Fibonacci polynomials are generated with the aid of the recurrence relation

$$\phi_i(x) = p(x)\,\phi_{i-1}(x) + q(x)\,\phi_{i-2}(x), \tag{2.1}$$

with the initial values: $\phi_0(x) = 0$ and $\phi_1(x) = 1$.
Note that $\phi_i(x)$ is of degree $(i-1)$. One of the fundamental properties of $\phi_i(x)$ is its power form representation (see [26])

$$\phi_i(x) = \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} \binom{i-k-1}{k} \, (p(x))^{i-2k-1} \, (q(x))^k \,,$$

where the notation $\lfloor z \rfloor$ denotes the largest integer less than or equal to z.
In this paper, we restrict $p(x)$ and $q(x)$ to take the values $p(x) = p\,x$ and $q(x) = q$, where p,q are real constants and denote the corresponding $(p,q)-$Fibonacci polynomials of degree i by $\bar{\phi}_i^{p,q}(x)$. Thus (2.1) implies that the following recurrence relation is satisfied by $\bar{\phi}_i^{p,q}(x)$:

$$\bar{\phi}_i^{p,q}(x) = p\,x\,\bar{\phi}_{i-1}^{p,q}(x) + q\,\bar{\phi}_{i-2}^{p,q}(x), \tag{2.2}$$

with the initial values: $\bar{\phi}_0^{p,q}(x) = 1$ and $\bar{\phi}_1^{p,q}(x) = p\,x$.
It is worthy to note here that the Fibonacci, Pell, Fermat, second kind Chebyshev, second kind Dickson polynomials can be obtained respectively as special cases of $\bar{\phi}_i^{p,q}(x)$ as follows:

$$F_{i+1}(x) = \bar{\phi}_i^{1,1}(x), \qquad P_{i+1}(x) = \bar{\phi}_i^{2,1}(x),$$
$$\mathcal{F}_{i+1}(x) = \bar{\phi}_i^{3,-2}(x), \qquad U_i(x) = \bar{\phi}_i^{2,-1}(x),$$
$$E_i(x,\alpha) = \bar{\phi}_i^{1,-\alpha}(x).$$

Now, we can define the $(i+1)-$th of Fibonacci, Pell and Mersenne numbers by the relations:

$$\begin{aligned} F_{i+1} &= F_{i+1}(1) = \bar{\phi}_i^{1,1}(1), \\ P_{i+1} &= P_{i+1}(1) = \bar{\phi}_i^{2,1}(1), \\ M_{i+1} &= \mathcal{F}_{i+1}(1) = \bar{\phi}_i^{3,-2}(1). \end{aligned}$$

The corresponding derivatives sequences of the Fibonacci, Pell and Mersenne numbers are denoted respectively by $F_{i+1}^{(r)}, P_{i+1}^{(r)}, M_{i+1}^{(r)}$. They are given explicitly as:

$$\begin{aligned} F_{i+1}^{(r)} &= D^r\, F_{i+1}|_{x=1} = D^r\, \bar{\phi}_i^{1,1}|_{x=1}, \\ P_{i+1}^{(r)} &= D^r\, P_{i+1}|_{x=1} = D^r\, \bar{\phi}_i^{2,1}|_{x=1}, \\ M_{i+1}^{(r)} &= D^r\, M_{i+1}|_{x=1} = D^r\, \bar{\phi}_i^{3,-2}|_{x=1}. \end{aligned}$$

The power form representation of $\bar{\phi}_i^{p,q}(x)$ is:

$$\bar{\phi}_i^{p,q}(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i-k}{k}\, (p\,x)^{i-2k}\, q^k.$$

Another important property is

$$\bar{\phi}_i^{p,q}(-x) = (-1)^i\, \bar{\phi}_i^{p,q}(x). \tag{2.3}$$

2.2. **An overview on Chebyshev polynomials of third and fourth kinds.** Chebyshev polynomials $V_i(x)$ and $W_i(x)$ are polynomials each of degree i, and can be defined by trigonometric functions as (see, Mason and Handscomb [20])

$$V_i(\cos\theta) = \frac{\cos(i+\frac{1}{2})\theta}{\cos\frac{\theta}{2}} \qquad \textit{and} \qquad W_i(\cos\theta) = \frac{\sin(i+\frac{1}{2})\theta}{\sin\frac{\theta}{2}},$$

or as special cases of the two nonsymmetric Jacobi polynomials $P_i^{(\alpha,\beta)}(x)$, namely

$$V_i(x) = \frac{2^{2i}}{\binom{2i}{i}}\, P_i^{(-\frac{1}{2},\frac{1}{2})}(x), \qquad \textit{and} \qquad W_i(x) = \frac{2^{2i}}{\binom{2i}{i}}\, P_i^{(\frac{1}{2},-\frac{1}{2})}(x).$$

It is clear that the polynomials $W_i(x)$ are linked with the polynomials $V_i(x)$ by the relation

$$W_i(x) = (-1)^i\, V_i(-x), \tag{2.4}$$

and hence it is sufficient to establish properties and relations for $V_i(x)$, and then deduce analogous properties for $W_i(x)$ (by replacing x by $-x$).

The polynomials $V_i(x)$ and $W_i(x)$ are orthogonal on $(-1,1)$ with respect to the weight functions $\sqrt{\dfrac{1+x}{1-x}}$ and $\sqrt{\dfrac{1-x}{1+x}}$, respectively, that is

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\; V_i(x)\, V_j(x)\, dx = \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\; W_i(x)\, W_j(x)\, dx = \begin{cases} 0, & i \neq j, \\ \pi, & i = j, \end{cases}$$

and they may be generated by using the two recurrence relations

$$V_i(x) = 2x\, V_{i-1}(x) - V_{i-2}(x), \quad i = 2, 3, \ldots,$$

with the initial values:

$$V_0(x) = 1, \quad V_1(x) = 2x - 1,$$

and

$$W_i(x) = 2x\, W_{i-1}(x) - W_{i-2}(x), \quad i = 2, 3, \ldots,$$

with the initial values:

$$W_0(x) = 1, \quad W_1(x) = 2x + 1.$$

The following special values are of important use later:

$$V_i(1) = 1, \quad W_i(1) = 2i + 1,$$

$$D^r V_i(1) = \frac{\sqrt{\pi}(i+r)!}{2^r\,(i-r)!\;\Gamma(r+\frac{1}{2})}, \quad r \geq 1, \tag{2.5}$$

and

$$D^r W_i(1) = \frac{\sqrt{\pi}(i+r)!}{2^{r+1}\,(i-r)!\;\Gamma(r+\frac{3}{2})}, \quad r \geq 1. \tag{2.6}$$

3. Connection formulae between Chebyshev polynomials of third and fourth kinds and $(p,q)-$Fibonacci polynomials

This section is interested in developing new connection formulae between Chebyshev polynomials of third and fourth kinds and $(p,q)-$Fibonacci polynomials. We prove that these connection formulae are expressed in terms of hypergeometric functions of the type ${}_2F_1(c)$ for certain values of c. The following two lemmas are needed in the sequel.

Lemma 1. *If i and m are two positive integers, and if we let* $A_{i,m} = {}_2F_1\left(\begin{array}{c} -m, i-m+1 \\ i-2m+2 \end{array}\middle| \frac{-4\,q}{p^2}\right)$, *then the following recurrence relation is satisfied by $A_{i,m}$:*

$$\binom{i-m-1}{m-1}\; A_{i-2,m-1} + \frac{4\,q}{p^2}\;\binom{i-m}{m-1}\; A_{i-1,m-1} + \binom{i-m-1}{m}\; A_{i-1,m} = \binom{i-m}{m}\; A_{i,m}. \tag{3.1}$$

Lemma 2. *If i and m are two positive integers, and if we let* $B_{i,m} = {}_2F_1\left(\begin{array}{c} -m, i-m \\ i-2m+1 \end{array}\middle| \frac{-4\,q}{p^2}\right)$, *then the following recurrence relation is satisfied by $B_{i,m}$:*

$$\binom{i-m-1}{m-1}\; B_{i-1,m-1} + \frac{4\,q}{p^2}\;\binom{i-m}{m-1}\; B_{i,m-1} + \binom{i-m-1}{m}\; B_{i,m} = \binom{i-m}{m}\; B_{i+1,m}.$$

Proof. The proofs of the two Lemmas 1 and 2 are similar, so it is sufficient to prove Lemma 1. Noting the definition of $A_{i,m}$, the left hand side of (3.1) can be written as

$$\begin{aligned}
&\binom{i-m-1}{m-1}\; A_{i-2,m-1} + \frac{4\,q}{p^2}\;\binom{i-m}{m-1}\; A_{i-1,m-1} + \binom{i-m-1}{m}\; A_{i-1,m} \\
=&\binom{i-m-1}{m-1}\sum_{k=0}^{m-1} \frac{(1-m)_k\,(i-m)_k}{(2+i-2m)_k\,k!}\left(\frac{-4\,q}{p^2}\right)^k + \frac{4\,q}{p^2}\;\binom{i-m}{m-1}\sum_{k=0}^{m-1} \frac{(1-m)_k\,(1+i-m)_k}{(3+i-2m)_k\,k!}\left(\frac{-4\,q}{p^2}\right)^k \\
&+\binom{i-m-1}{m}\sum_{k=0}^{m} \frac{(-m)_k\,(i-m)_k}{(1+i-2m)_k\,k!}\left(\frac{-4\,q}{p^2}\right)^k.
\end{aligned} \tag{3.2}$$

Relation (3.2) after performing some lengthy manipulations can be simplified to take the form

$$\binom{i-m-1}{m-1} A_{i-2,m-1} + \frac{4\,q}{p^2}\binom{i-m}{m-1} A_{i-1,m-1} + \binom{i-m-1}{m} A_{i-1,m} =$$

$$= \binom{i-m}{m} \sum_{k=0}^{m-1} \frac{(-m)_k\,(1+i-m)_k}{(2+i-2m)_k\,k!}\left(\frac{-4\,q}{p^2}\right)^k + \frac{\left(\frac{4\,q}{p^2}\right)^m \binom{i-m}{m}\,(i-m+1)_m}{(i-2m+2)_m}$$

$$= \binom{i-m}{m} \sum_{k=0}^{m} \frac{(-m)_k\,(1+i-m)_k}{(2+i-2m)_k\,k!}\left(\frac{-4\,q}{p^2}\right)^k = \binom{i-m}{m} A_{i,m}.$$

Lemma 1 is now proved. □

Based on Lemmas 1 and 2, the following connection formula can be obtained.

Theorem 1. *For every nonnegative integer i, the following connection formula holds:*

$$V_i(x) = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \left(\frac{p}{2}\right)^{2m-i} \binom{i-m}{m}\; {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix}\,\middle|\, -\frac{4\,q}{p^2}\right) \bar{\phi}^{p,q}_{i-2m}(x)$$
$$+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \left(\frac{p}{2}\right)^{2m-i+1} \binom{i-m-1}{m}\; {}_2F_1\left(\begin{matrix} i-m, -m \\ i-2m+1 \end{matrix}\,\middle|\, -\frac{4\,q}{p^2}\right) \bar{\phi}^{p,q}_{i-2m-1}(x). \qquad (3.3)$$

Proof. We will proceed by induction. Assume that formula (3.3) holds for all $j < i$, and we have to show that (3.3) itself holds, that is we wish to prove that

$$V_i(x) = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} g_{i,m}\; \bar{\phi}^{p,q}_{i-2m}(x) + \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} d_{i,m}\; \bar{\phi}^{p,q}_{i-2m-1}(x),$$

where

$$g_{i,m} = (-1)^m \left(\frac{p}{2}\right)^{2m-i} \binom{i-m}{m} {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix}\,\middle|\, -\frac{4\,q}{p^2}\right),$$

and

$$d_{i,m} = (-1)^{m+1} \left(\frac{p}{2}\right)^{2m-i+1} \binom{i-m-1}{m} {}_2F_1\left(\begin{matrix} i-m, -m \\ i-2m+1 \end{matrix}\,\middle|\, -\frac{4\,q}{p^2}\right).$$

We start from the recurrence relation satisfied by $V_i(x)$

$$V_i(x) = 2\,x\,V_{i-1}(x) - V_{i-2}(x),$$

and apply the induction hypothesis twice to $V_{i-1}(x)$ and $V_{i-2}(x)$ to obtain

$$V_i(x) = 2\,x\left\{\sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} g_{i-1,m}\; \bar{\phi}^{p,q}_{i-2m-1}(x) + \sum_{m=0}^{\lfloor \frac{i}{2}-1 \rfloor} d_{i-1,m}\; \bar{\phi}^{p,q}_{i-2m-2}(x)\right\}$$
$$-\left\{\sum_{m=0}^{\lfloor \frac{i}{2}-1 \rfloor} g_{i-2,m}\; \bar{\phi}^{p,q}_{i-2m-2}(x) + \sum_{m=0}^{\lfloor \frac{i-3}{2} \rfloor} d_{i-2,m}\; \bar{\phi}^{p,q}_{i-2m-3}(x)\right\}. \qquad (3.4)$$

Substitution of the recurrence relation (2.2) in the form

$$x\,\bar{\phi}_i^{p,q}(x)=\frac{1}{p}\bar{\phi}_{i+1}^{p,q}(x)-\frac{q}{p}\bar{\phi}_{i-1}^{p,q}(x),$$

into (3.4) and performing some manipulations enable one to express $V_i(x)$ in the form

$$V_i(x)=\sum_1+\sum_2,$$

where

$$\sum_1=-\frac{2\,q}{p}\sum_{m=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor} g_{i-1,m}\,\bar{\phi}_{i-2m-2}^{p,q}+\frac{2}{p}\sum_{m=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor} g_{i-1,m}\,\bar{\phi}_{i-2m}^{p,q}-\sum_{m=0}^{\left\lfloor\frac{i}{2}\right\rfloor-1} g_{i-2,m}\,\bar{\phi}_{i-2m-2}^{p,q},$$

and

$$\sum_2=-\frac{2\,q}{p}\sum_{m=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor} d_{i-1,m}\,\bar{\phi}_{i-2m-3}^{p,q}+\frac{2}{p}\sum_{m=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor} d_{i-1,m}\,\bar{\phi}_{i-2m-1}^{p,q}-\sum_{m=0}^{\left\lfloor\frac{i}{2}\right\rfloor-1} d_{i-2,m}\,\bar{\phi}_{i-2m-3}^{p,q}.$$

In fact, the latter two sums can be written in the following two alternative forms

$$\begin{aligned}\sum_1=&\sum_{m=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor}\left(\frac{-2\,q}{p}\,g_{i-1,m-1}+\frac{2}{p}\,g_{i-1,m}-g_{i-2,m-1}\right)\bar{\phi}_{i-2m}^{p,q}\\&-\frac{2\,q}{p}\,g_{i-1,\left\lfloor\frac{i-1}{2}\right\rfloor}\,\bar{\phi}_{i-2\left\lfloor\frac{i+1}{2}\right\rfloor}^{p,q}-g_{i-2,\left\lfloor\frac{i}{2}\right\rfloor-1}\,\bar{\phi}_{i-2\left\lfloor\frac{i}{2}\right\rfloor}^{p,q}\,\theta_i,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\sum_2=&\sum_{m=0}^{\left\lfloor\frac{i}{2}-1\right\rfloor}\left(\frac{-2\,q}{p}d_{i-1,m-1}+\frac{2}{p}d_{i-1,m}-d_{i-2,m-1}\right)\bar{\phi}_{i-2m-1}^{p,q}\\&-\frac{2\,q}{p}\,d_{i-1,\left\lfloor\frac{i}{2}\right\rfloor-1}\,\bar{\phi}_{i-2\left\lfloor\frac{i}{2}\right\rfloor-1}^{p,q}-d_{i-2,\left\lfloor\frac{i-1}{2}\right\rfloor-1}\,\bar{\phi}_{i-2\left\lfloor\frac{i-1}{2}\right\rfloor-1}^{p,q}\,\theta_{i+1},\end{aligned}\tag{3.6}$$

where

$$\theta_i=\begin{cases}1, & \text{if i even,}\\ 0, & \text{if i odd.}\end{cases}\tag{3.7}$$

Based on Lemmas 1 and 2, it can be shown that the following two identities hold:

$$\frac{-2\,q}{p}g_{i-1,m-1}+\frac{2}{p}\,g_{i-1,m}-g_{i-2,m-1}=g_{i,m},$$

and

$$\frac{-2\,q}{p}d_{i-1,m-1}+\frac{2}{p}\,d_{i-1,m}-d_{i-2,m-1}=d_{i,m}.$$

If we make use of the two latter identities along with performing some manipulations, then it can be shown that the two sums in (3.5) and (3.6) can be simplified to give

$$\sum_1=\sum_{m=0}^{\left\lfloor\frac{i}{2}\right\rfloor} g_{i,m}\;\bar{\phi}_{i-2m}^{p,q}(x),$$

and

$$\sum_2=\sum_{m=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor} d_{i,m}\;\bar{\phi}_{i-2m-1}^{p,q}(x).$$

This completes the proof of Theorem 1. □

Theorem 1 along with the two identities (2.3) and (2.4) lead to the following connection formula between Chebyshev polynomials of fourth kind $W_i(x)$ and $\bar{\phi}_i^{p,q}(x)$.

Theorem 2. *For every nonnegative integer i, the following connection formula holds:*

$$\begin{aligned} W_i(x) = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \left(\frac{p}{2}\right)^{2m-i} \binom{i-m}{m} {}_2F_1\left(\left. \begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix} \right| -\frac{4\,q}{p^2} \right) \bar{\phi}_{i-2m}^{p,q}(x) \\ + \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^m \left(\frac{p}{2}\right)^{2m-i+1} \binom{i-m-1}{m} {}_2F_1\left(\left. \begin{matrix} i-m, -m \\ i-2m+1 \end{matrix} \right| -\frac{4\,q}{p^2} \right) \bar{\phi}_{i-2m-1}^{p,q}(x). \end{aligned} \tag{3.8}$$

The following connection formulae between Chebyshev polynomials of third kind and the polynomials, namely second kind Chebyshev, Fibonacci, Pell, Fermat, second kind Dickson polynomials can be obtained as special cases of Theorem 1. These results are given in the following corollaries.

Corollary 1. *If we set $p = 2, q = -1$ in the connection formula* (3.3)*, then we get*

$$V_i(x) = U_i(x) - U_{i-1}(x), \tag{3.9}$$

Proof. The substitution of $p = 2, q = -1$ in the connection formula (3.3) yields

$$\begin{aligned} V_i(x) = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \binom{i-m}{m} \; {}_2F_1\left(\left. \begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix} \right| 1 \right) U_{i-2m}(x) \\ + \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \binom{i-m-1}{m} \; {}_2F_1\left(\left. \begin{matrix} i-m, -m \\ i-2m+1 \end{matrix} \right| 1 \right) U_{i-2m-1}(x). \end{aligned} \tag{3.10}$$

The two ${}_2F_1(1)$ which appear in (3.10) can be easily reduced by Chu-Vandermonde identity to give

$${}_2F_1\left(\left. \begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix} \right| 1 \right) = {}_2F_1\left(\left. \begin{matrix} i-m, -m \\ i-2m+1 \end{matrix} \right| 1 \right) = \begin{cases} 1, & m = 0, \\ 0, & \textit{otherwise}, \end{cases}$$

and therefore formula (3.10) reduces to the following simple formula

$$V_i(x) = U_i(x) - U_{i-1}(x).$$

□

Remark 1. *As a result of formula* (3.9) *and based on the identity* (2.4) *and the identity*

$$U_i(-x) = (-1)^i \, U_i(x), \tag{3.11}$$

it is easy to deduce the connection formula

$$W_i(x) = U_i(x) + U_{i-1}(x). \tag{3.12}$$

Remark 2. *The two relations* (3.9) *and* (3.12) *are in complete agreement with those given in Mason and Handscomb* [20] *(Eqs.(1.17) and (1.18)).*

Corollary 2. *If we set* $p = q = 1$ *in the connection formula* (3.3)*, then we get*

$$\begin{aligned} V_i(x) = &\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \, 2^{i-2m} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix} \middle| -4 \right) F_{i-2m+1}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \, 2^{i-2m-1} \binom{i-m-1}{m} \; {}_2F_1 \left(\begin{matrix} i-m, -m \\ i-2m+1 \end{matrix} \middle| -4 \right) F_{i-2m}(x). \end{aligned} \tag{3.13}$$

Corollary 3. *If we set* $p = 2, q = 1$ *in the connection formula* (3.3)*, then we get*

$$\begin{aligned} V_i(x) = &\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \binom{i-m}{m} \; {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix} \middle| -1 \right) P_{i-2m+1}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \binom{i-m-1}{m} \; {}_2F_1 \left(\begin{matrix} i-m, -m \\ i-2m+1 \end{matrix} \middle| -1 \right) P_{i-2m}(x). \end{aligned} \tag{3.14}$$

Corollary 4. *If we set* $p = 3, q = -2$ *in the connection formula* (3.3)*, then we get*

$$\begin{aligned} V_i(x) = &\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \; \left(\tfrac{3}{2}\right)^{2m-i} \binom{i-m}{m} \; {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix} \middle| \frac{8}{9} \right) \mathcal{F}_{i-2m+1}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \; \left(\tfrac{3}{2}\right)^{2m-i+1} \; \binom{i-m-1}{m} \; {}_2F_1 \left(\begin{matrix} i-m, -m \\ i-2m+1 \end{matrix} \middle| \frac{8}{9} \right) \mathcal{F}_{i-2m}(x). \end{aligned} \tag{3.15}$$

Corollary 5. *If we set* $p = 1, q = -\alpha$ *in the connection formula* (3.3)*, then we get*

$$\begin{aligned} V_i(x) = &\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \, 2^{i-2m} \; \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix} \middle| 4\,\alpha \right) E_{i-2m}(x, \alpha) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \, 2^{i-2m-1} \; \binom{i-m-1}{m} {}_2F_1 \left(\begin{matrix} i-m, -m \\ i-2m+1 \end{matrix} \middle| 4\,\alpha \right) E_{i-2m-1}(x, \alpha). \end{aligned} \tag{3.16}$$

Remark 3. *It is worthy to mention here that the connection formula* (3.13) *can be obtained as a direct spacial case of the connection formula* (3.16) *bey setting* $\alpha = -1$.

Remark 4. *The counterparts of the connection formulae* (3.13)-(3.16) *involving* $W_i(x)$ *can be easily deduced as special cases of the connection formula* (3.8).

4. Inversion formulae to the connection formulae (3.3) and (3.8)

In this section we are interested in developing the inversion formulae to those derived in Section 3. The proofs of the these formulae are similar to those obtained in Section 3, so they are omitted. Again the connection coefficients turn out to be expressions involving hypergeometric functions of the type ${}_2F_1$ for certain arguments.

Theorem 3. *For every nonnegative integer i, the following connection formula holds:*

$$\begin{aligned}\bar{\phi}_i^{p,q}(x) = &\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \left(\frac{p}{2}\right)^{i-2m} q^m \binom{i-m}{m} {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, -\frac{p^2}{4q}\right) V_{i-2m}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} \left(\frac{p}{2}\right)^{i-2m} q^m \binom{i-m}{m} {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, -\frac{p^2}{4q}\right) V_{i-2m-1}(x).\end{aligned} \tag{4.1}$$

Theorem 4. *For every nonnegative integer i, the following connection formula holds:*

$$\begin{aligned}\bar{\phi}_i^{p,q}(x) = &\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \left(\frac{p}{2}\right)^{i-2m} q^m \binom{i-m}{m} {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, -\frac{p^2}{4q}\right) W_{i-2m}(x) \\ &- \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} \left(\frac{p}{2}\right)^{i-2m} q^m \binom{i-m}{m} {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, -\frac{p^2}{4q}\right) W_{i-2m-1}(x).\end{aligned} \tag{4.2}$$

The following connection formulae can be deduced as special cases of Theorem 3.

Corollary 6. *If we set $p = 2, q = -1$ in relation* (4.1), *then the following connection formula is obtained:*

$$U_i(x) = \sum_{m=0}^{i} V_m(x). \tag{4.3}$$

Proof. The substitution of $p = 2, q = -1$ in the connection formula (4.1) yields

$$\begin{aligned}U_i(x) = &\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \binom{i-m}{m} \, {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, 1\right) V_{i-2m}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^m \binom{i-m}{m} \, {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, 1\right) V_{i-2m-1}(x).\end{aligned} \tag{4.4}$$

The ${}_2F_1(1)$ in the latter formula can be easily summed by Chu-Vandermonde identity to give

$${}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, 1\right) = \frac{(-1)^m}{\binom{i-m}{m}},$$

and so (4.4) can be written in the following reduced form

$$U_i(x) = \sum_{m=0}^{i} V_m(x). \tag{4.5}$$

□

Remark 5. *As a result of formula* (4.3) *and based on the two identities* (3.11) *and* (2.4), *it is easy to deduce the connection formula*

$$U_i(x) = \sum_{m=0}^{i} (-1)^{i+m} W_m(x). \tag{4.6}$$

Remark 6. *It should be noted here that the four connection formulae* (3.9),(3.12),(4.5) *and* (4.6) *can also follow from their trigonometric representations.*

Corollary 7. *If we set* $p=1, q=1$ *in relation* (4.1)*, then the following connection formula is obtained:*

$$\begin{aligned} F_{i+1}(x) &= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} 2^{2m-i} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, -\frac{1}{4} \right) V_{i-2m}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} 2^{2m-i} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, -\frac{1}{4} \right) V_{i-2m-1}(x). \end{aligned} \tag{4.7}$$

Corollary 8. *If we set* $p=2, q=1$ *in relation* (4.1)*, then the following connection formula is obtained:*

$$\begin{aligned} P_{i+1}(x) &= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, -1 \right) V_{i-2m}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, -1 \right) V_{i-2m-1}(x). \end{aligned} \tag{4.8}$$

Corollary 9. *If we set* $p=3, q=-2$ *in relation* (4.1)*, then the following connection formula is obtained:*

$$\begin{aligned} \mathcal{F}_{i+1}(x) &= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m\, 2^{3m-i}\, 3^{i-2m} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, \frac{9}{8} \right) V_{i-2m}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^m\, 2^{3m-i}\, 3^{i-2m} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, \frac{9}{8} \right) V_{i-2m-1}(x). \end{aligned} \tag{4.9}$$

Corollary 10. *If we set* $p=1, q=-\alpha$ *in relation* (4.1)*, then the following connection formula is obtained:*

$$\begin{aligned} E_i(x,\alpha) &= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-\alpha)^m\, 2^{2m-i} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, \frac{1}{4\,\alpha} \right) V_{i-2m}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-\alpha)^m\, 2^{2m-i} \binom{i-m}{m} {}_2F_1 \left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, \frac{1}{4\,\alpha} \right) V_{i-2m-1}(x). \end{aligned} \tag{4.10}$$

Remark 7. *It is worthy to mention here that the connection formula* (4.7) *can be obtained as a spacial case of the connection formula* (4.10) *bey setting* $\alpha=-1$*.*

Remark 8. *The counterparts of the connection formulae* (4.7)-(4.10) *involving* $W_i(x)$ *can be easily deduced as special cases of the connection formula* (4.2)*.*

5. Some applications to the derived connection formulae

the main aim of this section can be summarized in the following threefold:

- Introducing new expressions involving the celebrated numbers of Fibonacci, Pell, Mersenne and Fermat.
- Introducing some new expressions for the derivatives sequences of the above numbers.
- Evaluating some definite integrals involving certain products of $(p,q)-$Fibonacci polynomials and Chebyshev polynomials of third and fourth kinds.

5.1. **New $(p,q)-$Fibonacci numbers identities.** This section is interested in introducing new expressions involving $(p,q)-$Fibonacci numbers. More precisely, we give new expressions for Fibonacci, Pell, Mersenne and Fermat numbers. The following results are direct consequences of Corollaries 2-4 and Corollaries 7-9.

Corollary 11. *Let i be a nonnegative integer. The following two identities hold for Fibonacci numbers:*

$$F_{i+1} = {}_2F_1\left(\begin{array}{c} \frac{-i}{2}, 1+\frac{i}{2} \\ 1 \end{array}\middle| -\frac{1}{4}\right)\theta_i + \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} 2^{2m-i+1}\binom{i-m}{m}\, {}_2F_1\left(\begin{array}{c} i-m+1, -m \\ i-2m+1 \end{array}\middle| -\frac{1}{4}\right),$$

θ_i is as defined in (3.7),
and

$$\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor}(-1)^m\, 2^{i-2m}\binom{i-m}{m} {}_2F_1\left(\begin{array}{c} i-m+1, -m \\ i-2m+2 \end{array}\middle| -4\right) F_{i-2m+1}(x)$$
$$+\sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor}(-1)^{m+1}\, 2^{i-2m-1}\binom{i-m-1}{m} {}_2F_1\left(\begin{array}{c} i-m, -m \\ i-2m+1 \end{array}\middle| -4\right) F_{i-2m}(x) = 1.$$

Corollary 12. *Let i be a nonnegative integer. The following two identities hold for Pell numbers:*

$$P_{i+1} = {}_2F_1\left(\begin{array}{c} \frac{-i}{2}, 1+\frac{i}{2} \\ 1 \end{array}\middle| -1\right)\theta_i + 2\sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor}\binom{i-m}{m} {}_2F_1\left(\begin{array}{c} i-m+1, -m \\ i-2m+1 \end{array}\middle| -1\right),$$

and

$$\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor}(-1)^m\binom{i-m}{m} {}_2F_1\left(\begin{array}{c} i-m+1, -m \\ i-2m+2 \end{array}\middle| -1\right) P_{i-2m+1}$$
$$+\sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor}(-1)^{m+1}\binom{i-m-1}{m} {}_2F_1\left(\begin{array}{c} i-m, -m \\ i-2m+1 \end{array}\middle| -1\right) P_{i-2m} = 1.$$

Corollary 13. *Let i be a nonnegative integer. The following two identities hold for Mersenne numbers:*

$$M_{i+1} = 2^{\frac{i}{2}}\, {}_2F_1\left(\begin{array}{c} \frac{-i}{2}, 1+\frac{i}{2} \\ 1 \end{array}\middle| \frac{9}{8}\right)\theta_i + \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor}(-1)^m\, 2^{-2i+3m+2}\, 3^{2i-2m-1}\binom{i-m}{m}\times$$
$${}_2F_1\left(\begin{array}{c} i-m+1, -m \\ i-2m+1 \end{array}\middle| \frac{9}{8}\right),$$

and

$$\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \left(\frac{3}{2}\right)^{2m-i} \binom{i-m}{m} {}_2F_1\left(\begin{array}{c} i-m+1,-m \\ i-2m+2 \end{array} \middle| \frac{8}{9}\right) M_{i-2m+1}$$
$$+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \left(\frac{3}{2}\right)^{2m-i+1} \binom{i-m-1}{m} {}_2F_1\left(\begin{array}{c} i-m,-m \\ i-2m+1 \end{array} \middle| \frac{8}{9}\right) M_{i-2m} = 1.$$

Remark 9. *Noting that the well-known Fermat numbers defined as: $\mathcal{F}_i = 2^{2^i} + 1$ are linked with the Mersenne numbers by the relation: $\mathcal{F}_i = M_{2^i} + 2$, then the following expression for the Fermat numbers holds*

$$\mathcal{F}_i = 2 + \sum_{m=0}^{2^{i-1}-1} (-1)^m \, 2^{-2^i+3m+2} \, 3^{2^i-2m-1} \binom{2^i-m-1}{m} {}_2F_1\left(\begin{array}{c} -m, 2^i-m \\ 2^i-2m \end{array} \middle| \frac{9}{8}\right).$$

5.2. **New $(p,q)-$Fibonacci derivatives sequences identities.** Making use of the connection formulae introduced in Corollaries 2-4 and Corollaries 7-9, we may obtain new formulae for the derivatives sequences of Fibonacci, Pell and Mersenne numbers with the aid of the identities (2.5) and (2.6).

Corollary 14. *For all $r \geq 1$, the following two formulae hold*

$$F_{i+1}^{(r)} = \frac{\sqrt{\pi}}{\Gamma\left(r+\frac{1}{2}\right)} \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} \frac{\binom{i-m}{m}(i-2m)\, 2^{-i+2m-r+1}(i-2m+r-1)!}{(i-2m-r)!} \times$$
$${}_2F_1\left(\begin{array}{c} i-m+1,-m \\ i-2m+1 \end{array} \middle| -\frac{1}{4}\right),$$

and

$$\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \, 2^{i-2m} \binom{i-m}{m} {}_2F_1\left(\begin{array}{c} i-m+1,-m \\ i-2m+2 \end{array} \middle| -4\right) F_{i-2m+1}^{(r)}$$
$$+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \, 2^{i-2m-1} \binom{i-m-1}{m} {}_2F_1\left(\begin{array}{c} i-m,-m \\ i-2m+1 \end{array} \middle| -4\right) F_{i-2m}^{(r)} = \frac{\sqrt{\pi}(i+r)!}{2^r \, (i-r)! \, \Gamma(r+\frac{1}{2})}.$$

Corollary 15. *For all $r \geq 1$, the following two formulae hold*

$$P_{i+1}^{(r)} = \frac{2^{1-r}\sqrt{\pi}}{\Gamma\left(r+\frac{1}{2}\right)} \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} \frac{\binom{i-m}{m}(i-2m)(i-2m+r-1)!}{(i-2m-r)!} \times$$
$${}_2F_1\left(\begin{array}{c} i-m+1,-m \\ i-2m+1 \end{array} \middle| -1\right),$$

and

$$\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \binom{i-m}{m} {}_2F_1\left(\begin{array}{c} i-m+1,-m \\ i-2m+2 \end{array} \middle| -1\right) P_{i-2m+1}^{(r)}$$
$$+ \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m+1} \binom{i-m-1}{m} {}_2F_1\left(\begin{array}{c} i-m,-m \\ i-2m+1 \end{array} \middle| -1\right) P_{i-2m}^{(r)} = \frac{\sqrt{\pi}(i+r)!}{2^r \, (i-r)! \, \Gamma(r+\frac{1}{2})}.$$

Corollary 16. *For all $r \geq 1$, the following two formulae hold*

$$M_{i+1}^{(r)} = \frac{\sqrt{\pi}}{\Gamma\left(r+\frac{1}{2}\right)} \sum_{m=0}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \frac{(-1)^m\, 2^{1-i+3m-r}\, 3^{i-2m} \binom{i-m}{m} (i-2m)\,(i-2m+r-1)!}{(i-2m-r)!} \times {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+1 \end{matrix} \,\middle|\, \frac{9}{8} \right),$$

and

$$\sum_{m=0}^{\left\lfloor \frac{i}{2} \right\rfloor} (-1)^m \left(\tfrac{3}{2}\right)^{2m-i} \binom{i-m}{m}\, {}_2F_1\left(\begin{matrix} i-m+1, -m \\ i-2m+2 \end{matrix} \,\middle|\, \frac{8}{9} \right) M_{i-2m+1}^{(r)}$$
$$+ \sum_{m=0}^{\left\lfloor \frac{i-1}{2} \right\rfloor} (-1)^{m+1} \left(\tfrac{3}{2}\right)^{2m-i+1} \binom{i-m-1}{m}\, {}_2F_1\left(\begin{matrix} i-m, -m \\ i-2m+1 \end{matrix} \,\middle|\, \frac{8}{9} \right) M_{i-2m}^{(r)} = \frac{\sqrt{\pi}(i+r)!}{2^r\,(i-r)!\;\Gamma(r+\frac{1}{2})}.$$

5.3. **Several integrals formulae involving $(p,q)-$Fibonacci polynomials and Chebyshev polynomials of third and fourth kinds.** Based on the connection formulae which introduced in Theorems 3-4, the following integral formulae can be deduced.

Corollary 17. *For all $i \geq j$, the following two integral formulae are valid:*

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\; \bar{\phi}_i^{p,q}(x)\, V_j(x)\, dx = (-1)^{i+j} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\; \bar{\phi}_i^{p,q}(x)\, W_j(x)\, dx$$
$$= \pi \begin{cases} \left(\frac{p}{2}\right)^j q^{\frac{i-j}{2}} \binom{\frac{i+j}{2}}{\frac{i-j}{2}}\, {}_2F_1\left(\begin{matrix} \frac{j-i}{2}, \frac{i+j+2}{2} \\ j+1 \end{matrix} \,\middle|\, -\frac{p^2}{4\,q} \right), & (i+j) \text{ even}, \\ \left(\frac{p}{2}\right)^{j+1} q^{\frac{i-j-1}{2}} \binom{\frac{i+j+1}{2}}{\frac{i-j-1}{2}}\, {}_2F_1\left(\begin{matrix} \frac{-i+j+1}{2}, \frac{i+j+3}{2} \\ j+2 \end{matrix} \,\middle|\, -\frac{p^2}{4\,q} \right), & (i+j) \text{ odd}. \end{cases}$$

The following integrals formulae can be deduced as special cases of Corollary 17.

Corollary 18. *For all $i \geq j$, the following two integral formulae are valid:*

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\; U_i(x)\, V_j(x)\, dx = (-1)^{i+j} \sqrt{\frac{1-x}{1+x}}\; U_i(x)\, W_j(x)\, dx = \pi.$$

Corollary 19. *For all $i \geq j$, the following two integral formulae are valid:*

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\; F_{i+1}(x)\, V_j(x)\, dx = (-1)^{i+j} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\; F_{i+1}(x)\, W_j(x)\, dx$$
$$= \pi \begin{cases} \left(\frac{1}{2}\right)^j \binom{\frac{i+j}{2}}{\frac{i-j}{2}}\, {}_2F_1\left(\begin{matrix} \frac{j-i}{2}, \frac{i+j+2}{2} \\ j+1 \end{matrix} \,\middle|\, -\frac{1}{4} \right), & (i+j) \text{ even}, \\ \left(\frac{1}{2}\right)^{j+1} \binom{\frac{i+j+1}{2}}{\frac{i-j-1}{2}}\, {}_2F_1\left(\begin{matrix} \frac{-i+j+1}{2}, \frac{i+j+3}{2} \\ j+2 \end{matrix} \,\middle|\, -\frac{1}{4} \right), & (i+j) \text{ odd}. \end{cases}$$

Corollary 20. *For all $i \geq j$, the following two integral formulae are valid:*

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\; P_{i+1}(x)\, V_j(x)\, dx = (-1)^{i+j} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\; P_{i+1}(x)\, W_j(x)\, dx$$

$$= \pi \begin{cases} \binom{\frac{i+j}{2}}{\frac{i-j}{2}}\; {}_2F_1\left(\left. \begin{matrix} \frac{j-i}{2}, \frac{i+j+2}{2} \\ j+1 \end{matrix} \right| -1 \right), & (i+j) \text{ even}, \\ \binom{\frac{i+j+1}{2}}{\frac{i-j-1}{2}}\; {}_2F_1\left(\left. \begin{matrix} \frac{-i+j+1}{2}, \frac{i+j+3}{2} \\ j+2 \end{matrix} \right| -1 \right), & (i+j) \text{ odd}. \end{cases}$$

Corollary 21. *For all $i \geq j$, the following two integral formulae are valid:*

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\; \mathcal{F}_{i+1}(x)\, V_j(x)\, dx = (-1)^{i+j} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\; \mathcal{F}_{i+1}(x)\, W_j(x)\, dx$$

$$= \pi \begin{cases} \left(\frac{3}{2}\right)^{j} (-2)^{\frac{i-j}{2}} \binom{\frac{i+j}{2}}{\frac{i-j}{2}}\; {}_2F_1\left(\left. \begin{matrix} \frac{j-i}{2}, \frac{i+j+2}{2} \\ j+1 \end{matrix} \right| \frac{9}{8} \right), & (i+j) \text{ even}, \\ \left(\frac{3}{2}\right)^{j+1} (-2)^{\frac{i-j-1}{2}} \binom{\frac{i+j+1}{2}}{\frac{i-j-1}{2}}\; {}_2F_1\left(\left. \begin{matrix} \frac{-i+j+1}{2}, \frac{i+j+3}{2} \\ j+2 \end{matrix} \right| \frac{9}{8} \right), & (i+j) \text{ odd}. \end{cases}$$

Corollary 22. *For all $i \geq j$, the following two integral formulae are valid:*

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\; E_i(x,\alpha)\, V_j(x)\, dx = (-1)^{i+j} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\; E_i(x,\alpha)\, W_j(x)\, dx$$

$$= \pi \begin{cases} \left(\frac{1}{2}\right)^{j} (-\alpha)^{\frac{i-j}{2}} \binom{\frac{i+j}{2}}{\frac{i-j}{2}}\; {}_2F_1\left(\left. \begin{matrix} \frac{j-i}{2}, \frac{i+j+2}{2} \\ j+1 \end{matrix} \right| \frac{1}{4\,\alpha} \right), & (i+j) \text{ even}, \\ \left(\frac{1}{2}\right)^{j+1} (-\alpha)^{\frac{i-j-1}{2}} \binom{\frac{i+j+1}{2}}{\frac{i-j-1}{2}}\; {}_2F_1\left(\left. \begin{matrix} \frac{-i+j+1}{2}, \frac{i+j+3}{2} \\ j+2 \end{matrix} \right| \frac{1}{4\,\alpha} \right), & (i+j) \text{ odd}. \end{cases}$$

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Waleed Abd-Elhameed
Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia
Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
E-mail address: `walee_9@yahoo.com`