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OSTROWSKI AND JENSEN TYPE INEQUALITIES FOR HIGHER DERIVATIVES WITH APPLICATIONS

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Abstract. We consider inequalities which incorporate both Jensen and Ostrowski type inequalities for functions with absolutely continuous n-th derivatives. We provide applications of these inequalities for divergence measures. In particular, we obtain inequalities involving higher order χ-divergence.

1. Introduction

Jensen's inequality has been widely applied in many areas of research, e.g. probability theory, statistical physics, and information theory. The inequality was proved by Jensen in 1906 [13]: For a convex function $f : I \to \mathbb{R}$, the following inequality holds

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}, \quad a,b \in I. \tag{1.1}$$

Jensen's integral inequality takes the following form: for a μ-integrable function $g : \Omega \to [m,M] \subset \mathbb{R}$, and a convex function $f : [m,M] \to \mathbb{R}$, we have

$$f\left(\int_\Omega g\, d\mu\right) \le \int_\Omega f \circ g\, d\mu. \tag{1.2}$$

Here, $(\Omega, \mathcal{A}, \mu)$ is a measurable space with $\int_\Omega d\mu = 1$, consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω, and a countably additive and positive measure μ on $\mathcal{A}$ with values in the set of extended real numbers.

In 1938, Ostrowski proved the following inequality [12]:

Proposition 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] \|f'\|_\infty (b-a), \tag{1.3}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

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Dragomir [6] introduced some inequalities which combine the two aforementioned inequalities, referred to as the Jensen-Ostrowski type inequalities. We recall one of the results in the next proposition.

Proposition 1.2. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \in \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and $\Phi \circ g, g \in L(\Omega,\mu)$, then*

$$\begin{aligned}
&\left| \int_\Omega \Phi \circ g \, d\mu - \Phi(x) - \lambda \left(\int_\Omega g \, d\mu - x \right) \right| \\
&\le \int_\Omega |g - x| \|\Phi'((1-\ell)x + \ell g - \lambda\|_{[0,1],1} \, d\mu \\
&\le \begin{cases} \|g-x\|_{\Omega,\infty} \left\| \|\Phi'((1-\ell)x + \ell g - \lambda\|_{[0,1],1} \right\|_{\Omega,1}; \\ \|g-x\|_{\Omega,p} \left\| \|\Phi'((1-\ell)x + \ell g - \lambda\|_{[0,1],1} \right\|_{\Omega,q}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \|g-x\|_{\Omega,1} \left\| \|\Phi'((1-\ell)x + \ell g - \lambda\|_{[0,1],1} \right\|_{\Omega,\infty}; \end{cases}
\end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

Here, ℓ denotes the identity function on $[0,1]$, namely $\ell(t) = t$, for $t \in [0,1]$. We also use the notation

$$\|k\|_{\Omega,p} := \begin{cases} \left(\int_\Omega |k(t)|^p \, d\mu(t) \right)^{1/p}, & p \ge 1, \ k \in L_p(\Omega,\mu); \\ \operatorname*{ess\,sup}_{t \in \Omega} |k(t)|, & p = \infty, \ k \in L_\infty(\Omega,\mu); \end{cases}$$

and

$$\|f\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |f(s)|^p \, ds \right)^{1/p}, & p \ge 1, \ f \in L_p([0,1]); \\ \operatorname*{ess\,sup}_{s \in [0,1]} |f(s)|, & p = \infty, \ f \in L_\infty([0,1]). \end{cases}$$

Inequalities of Jensen and Ostrowski type are obtained by setting $x = \int_\Omega g \, d\mu$ and $\lambda = 0$, respectively, in Proposition 1.2. Further results on inequalities for functions with bounded derivatives and applications for f-divergence measures in information theory are also given in [6]. Similar inequalities are given for: (i) functions with derivatives that are of bounded variation and Lipschitz continuous in [7]; and (ii) functions which absolute values of the derivatives are convex in [8].

New inequalities of Jensen-Ostrowski type are given in the papers [2] and [3]. We recall one of the results in the following proposition:

Proposition 1.3 (Cerone, Dragomir, Kikianty [3])**.** *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a,b] \subset \mathring{I} \to \mathbb{C}$ is absolutely continuous on $[a,b]$, and $\zeta \in [a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω such that $f \circ g$, $g, (g-\zeta)^2 \in$*

$L(\Omega,\mu)$, with $\int_\Omega d\mu = 1$, then for any $\lambda \in \mathbb{C}$,

$$\left|\int_\Omega f\circ g d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta) - \frac{1}{2}\lambda\int_\Omega (g-\zeta)^2 d\mu\right|$$
$$\leq \frac{1}{2}\int_\Omega (g-\zeta)^2 \left\|f''((1-\ell)\zeta+\ell g)-\lambda\right\|_{[0,1],\infty} d\mu$$
$$\leq \begin{cases} \frac{1}{2}\|g-\zeta\|_{\Omega,\infty}^2 \left\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\right\|_{\Omega,1}; \\ \frac{1}{2}\|(g-\zeta)^2\|_{\Omega,p} \left\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\right\|_{\Omega,q},\ p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|(g-\zeta)^2\|_{\Omega,1} \left\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\right\|_{\Omega,\infty}. \end{cases}$$

In this paper, we generalise the results in [3] (including Proposition 1.3) for functions with absolutely continuous n-th derivative. We start with some identities in Section 2 to assist us in proving our main theorems. We obtain our main results in Section 3: inequalities with bounds involving the p-norms ($1 \leq p \leq \infty$), inequalities for functions with further assumptions of bounded $(n+1)$-th derivatives, and inequalities for functions where the absolute value of the $(n+1)$-th derivative satisfies some convexity conditions. The case of $n = 1$ recovers the results in [3]. Applications for f-divergence measure are provided in Section 4.

2. Identities

Throughout the paper, we denote $\mathring{I}$ to be the interior of the set I.

Lemma 2.1. *Let $f : I \in \mathbb{R} \to \mathbb{C}$ (I is an interval of $\mathbb{R}$) be such that $f^{(n)}$ is absolutely continuous on I, and $\zeta \in \mathring{I}$. If $g : \Omega \to I$ is Lebesgue μ-measurable on Ω and $f\circ g$, $(g-\zeta)^k$, $f^{(n+1)}((1-s)\zeta + sg) \in L(\Omega,\mu)$ for all $k \in \{1,...,n+1\}$ and $s \in [0,1]$, then we have*

$$\int_\Omega f\circ g\, d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!} d\mu - \lambda\frac{1}{(n+1)!}\int_\Omega (g-\zeta)^{n+1} d\mu \tag{2.1}$$
$$= \frac{1}{n!}\int_\Omega (g-\zeta)^{n+1}\left(\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta+sg)-\lambda\right] ds\right) d\mu$$
$$= \frac{1}{n!}\int_0^1 (1-s)^n \left(\int_\Omega (g-\zeta)^{n+1}\left[f^{(n+1)}((1-s)\zeta+sg)-\lambda\right] d\mu\right) ds,$$

for any $\lambda \in \mathbb{C}$.

Proof. For all $x, \zeta \in \mathring{I}$ we have the Taylor's formula with integral remainder

$$f(x) = f(\zeta) + \sum_{k=1}^{n}\frac{(x-\zeta)^k}{k!} f^{(k)}(\zeta) + \frac{1}{n!}\int_\zeta^x (x-t)^n f^{(n+1)}(t)\, dt. \tag{2.2}$$

If we make the change of variable $t = (1-s)\zeta + sx$, then $dt = (x-\zeta)\,ds$, and

$$x - t = x - (1-s)\zeta - sx = (1-s)(x-\zeta),$$

and from (2.2) we get

$$f(x) = f(\zeta) + \sum_{k=1}^{n} \frac{(x-\zeta)^k}{k!} f^{(k)}(\zeta) + \frac{1}{n!}(x-\zeta)^{n+1} \int_0^1 (1-s)^n f^{(n+1)}((1-s)\zeta + sx)\, ds. \tag{2.3}$$

On the other hand,

$$\begin{aligned}&\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sx) - \lambda\right] ds \\ &= \int_0^1 (1-s)^n f^{(n+1)}((1-s)\zeta + sx)\, ds - \lambda \int_0^1 (1-s)^n\, ds \\ &= \int_0^1 (1-s)^n f^{(n+1)}((1-s)\zeta + sx)\, ds - \lambda \frac{1}{n+1},\end{aligned}$$

therefore

$$\begin{aligned}&\int_0^1 (1-s)^n f^{(n+1)}((1-s)\zeta + sx)\, ds \\ &= \int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sx) - \lambda\right] ds + \lambda \frac{1}{n+1},\end{aligned}$$

and by (2.3) we get

$$\begin{aligned} f(x) &= f(\zeta) + \sum_{k=1}^{n} \frac{(x-\zeta)^k}{k!} f^{(k)}(\zeta) \\ &\quad + \frac{1}{n!}(x-\zeta)^{n+1} \left[\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sx) - \lambda\right] ds + \lambda \frac{1}{n+1}\right] \\ &= f(\zeta) + \sum_{k=1}^{n} \frac{(x-\zeta)^k}{k!} f^{(k)}(\zeta) + \lambda \frac{1}{(n+1)!}(x-\zeta)^{n+1} \\ &\quad + \frac{1}{n!}(x-\zeta)^{n+1} \int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sx) - \lambda\right] ds. \end{aligned} \tag{2.4}$$

If $g : \Omega \to I$ is Lebesgue μ-measurable on Ω then by (2.4) we have

$$\begin{aligned} f(g(u)) &= f(\zeta) + \sum_{k=1}^{n} \frac{(g(u)-\zeta)^k}{k!} f^{(k)}(\zeta) + \lambda \frac{1}{(n+1)!}(g(u)-\zeta)^{n+1} \\ &\quad + \frac{1}{n!}(g(u)-\zeta)^{n+1} \int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sg(u)) - \lambda\right] ds, \end{aligned} \tag{2.5}$$

for all $u \in \Omega$.

Since $f\circ g, (g-\zeta)^k$, and $f^{(n+1)}((1-s)\zeta+sg)\in L(\Omega,\mu)$ for $k\in\{1,...,n+1\}$, $s\in[0,1]$, we get the following by taking the integral in (2.5) and since $\int_\Omega d\mu=1$:

$$\begin{aligned}&\int_\Omega f\circ g\,d\mu-f(\zeta) \qquad (2.6)\\ &-\sum_{k=1}^{n}f^{(k)}(\zeta)\int_\Omega\frac{(g-\zeta)^k}{k!}\,d\mu-\lambda\frac{1}{(n+1)!}\int_\Omega(g-\zeta)^{n+1}\,d\mu\\ &=\frac{1}{n!}\int_\Omega(g-\zeta)^{n+1}\left(\int_0^1(1-s)^n\left[f^{(n+1)}((1-s)\zeta+sg)-\lambda\right]ds\right)d\mu\\ &=\frac{1}{n!}\int_0^1(1-s)^n\left(\int_\Omega(g-\zeta)^{n+1}\left[f^{(n+1)}((1-s)\zeta+sg)-\lambda\right]d\mu\right)ds,\end{aligned}$$

for any $\lambda\in\mathbb{C}$. We use Fubini's theorem for the last equality. □

Remark. *When $n=1$ we have*

$$\begin{aligned}&\int_\Omega f\circ g\,d\mu-f(\zeta)-f'(\zeta)\int_\Omega(g-\zeta)\,d\mu-\frac{1}{2}\lambda\int_\Omega(g-\zeta)^2\,d\mu\\ &=\int_\Omega(g-\zeta)^2\left(\int_0^1(1-s)\left[f''((1-s)\zeta+sg)-\lambda\right]ds\right)d\mu\\ &=\int_0^1(1-s)\left(\int_\Omega(g-\zeta)^2\left[f''((1-s)\zeta+sg)-\lambda\right]d\mu\right)ds,\end{aligned}$$

for any $\lambda\in\mathbb{C}$, which recover the identities obtained in [3, Lemma 1]. *Consequently, the results in this paper recover the associated ones in* [3] *by setting $n=1$.*

Corollary 2.2. *Under the assumptions of Lemma 2.1, we have*

$$\begin{aligned}&\int_\Omega f\circ g\,d\mu-f(\zeta)-\sum_{k=1}^{n}f^{(k)}(\zeta)\int_\Omega\frac{(g-\zeta)^k}{k!}\,d\mu \qquad (2.7)\\ &=\frac{1}{n!}\int_\Omega(g-\zeta)^{n+1}\left(\int_0^1(1-s)^n\left[f^{(n+1)}((1-s)\zeta+sg)\right]ds\right)d\mu\\ &=\frac{1}{n!}\int_0^1(1-s)^n\left(\int_\Omega(g-\zeta)^{n+1}\left[f^{(n+1)}((1-s)\zeta+sg)\right]d\mu\right)ds\end{aligned}$$

by setting $\lambda=0$.

Remark. *Another estimate one may obtain is to consider the mean value form of the remainder in* (2.2)

$$f(x)=f(\zeta)+\sum_{k=1}^{n}\frac{(x-\zeta)^k}{k!}f^{(k)}(\zeta)+\frac{(x-\zeta)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \qquad (2.8)$$

where ξ is between x and ζ. By setting $x=g(t)$ $(t\in\Omega)$ and integrate (2.8) *on Ω, we obtain*

$$\begin{aligned}&\int_\Omega f\circ g\,d\mu-f(\zeta)-\sum_{k=1}^{n}f^{(k)}(\zeta)\int_\Omega\frac{(g-\zeta)^k}{k!}\,d\mu \qquad (2.9)\\ &=\int_\Omega f^{(n+1)}(\xi)\frac{(g-\zeta)^{n+1}}{(n+1)!}\,d\mu\end{aligned}$$

where $\xi=\xi(t)$ is between $g(t)$ and ζ.

3. Main results

We denote by ℓ, the identity function on $[0,1]$, namely, $\ell(t)=t$ ($t\in[0,1]$). For $t\in\Omega$, $\zeta\in[a,b]$, and $\lambda\in\mathbb{C}$, we have

$$\operatorname*{ess\,sup}_{s\in[0,1]}|f^{(k)}\big((1-s)\zeta+sg(t)\big)-\lambda|=\|f^{(k)}\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty},$$

for all $k=1,\dots,n+1$.

Theorem 3.1. *Let $f:I\in\mathbb{R}\to\mathbb{C}$ (I interval of $\mathbb{R}$) be such that $f^{(n)}$ is absolutely continuous on I and $\zeta\in\mathring{I}$. If $g:\Omega\to I$ is Lebesgue μ-measurable on Ω and $f\circ g$, $(g-\zeta)^k$, $f^{(n+1)}((1-s)\zeta+sg)\in L(\Omega,\mu)$ for all $k\in\{1,\dots,n+1\}$ and $s\in[0,1]$, then we have*

$$\begin{aligned}&\Bigg|\int_\Omega f\circ g\,d\mu-f(\zeta)\\&-\sum_{k=1}^{n}f^{(k)}(\zeta)\int_\Omega\frac{(g-\zeta)^k}{k!}\,d\mu-\lambda\frac{1}{(n+1)!}\int_\Omega(g-\zeta)^{n+1}\,d\mu\Bigg|\\&\le\frac{1}{(n+1)!}\left(\int_\Omega|g-\zeta|^{n+1}\left\|f^{(n+1)}((1-\ell)\zeta+\ell g)-\lambda\right\|_{[0,1],\infty}d\mu\right)\\&\le\begin{cases}\frac{1}{(n+1)!}\|\,|g-\zeta|^{n+1}\|_{\Omega,\infty}\left\|\left\|f^{(n+1)}((1-\ell)\zeta+\ell g)-\lambda\right\|_{[0,1],\infty}\right\|_{\Omega,1},\\ \frac{1}{(n+1)!}\|\,|g-\zeta|^{n+1}\|_{\Omega,p}\left\|\left\|f^{(n+1)}((1-\ell)\zeta+\ell g)-\lambda\right\|_{[0,1],\infty}\right\|_{\Omega,q},\\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \frac{1}{(n+1)!}\|\,|g-\zeta|^{n+1}\|_{\Omega,1}\left\|\left\|f^{(n+1)}((1-\ell)\zeta+\ell g)-\lambda\right\|_{[0,1],\infty}\right\|_{\Omega,\infty},\end{cases}\end{aligned}\tag{3.1}$$

for any $\lambda\in\mathbb{C}$.

Proof. Taking the modulus in (2.6), we have

$$\begin{aligned}&\Bigg|\int_\Omega f\circ g\,d\mu-f(\zeta)\\&-\sum_{k=1}^{n}f^{(k)}(\zeta)\int_\Omega\frac{(g-\zeta)^k}{k!}\,d\mu-\lambda\frac{1}{(n+1)!}\int_\Omega(g-\zeta)^{n+1}\,d\mu\Bigg|\\&\le\frac{1}{n!}\int_0^1(1-s)^n\left(\int_\Omega|g-\zeta|^{n+1}\left|f^{(n+1)}((1-s)\zeta+sg)-\lambda\right|d\mu\right)ds\\&\le\frac{1}{n!}\int_0^1(1-s)^n\,ds\left(\int_\Omega|g-\zeta|^{n+1}\left\|f^{(n+1)}((1-\ell)\zeta+\ell g)-\lambda\right\|_{[0,1],\infty}d\mu\right)\\&\le\frac{1}{(n+1)!}\left(\int_\Omega|g-\zeta|^{n+1}\left\|f^{(n+1)}((1-\ell)\zeta+\ell g)-\lambda\right\|_{[0,1],\infty}d\mu\right),\end{aligned}$$

for any $\lambda\in\mathbb{C}$. We obtain the desired result by applying Hölder's inequality. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, we have the following Ostrowski type inequality:*

$$\left|\int_\Omega f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!}\,d\mu\right| \leq \frac{1}{(n+1)!}\|f^{(n+1)}\|_{I,\infty}\int_\Omega |g-\zeta|^{n+1}\,d\mu. \tag{3.2}$$

We also have the following Jensen type inequality:

$$\left|\int_\Omega f\circ g\,d\mu - f\left(\int_\Omega g\,d\mu\right) - \sum_{k=2}^{n} f^{(k)}\left(\int_\Omega g\,d\mu\right)\int_\Omega \frac{\left(g-\int_\Omega g\,d\mu\right)^k}{k!}\,d\mu\right| \leq \frac{1}{(n+1)!}\|f^{(n+1)}\|_{I,\infty}\int_\Omega \left|g-\int_\Omega g\,d\mu\right|^{n+1}\,d\mu. \tag{3.3}$$

Proof. We have from (3.1) with $\lambda = 0$

$$\left|\int_\Omega f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!}\,d\mu\right| \leq \frac{1}{(n+1)!}\left(\int_\Omega |g-\zeta|^{n+1}\left\|f^{(n+1)}((1-\ell)\zeta+\ell g)\right\|_{[0,1],\infty}d\mu\right).$$

For any $t\in\Omega$ and almost every $s\in[0,1]$, we have

$$|f^{(n+1)}((1-s)\zeta+sg(t))| \leq \operatorname*{ess\,sup}_{u\in I}|f^{(n+1)}(u)| = \|f^{(n+1)}\|_{I,\infty}.$$

Therefore, we have

$$\begin{aligned}\left\|f^{(n+1)}((1-\ell)\zeta+\ell g)\right\|_{[0,1],\infty} &\leq \operatorname*{ess\,sup}_{s\in[0,1],\ t\in\Omega}\|f^{(n+1)}((1-s)\zeta+sg(t))\| \\ &\leq \|f^{(n+1)}\|_{I,\infty}.\end{aligned}$$

Thus,

$$\left|\int_\Omega f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!}\,d\mu\right| \leq \frac{1}{(n+1)!}\|f^{(n+1)}\|_{I,\infty}\int_\Omega |g-\zeta|^{n+1}\,d\mu.$$

The proof is completed. □

Alternative proof for Corollary 3.2. From (2.9), we have the following for $\xi=\xi(t)$ is between $g(t)$ and ζ, where $t\in\Omega$:

$$\begin{aligned}&\left|\int_\Omega f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!}\,d\mu\right| \\ &= \left|\int_\Omega f^{(n+1)}(\xi)\frac{(g-\zeta)^{n+1}}{(n+1)!}\,d\mu\right| \\ &\leq \frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{I,\infty}\int_\Omega |g-\zeta|^{n+1}\,d\mu.\end{aligned}$$

This completes the proof. □

Remark (Ostrowski type inequality)**.** *Let $\Omega = [a,b]$, $g : [a,b] \to [a,b]$ defined by $g(t) = t$, and $\mu(t) = t/(b-a)$. We have*

$$\begin{aligned}
&\left|\int_\Omega f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!}\,d\mu\right| \\
&= \left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(\zeta) - \frac{1}{b-a}\sum_{k=1}^{n} f^{(k)}(\zeta)\int_a^b \frac{(t-\zeta)^k}{k!}\,dt\right| \\
&= \left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(\zeta) - \frac{1}{b-a}\frac{1}{(k+1)!}\sum_{k=1}^{n} f^{(k)}(\zeta)\left[(b-\zeta)^{k+1} - (a-\zeta)^{k+1}\right]\right| \\
&\le \frac{1}{(n+1)!}\|f^{(n+1)}\|_{[a,b],\infty}\frac{1}{b-a}\int_a^b |t-\zeta|^{n+1}\,dt \\
&= \frac{1}{(n+2)!}\|f^{(n+1)}\|_{[a,b],\infty}\frac{\left[(\zeta-a)^{n+2} + (b-\zeta)^{n+2}\right]}{b-a}.
\end{aligned}$$

For the next result, we need the following notation and proposition: for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions [6]

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{h : [a,b] \to \mathbb{C} \middle| \mathrm{Re}\left[(\Gamma - h(t))(\overline{h(t)} - \overline{\gamma})\right] \ge 0 \text{ for a.e. } t \in [a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{h : [a,b] \to \mathbb{C} \middle| \left|h(t) - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a,b]\right\}.$$

The following representation results may be stated [6].

Proposition 3.3. *For any $\gamma, \Gamma \in \mathbb{C}$ and $\gamma \ne \Gamma$, we have*

(*i*) *$\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets;*
(*ii*) *$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$; and*
(*iii*) $\bar{U}_{[a,b]}(\gamma,\Gamma) = \{h : [a,b] \to \mathbb{C} \mid (\mathrm{Re}(\Gamma) - \mathrm{Re}(h(t)))(\mathrm{Re}(h(t)) - \mathrm{Re}(\gamma)) + (\mathrm{Im}(\Gamma) - \mathrm{Im}(h(t)))(\mathrm{Im}(h(t)) - \mathrm{Im}(\gamma)) \ge 0$ *for a.e.* $t \in [a,b]\}$.

We have the following Jensen-Ostrowski inequality for functions with bounded higher $(n+1)$-th derivatives:

Theorem 3.4. *Let $f : I \in \mathbb{R} \to \mathbb{C}$ (I interval of $\mathbb{R}$) be such that $f^{(n)}$ is absolutely continuous on I and $\zeta \in \mathring{I}$. For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \ne \Gamma$, assume that $f^{(n+1)} \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. If $g : \Omega \to I$ is Lebesgue μ-measurable on Ω and $f \circ g$, $(g-\zeta)^k$, $f^{(n+1)}((1-s)\zeta + sg) \in L(\Omega,\mu)$ for all $k \in \{1,...,n+1\}$ and $s \in [0,1]$, then we have*

$$\begin{aligned}
&\left|\int_\Omega f\circ g\,d\mu - f(\zeta)\right. \\
&\left.\quad - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!}\,d\mu - \frac{\gamma+\Gamma}{2}\frac{1}{(n+1)!}\int_\Omega (g-\zeta)^{n+1}\,d\mu\right| \\
&\le \frac{1}{2(n+1)!}|\Gamma-\gamma|\int_\Omega |g-\zeta|^{n+1}\,d\mu. \qquad (3.4)
\end{aligned}$$

Proof. Let $\lambda = (\gamma + \Gamma)/2$ in (2.6), we have

$$\begin{aligned} & \int_\Omega f \circ g \, d\mu - f(\zeta) \\ & - \sum_{k=1}^{n} f^{(k)}(\zeta) \int_\Omega \frac{(g-\zeta)^k}{k!} \, d\mu - \frac{\gamma+\Gamma}{2} \frac{1}{(n+1)!} \int_\Omega (g-\zeta)^{n+1} \, d\mu \\ = \; & \frac{1}{n!} \int_\Omega (g-\zeta)^{n+1} \left(\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sg) - \frac{\gamma+\Gamma}{2} \right] ds \right) d\mu \end{aligned}$$

Since $f^{(n+1)} \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, we have

$$\left| f^{(n+1)}((1-s)\zeta + sg) - \frac{\gamma+\Gamma}{2} \right| \le \frac{1}{2} |\Gamma - \gamma|, \tag{3.5}$$

for almost every $s \in [0,1]$ and $t \in \Omega$. Multiply (3.5) with $(1-s)^n > 0$ and integrate over $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg) - \frac{\gamma+\Gamma}{2} \right| ds \\ & \le \frac{1}{2} |\Gamma - \gamma| \int_0^1 (1-s)^n \, ds = \frac{1}{2(n+1)} |\Gamma - \gamma|, \end{aligned}$$

for any $t \in \Omega$. Now, we have

$$\begin{aligned} & \left| \int_\Omega f \circ g \, d\mu - f(\zeta) \right. \\ & \left. - \sum_{k=1}^{n} f^{(k)}(\zeta) \int_\Omega \frac{(g-\zeta)^k}{k!} \, d\mu - \frac{\gamma+\Gamma}{2} \frac{1}{(n+1)!} \int_\Omega (g-\zeta)^{n+1} \, d\mu \right| \\ \le \; & \frac{1}{2(n+1)!} |\Gamma - \gamma| \int_\Omega |g-\zeta|^{n+1} \, d\mu. \end{aligned}$$

This completes the proof. □

Corollary 3.5. *When $\zeta = (a+b)/2$ in Theorem 3.4, we have the following Ostrowski inequality:*

$$\begin{aligned} & \left| \int_\Omega f \circ g \, d\mu - f\left(\frac{a+b}{2}\right) - \sum_{k=1}^{n} f^{(k)}\left(\frac{a+b}{2}\right) \int_\Omega \frac{\left(g - \frac{a+b}{2}\right)^k}{k!} \, d\mu \right. \\ & \left. - \frac{\gamma+\Gamma}{2} \frac{1}{(n+1)!} \int_\Omega \left(g - \frac{a+b}{2}\right)^{n+1} d\mu \right| \\ \le \; & \frac{1}{2(n+1)!} |\Gamma - \gamma| \int_\Omega \left| g - \frac{a+b}{2} \right|^{n+1} d\mu. \end{aligned}$$

When $\zeta = \int_\Omega g\,d\mu$ in Theorem 3.4, we have the following Jensen type inequality:

$$\begin{aligned} &\left|\int_\Omega f\circ g\,d\mu - f\left(\int_\Omega g\,d\mu\right)\right. \\ &\left. -\sum_{k=1}^{n} f^{(k)}\left(\int_\Omega g\,d\mu\right)\int_\Omega \frac{(g-\zeta)^k}{k!}\,d\mu - \frac{\gamma+\Gamma}{2}\frac{1}{(n+1)!}\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^{n+1}d\mu\right| \\ \leq\ & \frac{1}{2(n+1)!}|\Gamma-\gamma|\int_\Omega\left|g-\int_\Omega g\,d\mu\right|^{n+1}d\mu. \end{aligned}$$

We recall the following definition:

Definition. *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then,*

(1) *h is convex, if for any $x, y \in I$ and $s \in [0,1]$, we have*
$$h\big((1-s)x+sy\big) \leq (1-s)h(x)+sh(y).$$
(2) *h is quasi-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have*
$$h\big((1-s)x+sy\big) \leq \max\{h(x), h(y)\}.$$
(3) *h is log-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have*
$$h\big((1-s)x+sy\big) \leq h(x)^{1-s}h(y)^s.$$
(4) *for a fixed $q \in (0,1]$, h is q-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have*
$$h\big((1-s)x+sy\big) \leq (1-s)^q h(x)+s^q h(y).$$

We refer the reader to the paper by Dragomir [9], for further background on these notions of convexity.

We also need the following lemma to assist us in our calculations.

Lemma 3.6. *For $\alpha, \beta \in \mathbb{R}$ and $n \geq 1$, we have*

$$\int_0^1 (1-s)^n\left(\frac{\beta}{\alpha}\right)^s ds = -\frac{1}{\log(\frac{\beta}{\alpha})} - \sum_{i=1}^{n-1}\frac{\frac{n!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + n!\frac{\frac{\beta}{\alpha}-1}{(\log(\frac{\beta}{\alpha}))^{n+1}}. \tag{3.6}$$

Proof. For $n=1$, integrating by parts gives us

$$\begin{aligned} \int_0^1 (1-s)\left(\frac{\beta}{\alpha}\right)^s ds &= \frac{(1-s)}{\log(\frac{\beta}{\alpha})}\left(\frac{\beta}{\alpha}\right)^s\Bigg|_0^1 + \frac{1}{\log(\frac{\beta}{\alpha})}\int_0^1\left(\frac{\beta}{\alpha}\right)^s ds \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} + \frac{1}{(\log(\frac{\beta}{\alpha}))^2}\left(\frac{\beta}{\alpha}-1\right). \end{aligned}$$

For $n=2$, integrating by parts gives us

$$\begin{aligned} \int_0^1 (1-s)^2\left(\frac{\beta}{\alpha}\right)^s ds &= \frac{(1-s)^2}{\log(\frac{\beta}{\alpha})}\left(\frac{\beta}{\alpha}\right)^s\Bigg|_0^1 + \frac{2}{\log(\frac{\beta}{\alpha})}\int_0^1 (1-s)\left(\frac{\beta}{\alpha}\right)^s ds \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} - \frac{2}{(\log(\frac{\beta}{\alpha}))^2} + \frac{2}{(\log(\frac{\beta}{\alpha}))^3}\left(\frac{\beta}{\alpha}-1\right). \end{aligned}$$

We assume that for n, we have

$$\begin{aligned}&\int_0^1 (1-s)^n \left(\frac{\beta}{\alpha}\right)^s ds \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} - \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + n!\frac{\frac{\beta}{\alpha}-1}{(\log(\frac{\beta}{\alpha}))^{n+1}}.\end{aligned}$$

We have

$$\begin{aligned}&\int_0^1 (1-s)^{n+1} \left(\frac{\beta}{\alpha}\right)^s ds \\ &= \frac{(1-s)^{n+1}}{\log(\frac{\beta}{\alpha})}\left(\frac{\beta}{\alpha}\right)^s\Bigg|_0^1 + \frac{n+1}{\log(\frac{\beta}{\alpha})}\int_0^1 (1-s)^n \left(\frac{\beta}{\alpha}\right)^s ds \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} + \frac{n+1}{\log(\frac{\beta}{\alpha})}\left[-\frac{1}{\log(\frac{\beta}{\alpha})} - \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + n!\frac{\frac{\beta}{\alpha}-1}{(\log(\frac{\beta}{\alpha}))^{n+1}}\right] \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} - \frac{n+1}{\log(\frac{\beta}{\alpha})^2} - \sum_{i=1}^{n-1} \frac{\frac{(n+1)!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+2}} + (n+1)!\frac{\frac{\beta}{\alpha}-1}{(\log(\frac{\beta}{\alpha}))^{n+2}} \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} - \sum_{i=1}^{n} \frac{\frac{(n+1)!}{(n+1-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + (n+1)!\frac{\frac{\beta}{\alpha}-1}{(\log(\frac{\beta}{\alpha}))^{n+2}},\end{aligned}$$

and this completes the proof. □

In the next theorem, we assume that $|f^{(n+1)}|$ satisfies some convexity properties.

Theorem 3.7. *Let $f : I \in \mathbb{R} \to \mathbb{C}$ (I interval of $\mathbb{R}$) be such that $f^{(n)}$ is absolutely continuous on I and $\zeta \in \mathring{I}$. Suppose that $g : \Omega \to I$ is Lebesgue μ-measurable on Ω and $f \circ g,\ (g-\zeta)^k,\ f^{(n+1)}((1-s)\zeta + sg) \in L(\Omega,\mu)$ for all $k \in \{1, ..., n+1\}$.*

(i) If $|f^{(n+1)}|$ is convex, then we have

$$\begin{aligned}&\left|\int_\Omega f\circ g\, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!}\, d\mu\right| \\ &\leq \frac{1}{n!}\frac{1}{n+2}\left[|f^{(n+1)}(\zeta)|\int_\Omega |g-\zeta|^{n+1}\, d\mu + \frac{1}{(n+1)}\int_\Omega |g-\zeta|^{n+1}\,|f^{(n+1)}\circ g|\, d\mu\right].\end{aligned}$$

(ii) If $|f^{(n+1)}|$ is quasi-convex, then we have

$$\begin{aligned}&\left|\int_\Omega f\circ g\, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta)\int_\Omega \frac{(g-\zeta)^k}{k!}\, d\mu\right| \\ &\leq \frac{1}{(n+1)!}\max\left\{|f^{(n+1)}(\zeta)|\int_\Omega |g-\zeta|^{n+1}\, d\mu, \int_\Omega |g-\zeta|^{n+1}\,|f^{(n+1)}(g(t))|\, d\mu\right\}.\end{aligned}$$

(iii) If $|f^{(n+1)}|$ is log-convex, then we have

$$\left|\int_{\Omega} f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_{\Omega}\frac{(g-\zeta)^k}{k!}\,d\mu\right|$$
$$\leq \frac{1}{n!}\int_{\Omega}|g-\zeta|^{n+1}\left[-\frac{|f^{(n+1)}(\zeta)|}{\log(\frac{|f^{(n+1)}\circ g|}{|f^{(n+1)}(\zeta)|})}\right.$$
$$\left.-|f^{(n+1)}(\zeta)|\sum_{i=1}^{n-1}\frac{\frac{n!}{(n-i)!}}{(\log(\frac{|f^{(n+1)}\circ g|}{|f^{(n+1)}(\zeta)|}))^{i+1}} + n!\frac{|f^{(n+1)}\circ g| - |f^{(n+1)}(\zeta)|}{(\log(\frac{|f^{(n+1)}\circ g|}{|f^{(n+1)}(\zeta)|}))^{n+1}}\right]d\mu.$$

(iv) If $|f^{(n+1)}|$ is q-convex (for a fixed $q\in(0,1]$), then we have

$$\left|\int_{\Omega} f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_{\Omega}\frac{(g-\zeta)^k}{k!}\,d\mu\right|$$
$$\leq \frac{1}{n!}\frac{1}{n+q+1}\left[|f^{(n+1)}(\zeta)|\int_{\Omega}|g-\zeta|^{n+1}\,d\mu\right.$$
$$\left.+\frac{n}{(q+1)}\int_{\Omega}|g-\zeta|^{n+1}|f^{(n+1)}\circ g|\,d\mu\right].$$

Proof. (i) If $|f^{(n+1)}|$ is convex, then

$$\left|f^{(n+1)}\big((1-s)\zeta+sg(t)\big)\right| \leq (1-s)|f^{(n+1)}(\zeta)| + s|f^{(n+1)}(g(t))|,$$

for all $t\in\Omega$, which implies that

$$\int_0^1 (1-s)^n\left|f^{(n+1)}\big((1-s)\zeta+sg(t)\big)\right|ds$$
$$\leq \left[\int_0^1(1-s)^{n+1}\,ds\right]|f^{(n+1)}(\zeta)| + \left[\int_0^1 s(1-s)^n\,ds\right]|f^{(n+1)}(g(t))|$$
$$= \frac{1}{n+2}|f^{(n+1)}(\zeta)| + \frac{1}{(n+1)(n+2)}|f^{(n+1)}(g(t))|.$$

Thus,

$$\left|\int_{\Omega} f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_{\Omega}\frac{(g-\zeta)^k}{k!}\,d\mu\right|$$
$$\leq \frac{1}{n!}\int_{\Omega}|g-\zeta|^{n+1}\left(\int_0^1(1-s)^n\left|f^{(n+1)}((1-s)\zeta+sg)\right|ds\right)d\mu$$
$$\leq \frac{1}{n!}\frac{1}{n+2}\left[|f^{(n+1)}(\zeta)|\int_{\Omega}|g-\zeta|^{n+1}\,d\mu + \frac{1}{(n+1)}\int_{\Omega}|g-\zeta|^{n+1}|f^{(n+1)}\circ g|\,d\mu\right].$$

(ii) If $|f^{(n+1)}|$ is quasi-convex, then

$$\left|f^{(n+1)}\big((1-s)\zeta+sg(t)\big)\right| \leq \max\{|f^{(n+1)}(\zeta)|, |f^{(n+1)}(g(t))|\},$$

for all $t \in \Omega$, which implies that

$$\begin{aligned}
&\int_0^1 (1-s)^n \left| f^{(n+1)}\big((1-s)\zeta + sg(t)\big) \right| ds \\
&\leq \left[\int_0^1 (1-s)^n \, ds \right] \max\{|f^{(n+1)}(\zeta)|, |f^{(n+1)}(g(t))|\} \\
&= \frac{1}{n+1} \max\{|f^{(n+1)}(\zeta)|, |f^{(n+1)}(g(t))|\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left| \int_\Omega f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_\Omega \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\
&\leq \frac{1}{n!} \int_\Omega |g-\zeta|^{n+1} \left(\int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg) \right| ds \right) d\mu \\
&\leq \frac{1}{(n+1)!} \int_\Omega |g-\zeta|^{n+1} \max\{|f^{(n+1)}(\zeta)|, |f^{(n+1)}(g(t))|\} \, d\mu \\
&= \frac{1}{(n+1)!} \max\left\{ |f^{(n+1)}(\zeta)| \int_\Omega |g-\zeta|^{n+1} \, d\mu, \int_\Omega |g-\zeta|^{n+1} |f^{(n+1)}(g(t))| \, d\mu \right\}.
\end{aligned}$$

(iii) If $|f^{(n+1)}|$ is log-convex, then

$$\left| f^{(n+1)}\big((1-s)\zeta + sg(t)\big) \right| \leq |f^{(n+1)}(\zeta)|^{1-s} |f^{(n+1)}(g(t))|^s,$$

for all $t \in \Omega$, which implies that

$$\begin{aligned}
&\int_0^1 (1-s)^n \left| f^{(n+1)}\big((1-s)\zeta + sg(t)\big) \right| ds \\
&\leq \left[\int_0^1 (1-s)^n |f^{(n+1)}(\zeta)|^{1-s} |f^{(n+1)}(g(t))|^s ds \right].
\end{aligned}$$

Let $\alpha := |f^{(n+1)}(\zeta)|$ and $\beta = \beta(t) := |f^{(n+1)}(g(t))|$. Since α does not depend on t, we have

$$\int_0^1 (1-s)^n \alpha^{1-s} \beta^s ds = \alpha \int_0^1 (1-s)^n \left(\frac{\beta}{\alpha} \right)^s ds.$$

By Lemma 3.6, we have

$$\begin{aligned}
&\int_0^1 (1-s)^n \alpha^{1-s} \beta^s ds \\
&= \alpha \int_0^1 (1-s)^n \left(\frac{\beta}{\alpha} \right)^s ds \\
&= -\frac{\alpha}{\log(\frac{\beta}{\alpha})} - \alpha \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + n! \frac{\beta - \alpha}{(\log(\frac{\beta}{\alpha}))^{n+1}},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left|\int_{\Omega} f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_{\Omega}\frac{(g-\zeta)^k}{k!}\,d\mu\right| \\
&\le \frac{1}{n!}\int_{\Omega}|g-\zeta|^{n+1}\left(\int_0^1 (1-s)^n\left|f^{(n+1)}((1-s)\zeta+sg)\right|ds\right)d\mu \\
&\le \frac{1}{n!}\int_{\Omega}|g-\zeta|^{n+1}\left[\int_0^1 (1-s)^n|f^{(n+1)}(\zeta)|^{1-s}|f^{(n+1)}(g(t))|^s ds\right]d\mu \\
&\le \frac{1}{n!}\int_{\Omega}|g-\zeta|^{n+1}\left[-\frac{|f^{(n+1)}(\zeta)|}{\log(\frac{|f^{(n+1)}\circ g|}{|f^{(n+1)}(\zeta)|})}\right. \\
&\left.-|f^{(n+1)}(\zeta)|\sum_{i=1}^{n-1}\frac{\frac{n!}{(n-i)!}}{(\log(\frac{|f^{(n+1)}\circ g|}{|f^{(n+1)}(\zeta)|}))^{i+1}} + n!\frac{|f^{(n+1)}\circ g|-|f^{(n+1)}(\zeta)|}{(\log(\frac{|f^{(n+1)}\circ g|}{|f^{(n+1)}(\zeta)|}))^{n+1}}\right]d\mu.
\end{aligned}$$

(iv) If $|f^{(n+1)}|$ is q-convex (for a fixed $q\in(0,1]$), then

$$\left|f^{(n+1)}\big((1-s)\zeta+sg\big)\right| \le (1-s)^q|f^{(n+1)}(\zeta)| + s^q|f^{(n+1)}(g(t))|,$$

for all $t\in\Omega$, which implies that

$$\begin{aligned}
&\int_0^1 (1-s)^n\left|f^{(n+1)}\big((1-s)\zeta+sg(t)\big)\right|ds \\
&\le \left[\int_0^1 (1-s)^{n+q}\,ds\right]|f^{(n+1)}(\zeta)| + \left[\int_0^1 (1-s)^n s^q\,ds\right]|f^{(n+1)}(g(t))| \\
&= \frac{1}{n+q+1}|f^{(n+1)}(\zeta)| + \frac{n}{(q+1)(n+q+1)}|f^{(n+1)}(g(t))|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\int_{\Omega} f\circ g\,d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta)\int_{\Omega}\frac{(g-\zeta)^k}{k!}\,d\mu\right| \\
&\le \frac{1}{n!}\int_{\Omega}|g-\zeta|^{n+1}\left(\int_0^1 (1-s)^n\left|f^{(n+1)}((1-s)\zeta+sg)\right|ds\right)d\mu \\
&\le \frac{1}{n!}\frac{1}{n+q+1}\left[|f^{(n+1)}(\zeta)|\int_{\Omega}|g-\zeta|^{n+1}\,d\mu\right. \\
&\left.+\frac{n}{(q+1)}\int_{\Omega}|g-\zeta|^{n+1}|f^{(n+1)}\circ g|\,d\mu\right].
\end{aligned}$$

This completes the proof. □

4. Applications for f-Divergence

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be

$$\mathcal{P} := \left\{p | p:\Omega\to\mathbb{R},\ p(t)\ge 0,\ \int_{\Omega} p(t)\,d\mu(t) = 1\right\}.$$

We recall the definition of some divergence measures which we use in this text. The Kullback-Leibler divergence [10] is defined as:

$$D_{KL}(p,q) := \int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right]d\mu(t), \quad p,q\in\mathcal{P}. \tag{4.1}$$

Following is the definition of χ^2-divergence:

$$D_{\chi^2}(p,q) := \int_\Omega p(t)\left[\left(\frac{q(t)}{p(t)}\right)^2-1\right]d\mu(t), \quad p,q\in\mathcal{P}. \tag{4.2}$$

Following is the definition of the higher order χ-divergence [1]:

$$D_{\chi^k}(p,q) := \int_\Omega \frac{(q(t)-p(t))^k}{p^{k-1}(t)}d\mu(t), \quad p,q\in\mathcal{P}; \tag{4.3}$$

$$D_{|\chi|^k}(p,q) := \int_\Omega \frac{|q(t)-p(t)|^k}{p^{k-1}(t)}d\mu(t), \quad p,q\in\mathcal{P}. \tag{4.4}$$

The above definition(s) can be generalised as follows [11]:

$$D_{\chi^k,\lambda}(p,q) := \int_\Omega \frac{(q(t)-\lambda p(t))^k}{p^{k-1}(t)}d\mu(t), \quad p,q\in\mathcal{P}; \tag{4.5}$$

$$D_{|\chi|^k,\lambda}(p,q) := \int_\Omega \frac{|q(t)-\lambda p(t)|^k}{p^{k-1}(t)}d\mu(t), \quad p,q\in\mathcal{P}. \tag{4.6}$$

Csiszár f-divergence is defined as follows [4]

$$I_f(p,q) := \int_\Omega p(t)f\left[\frac{q(t)}{p(t)}\right]d\mu(t), \quad p,q\in\mathcal{P}, \tag{4.7}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. The Kullback-Leibler divergence and the χ^2-divergence are particular instances of Csiszár f-divergence. For the basic properties of Csiszár f-divergence, we refer the readers to [4], [5], and [14].

Proposition 4.1. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Assume that $p,q\in\mathcal{P}$ and there exists constants $0<r<1<R<\infty$ such that*

$$r\le\frac{q(t)}{p(t)}\le R, \quad \text{for } \mu\text{-a.e. } t\in\Omega. \tag{4.8}$$

If $\zeta\in[r,R]$ and $f^{(n)}$ is absolutely continuous on $[r,R]$, then we have the inequalities

$$\left|I_f(p,q)-f(\zeta)-\sum_{k=1}^n\frac{1}{k!}f^{(k)}(\zeta)D_{\chi^k,\zeta}(p,q)\right| \le \frac{1}{(n+1)!}\|f^{(n+1)}\|_{I,\infty}D_{|\chi|^{n+1},\zeta}(p,q).$$

In particular, when $\zeta=1$, we have

$$\left|I_f(p,q)-\sum_{k=1}^n\frac{1}{k!}f^{(k)}(1)D_{\chi^k}(p,q)\right|\le\frac{1}{(n+1)!}\|f^{(n+1)}\|_{I,\infty}D_{|\chi|^{n+1}}(p,q). \tag{4.9}$$

We remark that we recover Theorem 1 of [1] in (4.9), with the assumption that $f(1)=0$.

Proof. We choose $g(t) = q(t)/p(t)$ in (3.2), and note that $\int_\Omega p(t)d\mu = 1$. Therefore, we have

$$\begin{aligned} & \left| \int_\Omega f\left(\frac{q(t)}{p(t)}\right) p(t)\, d\mu - f(\zeta) - \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(\zeta) \int_\Omega \left(\frac{q(t)}{p(t)} - \zeta\right)^k p(t)\, d\mu \right| \\ = & \left| I_f(p,q) - f(\zeta) - \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(\zeta) \int_\Omega \frac{(q(t)-\zeta p(t))^k}{p(t)^{k-1}}\, d\mu \right| \\ = & \left| I_f(p,q) - f(\zeta) - \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(\zeta) D_{\chi^k,\zeta}(p,q) \right| \\ & \le \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I,\infty} \int_\Omega \left| \frac{q(t)-\zeta p(t)}{p(t)} \right|^{n+1} p(t)\, d\mu \\ & \le \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I,\infty} D_{|\chi|^{n+1},\zeta}\, d\mu. \end{aligned}$$

This completes the proof. □

Example. *If we consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t\log(t)$, then*

$$I_f(p,q) = \int_\Omega p(t) \frac{q(t)}{p(t)} \log\left(\frac{q(t)}{p(t)}\right) d\,\mu(t) = \int_\Omega q(t) \log\left(\frac{q(t)}{p(t)}\right) d\,\mu(t) = D_{KL}(q,p).$$

We have $f'(t) = \log(t)+1$ and $f^{(k)}(t) = (-1)^k t^{-(k-1)}$, for $k \ge 2$. By Proposition 4.1, we have

$$\begin{aligned} & \left| D_{KL}(q,p) - \zeta\log(\zeta) - (1-\zeta)(\log(\zeta)+1) - \sum_{k=2}^{n} \frac{1}{k!} (-1)^k \zeta^{-(k-1)} D_{\chi^k,\zeta}(p,q) \right| \\ & \le \frac{1}{(n+1)!} r^{-n} D_{|\chi|^{n+1},\zeta}(p,q), \end{aligned}$$

for all $\zeta \in [r,R]$. When $\zeta = 1$, we have

$$\left| D_{KL}(q,p) - \sum_{k=2}^{n} \frac{1}{k!} (-1)^k D_{\chi^k}(p,q) \right| \le \frac{1}{(n+1)!} r^{-n} D_{|\chi|^{n+1}}(p,q).$$

Example. *If we consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\log(t)$, then*

$$I_f(p,q) = -\int_\Omega p(t) \log\left(\frac{q(t)}{p(t)}\right) d\,\mu(t) = \int_\Omega p(t) \log\left(\frac{p(t)}{q(t)}\right) d\,\mu(t) = D_{KL}(p,q).$$

We have $f^{(k)}(t) = (-1)^k t^{-k}$ for $k \ge 1$. By Proposition 4.1, we have

$$\begin{aligned} & \left| D_{KL}(p,q) + \log(\zeta) - \sum_{k=1}^{n} \frac{1}{k!} (-1)^k \zeta^{-k} D_{\chi^k,\zeta}(p,q) \right| \\ & \le \frac{1}{(n+1)!} r^{-(n+1)} D_{|\chi|^{n+1},\zeta}(p,q), \end{aligned}$$

for all $\zeta \in [r,R]$. When $\zeta = 1$, we have

$$\left| D_{KL}(p,q) - \sum_{k=1}^{n} \frac{1}{k!} (-1)^k D_{\chi^k}(p,q) \right| \le \frac{1}{(n+1)!} r^{-(n+1)} D_{|\chi|^{n+1}}(p,q).$$

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