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CLOSED FORM GENERAL SOLUTION OF THE HYPERGEOMETRIC k-MATRIX DIFFERENTIAL EQUATION

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ABSTRACT. The main goal of this present paper is to deal with the hypergeometric k-matrix differential equation $kz(1-kz)W'' - kzAW' + (C - kz(B + kI))W' - ABW = 0$, where $k > 0$. First we prove that, if a matrix C is invertible and no non negative integer is one of its eigenvalue, then the hypergeometric matrix k-function $F_k(A, B; C; z)$ is an analytic solution in the unit disc. Also, if the matrices A and B commutes with matrix C, then a closed form general solution is expressed in terms of $F_k(A, B; C; z)$ and $F_k(A+kI-C, B+kI-C; 2kI-C; z)z^{I-\frac{C}{k}} \in \Omega(\alpha) = \{z \in D_0, 0 < |z| < \alpha\}$, where D_0 is the complex plane cut along the negative real axis and $\alpha > 0$ is a positive number.

1. INTRODUCTION

Most of the special functions encountered in physics, engineering and probability theory are special cases of hypergeometric functions ([1]-[3], [4]). A function of matrix arguments a real or complex valued function of the elements of a matrix. Special matrix functions appear in the literature related to Statistics [5], Lie groups theory [6], and more recently in connection with matrix analogues of Laguerre, Hermite and Legendre differential equations and the corresponding polynomial families ([7]-[9]). Apart from the close relationship with the well-known k-beta and k-gamma matrix functions, the emerging theory of orthogonal matrix polynomials ([10]-[12]) and its operational calculus suggest the study of hypergeometric matrix function. Recently Mubeen *et al.* [13], have defined beta matrix k-function

$$B_k(P, Q) = \frac{\Gamma_k(P)\Gamma_k(Q)}{\Gamma_k(P+Q)}. \tag{1.1}$$

In this paper, we investigate some results on Gauss hypergeometric matrix k-function $F_k(A, B; C; z)$. Conditions for the convergence on $|z| < 1$ of the unit disc are treated. Recently, the researchers have worked on k-special functions ([14],[15],[16])and Mubeen *et al.* [17] have defined the solution of hypergeometric k-differential equations. In this paper, we define that if matrices B and C commutes and are stable then the hypergeometric k-matrix $F_k(A, B; C; z)$ is a solution

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of the differential equation and is called the hypergeometric k-matrix differential equation.

$$kz(1-kz)W'' - kzAW' + (C-(B+kI)kz)W' - AWB = 0.$$

If A is an arbitrary matrix in $C^{r\times r}$ and C is an invertible matrix whose eigenvalues are not negative integers then we prove that equation

$$kz(1-kz)W'' - kzAW' + W'(C+(n-k)kIz) - AWB = 0$$

has k-matrix polynomial solutions of degree n for all integer $n \geq 1$.
Throughout in this paper, for a matrix A in $C^{r\times r}$ its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. The 2-norm of A will be denoted by $\parallel A \parallel$ and it is defined by

$$\parallel A \parallel = \sup_{x\neq 0} \frac{\parallel Ax \parallel_2}{\parallel x \parallel_2} \tag{1.2}$$

where for a y in $C^{r\times r}$, $\parallel Y \parallel_2 = (yTy)^{\frac{1}{2}}$ is the Euclidean norm of y. Let us denote $\sigma(A)$ and $\beta(A)$ the real numbers

$$\alpha(A) = \max\{Re(z) : z \in \alpha(A)\}, \qquad \beta(B) = \min\{Re(z) : z \in \alpha(B)\}. \tag{1.3}$$

Let $f(z)$ and $g(z)$ be two holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $C^{r\times r}$ with $\alpha(A) \subset \Omega$, then from the properties of matrix functional calculus [[18],p.558], it follows that

$$f(A)g(B) = g(B)f(A). \tag{1.4}$$

If P is a positive stable matrix in $C^{r\times r}$, $n \geq 1$ is an integer and $k > 0$, then

$$\Gamma_k(P) = \lim_{n\to\infty} n!k!(P)^{-1}_{n,k}(nk)^{\frac{P}{k}-I}. \tag{1.5}$$

The reciprocal of gamma k-function denoted by $\Gamma_k^{-1} = \frac{1}{\Gamma_k}$ is an entire function of the complex variable. Like wise the image of the inverse gamma k-matrix acting on the matrix A, denoted by $\Gamma_k^{-1}(A)$ is a well defined matrix for $k > 0$. Now, if
$A + nkI$ is invertible matrix for every integer $n \geq 0$ and $k > 0$,
then Γ_k is invertible and its inverse coincides with $\Gamma_k^{-1}(A)$, and recently we have defined

$$A(A+kI)(A+2kI)\cdots(A+(n-1)kI)\Gamma_k^{-1}(A+nkI) = \Gamma_k^{-1}(A), \quad n \geq 1, k > 0. \tag{1.6}$$

Hence by using the condition that $A + nkI$ is invertible matrix, equation (1.5) can be written as

$$A(A+kI)(A+2kI)\cdots(A+(n-1)kI) = \Gamma_k(A+nkI)\Gamma_k^{-1}(A), \quad n \geq 1, k > 0. \tag{1.7}$$

Like the Pochhammer k-symbol the researchers also defined see [13]

$$(A)_{n,k} = A(A+kI)(A+2kI)\cdots(A+(n-1)kI), \quad n > 0, \quad (A)_0 = I, \tag{1.8}$$

for any matrix A in $C^{r\times r}$ by application of the matrix functional calculus. The Schur deposition of a matrix A is given by

$$\|e^{tA}\| \leq \sum_{i=0}^{r-1} \frac{(\|A\| r^{\frac{1}{2}} t)}{i!}, \quad t \geq 0. \tag{1.9}$$

By the perturbation lemma [[18], p. 584.], if P and Q are matrices in $C^{r\times r}$ with Q invertible and if

$$||P-Q|| < ||Q^{-1}||^{-1}, \tag{1.10}$$

then

$$||P^{-1}-Q^{-1}|| \le \frac{||Q^{-1}||^2||P-Q||}{1-||Q^{-1}|| \quad ||P-Q||}. \tag{1.11}$$

Similarly, if P and Q are positive stable matrices in $C^{r\times r}$, then beta matrix k-function is defined by

$$B_k(P,Q) = \frac{1}{k}\int_0^1 t^{\frac{P}{k}-I}(1-t)^{\frac{Q}{k}-I}dt. \tag{1.12}$$

Recently Mubeen *et al.* [13] have defined that if P and Q are commuting positive stable matrices then $B_k(P,Q) = B_k(Q,P)$, and commutativity is a necessary condition for the symmetry of beta matrix k-function.

2. On Bilateral Second Order Linear Matrix Differential Equations

In this section, we summarize some results whose proofs can be found in ([19], chapter 10).

Theorem 2.1 [see ([19], p. 287).] Let $B(z_0;r)$ be an open disc of the complex plane of radius r centered at z_0. Let E be the Banach space of all $C^{r\times r}$ matrices endowed with the two norms. Let $f : B(z_0;r)\times E \to E$ be a continuous function such that

$$||f(z,X_1)-f(z,X_2)|| \le K(|z-z_0|)||X_1-X_2||, \tag{2.1}$$

where $z\in B(z_0;r)$, X_i lies in E and $\xi \to K(\xi)$ is a real valued continuous function on $[0,r]$. Then for each X_0 in E, there exist one and one solution U of

$$X' = f(z,X)$$

defined in $B(z_0;r)$, such that $U(z_0) = X_0$.

In particular, if we consider a bilateral linear differential equation of the form

$$X' = A_1(z)X + XB_1(z) + A_2(z)XB_2(z) + A_3(z)XB_3(z) + C(z), \tag{2.2}$$

where A_i, B_i for $i = 1,2,3$ and C are continuous functions from $B(z_0;r)$ into E, then

$$f(z,X) = A_1(z)X + XB_1(z) + A_2(z)XB_2(z) + A_3(z)XB_3(z) + C(z),$$

and f satisfies a Lipschitz condition of type (2.1). By theorem 2.1, for each X_0 in E, there exist one and only one solution U of (2.1) for $z\in B(z_0;r)$ such that $U(z_0) = X_0$.

The following concept is an extension of the concept of the fundamental set of solution introduced in [20] for a particular case.

Definition 2.2. Let $f_j : B(z_0;r)\to E$ be bounded continuous function in $B(z_0;r)$ for $1\le j\le 4$ and let U_1, U_2 be two solutions of the second order differential equation

$$X'' = f_1(z)X' + f_2(z)Xf_3(z) + X'f_4(z). \tag{2.3}$$

We say that $\{U_1, U_2\}$ is a fundamental set of solution of (2.3) in $B(z_0; r)$, if for any solution U of (2.3) admits a unique representation of the form

$$U(z) = U_1(z)P + U_2(z)Q, \quad z \in B(z_0; r), \tag{2.4}$$

where P, Q are matrices in $C^{r\times r}$ uniquely determined by U.
Theorem 2.3 (see [21].) If $\{U_1, U_2\}$ is a pair of solutions of equation (2.3) in $B(z_0; r)$ such that $C^{2r\times 2r}$ matrix

$$W(U_1, U_2, z_0) = \begin{bmatrix} U_1(z_0) & U_2(z_0) \\ U_1'(z_0) & U_2'(z_0) \end{bmatrix} \tag{2.5}$$

is invertible, then $\{U_1, U_2\}$ is a fundamental set of solutions of (2.3) in $B(z_0; r)$.
Example 2.4 Let z_0 be a complex number with $0 < |z_0| < 1$ and let A, B and C be complex matrices in $C^{r\times r}$. Let $0 < \delta_0 < |z_0| < 1 - \delta_1$, with $0 < \delta_1 < 1$ and $r = \min(\delta_0, \delta_1)$. Then in $B(z_0; r)$ equation

$$W'' = \frac{A}{1-kz}W' + \frac{A}{kz(1-kz)}WB - [\frac{C - kz(B+kI)}{kz(1-kz)}] \tag{2.6}$$

is of type (2.3) with

$$f_1(z) = \frac{A}{1-kz}, \quad f_2(z) = \frac{A}{kz(1-kz)}, \quad f_3(z) = B, \quad f_4(z) = \frac{-C + kz(B+kI)}{kz(1-kz)}. \tag{2.7}$$

3. On The Hypergeometric Matrix k-Functions

In this section, we begin with the definition of the hypergeometric matrix k-functions where $k > 0$
Definition 3.1 Let A, B and C be matrices in $C^{r\times r}$ where $C + nkI$ is invertible for all integer $n \geq 0$. Then we defined the hypergeometric matrix k-function by

$$F_k(A, B; C; z) = \sum_{n=0}^{\infty} \frac{1}{n!}(A)_{n,k}(B)_{n,k}[(C)_{n,k}]^{-1}z^n. \tag{3.1}$$

Now we prove that the hypergeometric matrix k-function converges for $|z| < 1$. Note that if $n > ||C||$, then by perturbation lemma see (1.11), we can write

$$||(\frac{C}{nk} + I)^{-1}|| \leq \frac{1}{1 - \frac{||C||}{nk}} = \frac{nk}{nk - ||C||}$$

and

$$||(C + nkI)^{-1}|| \leq ||\frac{1}{nk}(\frac{C}{nk} + I)^{-1}|| = \frac{1}{nk}||(\frac{C}{nk} + I)^{-1}|| \leq \frac{1}{nk - ||C||}. \tag{3.2}$$

Now, let us denote

$$\Upsilon(n) = ||C^{-1}|| \quad ||(C + kI)^{-1}|| \cdots ||(C + (n+1)kI)^{-1}||, n \geq 0 \tag{3.3}$$

and note that for $k > 0$ the following relation holds true

$$||(A)_{n,k}|| \leq (||A||)_{n,k}, ||(B)_{n,k}|| \leq (||B||)_{n,k}. \tag{3.4}$$

With the aid of (3.3) and (3.4), we get

$$\begin{aligned} ||\frac{(A)_{n,k}(B)_{n,k}[(C)_{n,k}]^{-1}}{n!}z^n|| &\leq \frac{||(A)_{n,k}||||(B)_{n,k}||\Upsilon(n)}{n!}|z^n| \\ &\leq \frac{(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)}{n!}|z^n|. \end{aligned}$$

Now we prove the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}(A)_{n,k}(B)_{n,k}\Upsilon(n)z^n.$$

converges for $|z| < \frac{1}{k}$ using ratio test. In fact by using (3.1), it follows

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{(||A||_{n+1,k})(||B||)_{n+1,k}\Upsilon(n+1)|z|^{n+1}}{(||A||)_{n,k}(||B||_{n,k})\Upsilon(n)|z|^n(n+1)}\right| &= \lim_{n\to\infty} \left|\frac{(||A||+nk)(||B||+nk)||(C+nkI)^{-1}||}{n+1}z\right| \\ &\leq \lim_{n\to\infty} \frac{(||A||+nk)(||B||+nk)}{(n+1)(nk-C)}|z| = k|z|.\end{aligned}$$

Thus, the series (3.1) is absolutely convergent for $|z| < \frac{1}{k}$.
Now, we are interested in the determination of domain where $U_1(z) = F_k(A, B; C; z)$ is invertible. Let $0 < \gamma < 1$ and note that

$$\lim_{n\to\infty} \frac{(||A||+nk)(||B||+nk)}{(n+1)(nk-C)} = k.$$

Let $n_0 \geq \max(1, ||C||)$ be a positive integer such that

$$\frac{(||A||+nk)(||B||+nk)}{(n+1)(nk-C)} \leq k+\gamma, n \geq n_0. \tag{3.5}$$

Now let us denote

$$\Lambda = \sum_{n=1}^{n_0-1} \frac{(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)}{n!}, \quad L = \frac{(||A||)_{n_0,k}(||B||)_{n_0,k}\Upsilon(n_0)}{n_0!}, \tag{3.6}$$

where $\Upsilon(n)$ is defined by (3.3). Now, since the function

$$f(x) = \Lambda x + L\frac{x}{1-x}, 0 \leq x < 1 \tag{3.7}$$

increases, $f(0) = 0$ and $\lim_{x\to 1^-} f(x) = +\infty$, so there exist only one value ρ_0 such that

$$f(\rho_0) = 1-\gamma. \tag{3.8}$$

Let us take ρ_1 with $0 < \rho_1 < \rho_0$, such that $(k+\gamma)\rho_1 < \rho_0$. By (3.5), we can write

$$\frac{(\frac{1}{(n+1)!})(||A||)_{n+1,k}(||B||)_{n+1,k}\Upsilon(n+1)}{(\frac{1}{n!})(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)} \leq \frac{(||A||+nk)(||B||+nk)}{(n+1)(nk-||C||)} \leq k+\gamma$$

$$\frac{1}{(n+1)!}(||A||)_{n+1,k}(||B||)_{n+1,k}\Upsilon(n+1)\rho_1 \leq \rho_0\frac{1}{n!}(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n), n \geq n_0. \tag{3.9}$$

Hence,

$$\begin{aligned}||U_1(\rho_1) - U_1(0)|| &= ||U_1(\rho_1) - I|| \\ &\leq \sum_{n\geq 1} \frac{1}{n!}(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)\rho_1^n \\ &\leq \left\{ \sum_{n=1}^{n_0-1} \tfrac{1}{n!}(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n) \right\}\rho_1^n + \sum_{n\geq n_0} \frac{1}{n!}(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)\rho_1^n \\ &\leq \Lambda\rho_0 + \sum_{n\geq n_0} \frac{1}{n!}(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)\rho_1^n.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n\geq n_0}\frac{1}{n!}(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)\rho_1^n &\leq \left[\ \tfrac{1}{n_0!}(||A||)_{n_0,k}(||B||)_{n_0,k}\Upsilon(n_0)\ \right]\sum_{n\geq n_0}\rho_0^n\\ &= \frac{1}{n_0!}(||A||)_{n_0,k}(||B||)_{n_0,k}\frac{\rho_0^n}{1-\rho_0} < L\frac{\rho_0}{1-\rho_0}.\end{aligned}$$

Thus, we obtain

$$||U_1(\rho_1)-I|| \leq \Lambda\rho_0 + L\frac{\rho_0}{1-\rho_0} = f(\rho_0) = 1-\gamma.$$

Hence, $||U_1(\rho_1)-I|| < 1-\gamma$, for $0\leq\rho\leq\rho_1$ and by the perturbation lemma, we get that

$$U_1(z) = F_k(A,B;C;z)$$

is invertible and

$$||U_1(\rho_1)-I|| < 1-\gamma, \tag{3.10}$$

for $|z|\leq\rho_1$. By (1.10) and (3.10), we have

$$\begin{aligned}||(U_1(z)-I|| &\leq \frac{||U_1(z)-I||}{1-||U_1(z)-I||} < \frac{1-\gamma}{\gamma}, |z|\leq\rho_1,\\ ||(U_1(z))^{-1}|| &\leq 1+\frac{1-\gamma}{\gamma} = \frac{1}{\gamma}, |z|\leq\rho_1.\end{aligned}$$

Furthermore, for $n\geq n_0$ and $|z|\leq\rho_1$, we obtain

$$\begin{aligned}||U_1(z)-\sum_{j=1}^{n-1}\frac{1}{j!}(A)_{j,k}(B)_{j,k}[(C)_{j,k}]^{-1}z^j|| &\leq \sum_{m\geq n}\frac{1}{m!}(||A||)_{m,k}(||B||)_{m,k}\Upsilon(m)\rho_1^m\\ &\leq L\sum_{m\geq n}\rho_0^m = L\frac{\rho_0^n}{1-\rho_0}. \end{aligned}\tag{3.11}$$

Hence, given any $\epsilon>0$, taking $n\geq n_0$ so that $\rho_0^n < \frac{\epsilon(1-\rho_0)}{L}$, or equivalently

$$n \geq \max\left\{\ \frac{\ln[\frac{\epsilon(1-\rho_0)}{L}]}{\ln(\rho_0)}, n_0\ \right\} = n_\epsilon, \tag{3.12}$$

if we denote

$$F_{n,k}(A,B;C;z) = \sum_{j=0}^{n}\frac{(A)_{j,k}(B)_{j,k}[(C)_{j,k}]^{-1}z^j}{j!}, \tag{3.13}$$

by the truncation of order n of $F_k(A,B;C;z)$, then it follows

$$||U_1(z)-F_{n-1,k}(A,B;C;z)|| \leq \epsilon, |z|\leq\rho_1, n\geq n_\epsilon.$$

Summarizing, we established the following result.

Theorem 3.2 Let $0<\gamma<1$ and $\epsilon>0$ and let n_0 and n_ϵ be positive integers satisfying (3.9) and (3.11) respectively. Let ρ_0 be the solution of (3.8) where f is defined by (3.7) and let ρ_1 be such that $\rho_1 < (k+\gamma)^{-1}\rho_0$, $U_k(z) = F_k(A,B;C;z)$ be defined by (3.13). Then

(i) $U_1(z)$ is invertible and $||(U_1(z))^{-1}|| \leq \gamma^{-1}$ for $|z|\leq\rho_1$,

(ii) $||U_1(z)-F_{n,k}(A,B;C;z)|| < \epsilon$, $|z|\leq\rho_1$, $n\geq n_\epsilon$.

Now, let us consider the boundness of $U_1'(z)$ for $|z| \leq \rho_1$, where ρ_1 is given by theorem 3.2. Note that for $|z| \leq \rho_1$, we get

$$||U_1'(z)|| \leq \sum_{n\geq 1} \frac{(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)}{(n-1)!}\rho_1.$$

Given γ with $0 < \gamma < 1$, let $n_1 \geq n_0$ such that

$$\frac{(||A|| + nk)(||B|| + nk)}{n(nk - ||C||)} < k + \gamma, \quad n \geq n_1 > \max(1, ||C||). \tag{3.14}$$

Let ρ_2 be such that $(k+\gamma)\rho_2 < \rho_1$ and

$$\Lambda_1 = \sum_{n=1}^{n_1-1} \frac{(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)}{(n-1)!}, \quad L_1 = \frac{(||A||)_{n_1,k}(||B||)_{n_1,k}\Upsilon(n_1)}{(n-1)!}, \tag{3.15}$$

where $\Upsilon(n)$ is defined by (3.3). Also note that for $|z| \leq \rho_2$, we can write

$$||U_1'(z)|| \leq \sum_{n=1}^{n_1-1} \frac{(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)}{(n-1)!}\rho_2^{n-1} + \sum_{n\geq n_1} \frac{(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)}{(n-1)!}\rho_2^{n-1}, \tag{3.16}$$

and by (3.14), we have

$$\frac{1}{n!}(||A||)_{n+1,k}(||B||)_{n+1,k}\Upsilon(n+1) < (k+\gamma)\frac{1}{(n-1)!}(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n), n \geq n_1,$$

it follows

$$\begin{aligned}
\sum_{n\geq n_1} \frac{(||A||)_{n,k}(||B||)_{n,k}\Upsilon(n)}{(n-1)!}\rho_2^{n-1} \quad &\leq \quad \frac{(||A||)_{n_1,k}(||B||)_{n_1,k}\Upsilon(n_1)}{(n-1)!}\sum_{n\geq n_1}[\rho_2(k+\gamma)]^{n-1} \\
&\leq \quad L_1 \sum_{n\geq n_1} \rho_1^{n-1} = \frac{L_1\rho_1^{n-1}}{1-\rho_1} \\
&\leq \quad \frac{L_1}{1-\rho_1}.
\end{aligned} \tag{3.17}$$

If $F_{n,k}(A, B; C; z)$ is defined by (3.13), from (3.15)-(3.17), we get

$$\begin{aligned}
||U_1'(z) - F_{n,k}'(A, B; C; z)|| \quad &\leq \quad \frac{L_1}{1-\rho_1}\rho_1^{n-1}, n \geq n_1, |z| \leq \rho_2, \\
||U_1'(z)|| \quad &\leq \quad \Lambda_1 + \frac{L_1}{1-\rho_1}, |z| \leq \rho_2.
\end{aligned} \tag{3.18}$$

Summarizing, the following result has been established.

Corollary 3.3 With the notation of theorem 3.2, let $\rho_2 > 0$ be chosen so that $\rho_2(k+\gamma) < \rho_1$. Let n_1 be a positive integer satisfying (3.14) and

$$n_\epsilon' = 1 + \max\left\{ \frac{\ln[\frac{\epsilon(1-\rho_1)}{L}]}{\ln(\rho_1)}, n_1 \right\}, \tag{3.19}$$

where Λ_1 and L_1 are defined by (34). Then
(i) $||U_1'(z)|| \leq \Lambda_1 + (1-\rho_1)^{-1}L_1$, $|z| \leq \rho_2$,
(ii) $||U_1'(z) - F_{n,k}'(z)(A, B : C; z)|| \leq \epsilon$, $|z| \leq \rho_2$, $n \geq n_\epsilon'$.

4. The Hypergeometric k-Matrix Differential equation

Let us consider the hypergeometric k-Matrix differential equation

$$kz(1-kz)W'' - kzAW' + W'(C - kz(B+kI)) - ABW = 0, \quad 0 \leq |z| < 1 \quad (4.1)$$

where

$$CB = BC \quad (4.2)$$

and $C + nkI$ is invertible matrix. Note that equation (4.1) can be written in the form of (2.6). Now, let us seek a solution W of (3.19) in the form

$$W(z) = \sum_{n\geq 0} W_n z^n, \quad |z| < 1. \quad (4.3)$$

Taking formal derivatives of (4.3) and substituting into (4.1), we get

$$W'(z) = \sum_{n=1}^{\infty} nW_n z^{n-1}, \quad W''(z) = \sum_{n=2}^{\infty} n(n-1)W_n z^{n-2}, \quad |z| < 1.$$

Hence

$$\begin{aligned} & kz(1-kz)W'' - kzAW' + W'(C - kz(B+kI)) - ABW \\ &= \sum_{n=2}^{\infty} nk(n-1)W_n z^{n-1} - \sum_{n=2}^{\infty} nk^2(n-1)W_n z^n - A\sum_{n=1}^{\infty} nkW_n z^n + \sum_{n=1}^{\infty} nW_n C z^{n-1} \\ &\qquad - \sum_{n=1}^{\infty} nkW_n(B+kI)z^n - \sum_{n=0}^{\infty} AW_n B z^n, \end{aligned}$$

replacing $n = n+1$ in the first and fourth summation, we obtain

$$\begin{aligned} & kz(1-kz)W'' - kzAW' + W'(C - kz(B+kI)) - ABW \\ &= \sum_{n=1}^{\infty} nk(n+1)W_{n+1} z^{n-1} - \sum_{n=2}^{\infty} nk^2(n-1)W_n z^n - A\sum_{n=1}^{\infty} nkW_n z^n + \sum_{n=1}^{\infty} nW_n C z^{n-1} \\ &\qquad - \sum_{n=1}^{\infty} nkW_n(B+kI)z^n - \sum_{n=0}^{\infty} AW_n B z^n, \end{aligned}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \{nk(n+1)W_n - nk^2(n-1)W_n - nkAW_n + (n+1)W_{n+1}C - nkW_n(B+kI) - AW_n B\} z^n + 2kW_2 z \\ &\qquad - kAW_1 z + W_1 C + 2W_2 C z - W_1(B+kI)kz - AW_0 - AW_1 B z = 0. \end{aligned}$$

By equating the coefficients of each power z^n, we get

$$\begin{aligned} z^0 \quad &: \quad W_1 C - AW_0 B = 0, \\ z^1 \quad &: \quad 2kW_2 - kAW_1 + 2W_2 C - W_1(B+kI)kz - AW_1 B \\ &= \quad 2W_2(kI + C) - AW_1(kI + B) - W_1(B+kI)k = 0 \\ \vdots \quad &\vdots \\ z^n \quad &: \quad nk(n+1)W_{n+1} - nk^2(n-1)W_n - nkAW_n - nkW_n B - nk^2 W_n + (n+1)W_{n+1}C - AW_{n,k} \\ &= \quad W_{n+1}(n+1)(nkI + C) - (A + nkI)W_n(B + nkI) = 0. \end{aligned}$$

Hence,

$$W_1 = AW_0BC^{-1} \tag{4.4}$$

and

$$W_{n+1} = \frac{(A+nkI)W_n(B+nkI)(C+nkI)^{-1}}{n+1}, \quad n \geq 0. \tag{4.5}$$

Note that from (4.5) and (4.2), taking into account $W_0 = I$, it follows that

$$\begin{aligned} W_{n+1} &= \frac{A(A+kI)\cdots(A+nkI)B(B+kI)\cdots(B+nkI)(C+nkI)^{-1}\cdots(C+kI)^{-1}C^{-1}}{(n+1)!} \\ &= \frac{(A)_{n+1,k}(B)_{n+1,k}[(C)^{-1}_{n+1,k}}{(n+1)!}, \quad n \geq 0. \end{aligned}$$

Hence, there exist a solution of (4.1) satisfying $W_1(0) = I$, given by

$$W_1(z) = F_k(A, B; C; z). \tag{4.6}$$

Now, we derive a second solution W_2 of equation (4.1), under the hypothesis that $C + nkI$ is invertible for all $n \geq 0$, (4.2) and together with $AC = CA$.
Let D_0 be the complex plane cut along the negative real axis, and let us denote $z^{I-\frac{C}{k}} = \exp((I - \frac{C}{k})\log z)$ where log represent the principal logarithm [[22], p. 72]. Let us seek a solution of the form

$$W_2(z) = V(z)z^{I-\frac{C}{k}}, \quad |z| < 1, z \in D_0, \tag{4.7}$$

where V is a function to be determined. By taking derivative of W_2, we obtain

$$\begin{aligned} W_2'(z) &= V'(z)z^{I-\frac{C}{k}} + \frac{1}{k}V(z)z^{-\frac{C}{k}}(kI - C), \\ W_2''(z) &= V''(z)z^{I-\frac{C}{k}} + \frac{2}{k}V'(z)z^{-\frac{C}{k}}(kI - C) - \frac{1}{k^2}V(z)z^{-\frac{C}{k}-I}C(kI - C). \end{aligned}$$

Substituting these expression in (4.1) and after simplification, we get

$$\left\{ \begin{array}{c} kz(1-kz)V' + \{2kV' - V' - 3k^2zV' + 2kzV'C - kzAV' - kzBV'\} \\ +\{kVC - V^2C - kAV + AVC - k^2VB + VCB - k^2V + kVC - AVB\} \end{array} \right\} z^{I-\frac{C}{k}} = 0,$$

this implies that

$$\begin{aligned} &kz(1-kz)V' + \{2kV' - V' - 3k^2zV' + 2kzV'C - kzAV' - kzBV'\} \\ &+ \{kVC - V^2C - kAV + AVC - k^2VB + VCB - k^2V + kVC - AVB\} = 0. \end{aligned} \tag{4.8}$$

Now, Suppose that

$$V(z)C = CV(z), \quad V'(z)C = CV'(z). \tag{4.9}$$

Then (4.8) is equivalent to

$$kz(1-kz)V'' - kz(A + kI - C)V' \tag{4.10}$$

$$+V'[(2kI - C) - kz(B + 2kI - C)] - (A + kI - C)V(B + kI - C) = 0. \tag{4.11}$$

Note that (4.10) is a hypergeometric k-matrix differential equation of the form (4.1) with parameters

$$A' = A + kI - C, \quad B' = B + kI - C, \quad C' = 2kI - C. \tag{4.12}$$

As we proved above

$$V(z) = F_k(A + kI - C, B + kI - C; 2kI - C; z), \quad |z| < 1 \tag{4.13}$$

is a solution of (4.10) as $BC = CB$. Furthermore, under the hypothesis (4.1) and (4.9), in fact $F_k(A+kI-C, B+kI-C; 2kI-C; z)$ and its derivative commutes with matrix C because the k-matrix coefficient of $F_k(A+kI-C, B+kI-C; 2kI-C; z)$ are

$$\frac{1}{n!}(A+kI-C)_{n,k}(B+kI-C)_{n,k}[(2kI-C)_{n,k}]^{-1}, n \geq 0.$$

Hence,

$$W_2(z) = F_k(A+kI-C, B+kI-C; 2kI-C; z)z^{I-\frac{C}{k}} \tag{4.14}$$

is a solution of (4.1) for $z \in D_0$, $|z| < 1$.
Now, we construct a closed form general solution of the hypergeometric k-matrix equation (4.1) under the hypothesis $C+nkI$ is invertible,(4.2) and $AC+CA$. Note that

$$W_1(z) \quad = \quad U_1(z), \quad W_2(z) = U_2(z)z^{I-\frac{C}{k}}, 0 < |z| < 1, z \in D_0, \tag{4.15}$$

$$U_1(z) \quad = \quad F_k(A, B; C; z), \quad U_2(z) = F_k(A+kI-C, B+kI-C; 2kI-C; z) \tag{4.16}$$

are solutions of (4.1). By theorem 2.3, the pair $\{W_1, W_2\}$ is a set of solutions of (4.1) in the domain $\Omega(\alpha) = \{z \in D_0, 0 < |z| < \alpha\}$ if the matrix

$$S(z) = \begin{bmatrix} W_1(z) & W_2(z) \\ W_1'(z) & W_2'(z) \end{bmatrix} \tag{4.17}$$

is invertible in $\Omega(\alpha)$. Now, by the properties of the Schur complement of a matrix see [23], the matrix $S(z)$ given by (4.17) is invertible if and only if

$$M(z) = W_2'(z) - W_1'(z)[W_1(z)]^{-1}W_2(z)$$

is invertible. Note that

$$\begin{aligned} M(z) \quad &= \quad U_2'(z)z^{I-\frac{C}{k}} + U_2 z^{-\frac{C}{k}}(kI-C) - U_1'[U_1(z)]^{-1}U_2(z)z^{I-\frac{C}{k}}, \\ &= \quad \left\{ \ U_2'(z)z + U_2(kI-C) - U_1'[U_1(z)]^{-1}U_2(z)z \ \right\} z^{-\frac{C}{k}}. \end{aligned} \tag{4.18}$$

By (4.18), $M(z)$ is invertible if and only if

$$N(z) = z[U_2'(z) - U_1'(z)[U_1(z)]^{-1}U_2(z)] + U_2(z)(kI-C) \tag{4.19}$$

is invertible. By the proofs of theorem 3.2 and corollary 3.3, we know that if $0 < \gamma < 1$, and n_2 is a positive integer such that

$$\frac{(||A+kI-C|| + nk)(||B+kI-C|| + nk)}{(n+1)(nk - ||2kI-C||)} < k + \gamma, \tag{4.20}$$

$$n \geq n_2 \geq \max(1, ||2kI-C||), \tag{4.21}$$

with

$$L_2 = \frac{(||A+kI-C||)_{n_2,k}(B+kI-C)_{n_2,k}\Phi(n_2)}{n_2!}, \tag{4.22}$$

$$\Lambda_2 = \sum_{n=1}^{n_2-1} \frac{(||A+kI-C||)_{n,k}(||B+kI-C||)_{n,k}\Phi(n)}{n!}, \tag{4.23}$$

where

$$\Phi(n) = ||(2kI-C)^{-1}|| \quad ||(kI-C)^{-1}|| \cdots ||(-C+(n-1)kI)^{-1}||, \quad n \geq 0, \tag{4.24}$$

and ρ_2 is the solution of

$$h_1(\rho_2) = 1 - \gamma, \tag{4.25}$$

with

$$h_1(x) = \Lambda_2 x + L_2 \frac{x}{1-x}, \quad 0 \le x < 1. \tag{4.26}$$

If $(k+\gamma)\rho_3 < \rho_2$, then $U_2(z) = F_k(A + kI - C, B + kI - C; 2kI - C; z)$ satisfies $||U_2(z) - I|| < 1 - \gamma$ and $||(U_2(z))^{-1}|| \le \gamma^{-1}$, $|z| \le \rho_3$.
Furthermore, if $n \ge n_2$ is chosen so that

$$\frac{(||A + kI - C|| + nk)(||B + kI - C|| + nk)}{(n+1)(nk - ||2kI - C||)} < k + \gamma, \tag{4.27}$$

$$n \ge n_2 \ge \max(1, ||2kI - C||), k > 0 \tag{4.28}$$

with

$$L_3 = \frac{(||A + kI - C||)_{n_3,k}(B + kI - C)_{n_3,k}\Phi(n_3)}{(n_3 - 1)!}, \tag{4.29}$$

$$\Lambda_3 = \sum_{n=1}^{n_3-1} \frac{(||A + kI - C||)_{n,k}(||B + kI - C||)_{n,k}\Phi(n)}{(n-1)!}, \tag{4.30}$$

by theorem 3.2 and corollary 3.3, we get

$$||U_2(z) - I|| < 1 - \gamma, \quad ||(U_2(z))^{-1}|| \le \gamma^{-1} \tag{4.31}$$

and

$$||U_2'(z)|| \le \Lambda_3 + \frac{L}{1-\rho_3}, \quad |z| \le \rho_3. \tag{4.32}$$

Thus, the following result has been established.
Corollary 4.1 Let A, B and C be matrices in $C^{r\times r}$ with hypothesis $BC = CB$, $AC = CA$ and together with (4.2) and let $U_2(z) = F_{n,k}(A + kI - C, B + kI - C; 2kI - C; z)$. Let $0 < \gamma < 1$ and let n_3 be a positive integer satisfying (4.27) and $(k+\gamma)\rho_3 < \rho_2$, then (4.32) holds true.

Now, since $N(z)$ given by (4.19) is well defined in the disc $|z| \le \rho_1$ where $U_1(z) = F_k(A, B; C; z)$ is invertible. Also, note that $N(0) = kI - C$ is invertible. By perturbation lemma, $N(z)$ is invertible in the domain $|z| < \gamma \le \rho_1$ where

$$||N(z) - (kI - C)|| < ||(kI - C)^{-1}||^{-1}. \tag{4.33}$$

Now, let us write for $|z| < \rho_1$,

$$N(z) - N(0) = N(z) - (kI - C) = z[U_2'(z) - U_1'(z)[U_1(z)]^{-1}U_2(z)] \tag{4.34}$$

$$+(U_2(z) - kI)(kI - C). \tag{4.35}$$

Taking into an account that $0 < \epsilon < 1$, $\gamma = 1 - \epsilon$ in the proof of corollary 4.2 and considering $n_4 \ge \max(1, ||2kI - C||)$ such that

$$\frac{(||A + kI - C|| + nk)(||B + kI - C|| + nk)}{n(nk - ||2kI - C||)} < 2 - \epsilon, \quad n \ge n_4, \tag{4.36}$$

and taking

$$\Lambda_4 = \sum_{n=1}^{n_4-1} \frac{(||A+kI-C||)_{n,k}(||B+kI-C||)_{n,k}\Phi(n)}{(n-1)!}, \tag{4.37}$$

$$L_4 = \frac{(||A+kI-C||)_{n_4,k}(||B+kI-C||)_{n_4,k}\Phi(n_4)}{(n_4-1)!}, \tag{4.38}$$

$$h_2(x) = \Lambda_4(x) + L_4\frac{x}{1-x}, \quad 0 \leq x < 1, \tag{4.39}$$

if ρ_4 is the only solution of the equation

$$h_2(\rho_4) = \epsilon, \tag{4.40}$$

now, taking ρ_5 such that

$$(k+\gamma)\rho_5 = \rho_5(2-\epsilon) < \rho_4, \tag{4.41}$$

then

$$||U_2(z) - I|| < \epsilon, \quad |z| \leq \rho_5. \tag{4.42}$$

By theorem 3.2, corollary 3.3 and 4.1, and by (4.24)-(4.42), taking $|z| < \min(\rho_1, \rho_3, \rho_5)$, it follows that

$$||N(z) - N(0)|| \quad \leq \quad |z| \Big[\ ||U_2'(z)|| + ||(U_1(z))^{-1}|| \quad ||U_1'(z)|| \quad ||U_2(z)|| \ \Big]$$
$$+||U_2(z) - I|| \quad ||kI - C||$$

$$\leq \quad |z| \Big[\ (\Lambda_3 + \tfrac{L_3}{1-\rho_3}) + (\tfrac{1+\epsilon}{1-\epsilon})(\Lambda_1 + \tfrac{L_1}{1-\rho_1}) \ \Big] + \epsilon||kI - C||. \tag{4.43}$$

Hence, taking

$$\epsilon = \frac{1}{2}\min\Big\{ \ ||(kI-C)^{-1}||^{-1}(||kI-C||)^{-1}, 1 \ \Big\} \tag{4.44}$$

and $|z| < \min(\rho_1, \rho_3, \rho_5, \rho_6) = \rho^*$, where

$$\rho_6 = \frac{1}{2}\Big\{ \ \Big[\ (\Lambda_3 + \tfrac{L_3}{1-\rho_3}) + (\tfrac{1+\epsilon}{1-\epsilon})(\Lambda_1 + \tfrac{L_1}{1-\rho_1}) \ \Big] (||(kI-C)^{-1}||)^{-1} \ \Big\}, \tag{4.45}$$

by (4.43), we get

$$||N(z) - N(0)|| < ||(kI-C)^{-1}||^{-1}, \quad |z| < \rho^*, \tag{4.46}$$

and thus

$$N(z) \text{ is invertible for } |z| < \min(\rho_1, \rho_3, \rho_5, \rho_6) = \rho^*.$$

Summarizing, by Theorem 2.3 and 3.2, Corollaries 3.3 and 4.1, and by previous comments the following result has been established.

Theorem 4.2 Let A, B and C be matrices in $C^{r\times r}$ satisfying $C + nkI$ is invertible, $AC = CA$ and (4.2) and let $W_1(z)$ and $W_2(z)$ be defined by (4.15) for z in D_0 with $|z| < 1$. Then there exist a positive number ρ^* such that the general solution of equation (4.1) in $\Omega(\rho^* = \{z \in D_0; 0 < |z| < \rho^*\}$ is given by

$$W(z) = W_1(z)P + W_2(z)Q, \quad P, Q \in C^{r\times r}.$$

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