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PACHPATTE'S INEQUALITIES ASSOCIATED WITH POWER QUANTUM DIFFERENCE OPERATORS

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Abstract. In this paper, we establish many inequalities of Pachpatte's type associated with a power quantum difference operator which is defined by

$$D_{n,q}f(t) = (f(qt^n) - f(t))/(qt^n - t),$$

where n is an odd natural number, $0 < q \leq 1$ is fixed such that $nq \neq 1$. This operator yields Jackson q-difference operator when $n = 1$.

1. Introduction

The power quantum difference operator [5] is defined by

$$D_{n,q}\,f(t) = \begin{cases} \dfrac{f(qt^n) - f(t)}{qt^n - t} & t \in \mathbb{R} \setminus \{-\theta, 0, \theta\}, \\ f^{'}(t) & t \in \{-\theta, 0, \theta\}, \end{cases}$$

where n is a fixed odd positive integer, $0 < q \leq 1$ is a fixed number, $nq \neq 1$ and $\theta = \infty$ for $n = 1$ and $\theta = q^{\frac{1}{1-n}}$ for $n > 1$, See [3, 4]. If $D_{n,q}\,f(t)$ exists, $t \in \mathbb{R}$, then f is called n,q-differentiable for all $t \in \mathbb{R}$. Here f is normally supposed to be defined on a set $A \subseteq \mathbb{R}$ for which $qt^n \in A$ whenever $t \in A$. This operator unifies and generalizes two difference operators. The first is the well-known and the most used Jackson q-difference operator defined by

$$D_q\,f(t) = \frac{f(qt) - f(t)}{t(q-1)}, \quad t \neq 0,$$

where $0 < q < 1$ is fixed. Here f is supposed to be defined on a q-geometric set A, i.e., A is a subset of $\mathbb{R}$ (or $\mathbb{C}$) for which $qt \in A$ whenever $t \in A$. The derivative at zero is normally defined to be $f'(0)$, provided that $f'(0)$ exists [6, 8, 10, 11]. The second operator is the n-power difference operator defined by

$$D_n\,f(t) = \begin{cases} \dfrac{f(t^n) - f(t)}{t^n - t} & t \in \mathbb{R} \setminus \{-1, 0, 1\}, \\ f^{'}(t) & t \in \{-1, 0, 1\}, \end{cases}$$

where $n > 1$ is a fixed odd positive integer [5].

In [4], Aldwoah et al. gave a rigorous analysis of the calculus associated with $D_{n,q}$. They proved some basic properties of such a calculus. For instance, they

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defined the inverse of $D_{n,q}$ which includes the right inverse of D_q and the right inverse of D_n. Then, they proved a fundamental lemma of the power quantum variational calculus.

In [2], M. H. Al-Ashwal et al. defined the n,q-exponential and trigonometric (hyperbolic) functions. They proved that these functions are solutions of power quantum difference equations of first and second order respectively.

In this paper we establish many Pachpatte's inequalities based on the power quantum difference operator $D_{n,q}$. For Pachpatte's inequalities in the time scales setting, see [1]. We organize this paper as follows. Section 2 gives an introduction to power quantum difference calculus. In Section 3, we introduce Gronwall's and Bernoulli's inequalities associated with the operator $D_{n,q}$. Then, we obtain the main results of the paper and prove many inequalities of Pachpatte's type associated with this operator.

2. Preliminaries

Throughout this paper, I is the interval $(-\theta,\theta)$, $\mathbb{R}_+ = [0,\infty[$ and $\mathbb{X}$ is a Banach space, endowed with a norm $\|\cdot\|$, where $\theta=\infty$ for $n=1$ and $\theta = q^{\frac{1}{1-n}}$ for $n>1$. An essential function which plays an important role in this calculus is $h(t) := qt^n$, $t\in I$. The set of fixed points of $h(t)$ is $\{0\}$ when $n=1$ and is $\{-\theta,0,\theta\}$ for $n>1$. One can see that the k-th order iteration of $h(t)$ is given by

$$h^k(t) = q^{[k]_n} t^{n^k}, \quad t\in I,$$

where, for $\alpha\in\mathbb{C}$, $[k]_\alpha$ is defined by

$$[k]_\alpha = \begin{cases} \sum_{i=0}^{k-1}\alpha^i & k\in\mathbb{N},\\ 0 & k=0.\end{cases}$$

The sequence $\{h^k(t)\}_{k=0}^{\infty}$ is uniformly convergent to 0 on I [2], see Figure 1.

Lemma 2.1 ([4])**.** *Let $f,g : I\to\mathbb{R}$ be n,q-differentiable at $t\in I$ and $c_1,c_2\in\mathbb{R}$. Then,*

(i) $D_{n,q}\big(c_1 f + c_2\, g\big)(t) = c_1 D_{n,q}\, f(t) + c_2 D_{n,q}\, g(t)$.

(ii) $D_{n,q}\big(fg\big)(t) = g(t) D_{n,q}\, f(t) + f(qt^n) D_{n,q}\, g(t)$.

(iii) $D_{n,q}\left(\dfrac{f}{g}\right)(t) = \dfrac{g(t)D_{n,q}\, f(t) - f(t) D_{n,q}\, g(t)}{g(t)\, g(qt^n)}$, *provided that* $g(t)\, g(qt^n)\neq 0$.

We notice that (ii) and (iii) are also true for $f : I\to\mathbb{X}$. Also, (i) is true if $f,g : I\to\mathbb{X}$.

Definition 2.2 ([4])**.** *Assume that $f : I\to\mathbb{X}$ and $a,b\in I$. The n,q-integral of f from a to b is defined by*

$$\int_a^b f(t)\, d_{n,q}t = \int_0^b f(t)\, d_{n,q}t - \int_0^a f(t)\, d_{n,q}t,$$

where

$$\int_0^x f(t)\, d_{n,q}t = -\sum_{k=0}^{\infty} q^{[k]_n} x^{n^k} (q^{n^k} x^{n^k(n-1)} - 1) f(q^{[k]_n} x^{n^k}), \quad x\in I, \tag{2.1}$$

provided that the series converges at $x=a$ and $x=b$. In this case, we say that f is n,q-integrable on $[a,b]$.

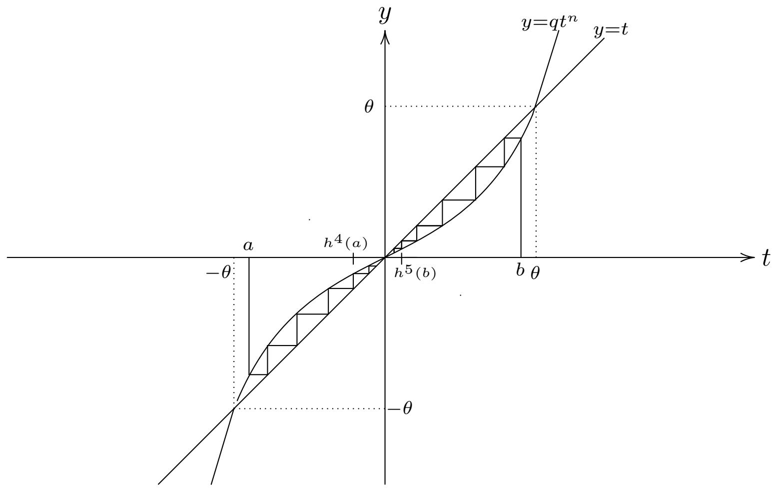

FIGURE 1. The iteration of $h(t)=qt^n,\ t\in I$ for fixed $n>1$ and $0<q\le 1$.

If f is continuous at 0, then the series (2.1) is convergent, see [2]. The following results were proved in [2, 4].

Lemma 2.3 ([4])**.** *Let $f,g:I\to\mathbb{R}$ be n,q-integrable, $k_1,k_2\in\mathbb{R}$, and $a,b,c\in I$. Then,*

(i) $\displaystyle\int_a^a f(t)\,d_{n,q}t=0.$

(ii) $\displaystyle\int_a^b f(t)\,d_{n,q}t=-\int_b^a f(t)\,d_{n,q}t.$

(iii) $\displaystyle\int_a^b \big(k_1f+k_2g\big)(t)\,d_{n,q}t=k_1\int_a^b f(t)\,d_{n,q}t+k_2\int_a^b g(t)\,d_{n,q}t.$

(iv) $\displaystyle\int_a^b f(t)\,d_{n,q}t=\int_a^c f(t)\,d_{n,q}t+\int_c^b f(t)\,d_{n,q}t \quad \text{for } a\le c\le b.$

Lemma 2.4 ([4])**.** *Let $s\in J\subseteq[0,\theta)$, $f,g:I\to\mathbb{R}$ be n,q-integrable on J and $|f(t)|\le g(t)$ for all $t\in\{q^{[k]_n}s^{n^k}:k\in\mathbb{N}_0\}$. Then, for $a,b\in\{q^{[k]_n}s^{n^k}:k\in\mathbb{N}_0\}$ with $a<b$, we have*

$$\left|\int_0^b f(t)\,d_{n,q}t\right|\le\int_0^b g(t)\,d_{n,q}t \quad \text{and} \quad \left|\int_a^b f(t)\,d_{n,q}t\right|\le\int_a^b g(t)\,d_{n,q}t.$$

Theorem 2.5 ([4])**.** *Assume that $f:I\to\mathbb{R}$ is continuous at 0. Define*

$$F(x):=\int_0^x f(t)\,d_{n,q}t,\quad x\in I.$$

Then, F is continuous at 0. Furthermore, $D_{n,q}F(x)$ exists for every $x\in I$ and

$$D_{n,q}F(x)=f(x).$$

Conversely,

$$\int_a^b D_{n,q}\, f(t)\, d_{n,q}t = f(b) - f(a) \quad \textit{for all } a, b \in I.$$

Lemma 2.4 and Theorem 2.5 are also true if f is a function with values in $\mathbb{X}$ with replacing the norm $\|\cdot\|$ instead of the modulus $|\cdot|$.

Definition 2.6 ([2])**.** *Let f be a complex valued function and continuous at 0. The n,q-exponential functions $e_{n,q}(f\,;t)$ and $E_{n,q}(f\,;t)$ are defined by*

$$e_{n,q}(f\,;t) = \frac{1}{\prod_{k=0}^{\infty}\Big(1 + f(q^{[k]_n}t^{n^k})\, q^{[k]_n}t^{n^k}(q^{n^k}t^{n^k(n-1)} - 1)\Big)}, \quad t \in I, \tag{2.2}$$

$$E_{n,q}(f\,;t) = \prod_{k=0}^{\infty}\Big(1 - f(q^{[k]_n}t^{n^k})\, q^{[k]_n}t^{n^k}(q^{n^k}t^{n^k(n-1)} - 1)\Big), \quad t \in I. \tag{2.3}$$

It is worthwhile that the infinite products in (2.2) and (2.3) are convergent since the series in definition 2.2 is convergent. One can see that $e_{n,q}(f\,;t)E_{n,q}(-f\,;t) = 1$.

Lemma 2.7 ([2])**.** *The n,q-exponential functions $e_{n,q}(f\,;t)$ and $E_{n,q}(f\,;t)$ are the unique solutions of the first order initial value problems*

$$D_{n,q}\, u(t) = f(t)\, u(t), \quad u(0) = 1, \tag{2.4}$$

$$D_{n,q}\, u(t) = f(t)\, u(qt^n), \quad u(0) = 1, \tag{2.5}$$

respectively.

3. n,q-Pachpatte's Inequalities

In this section we establish a variety of n,q-integral inequalities of Pachpatte type which will be equally important to achieve a diversity of desired goals in some applications. In what follows, $\sigma \in I$ is a given number.

Lemma 3.1. *Let a be a continuous function at 0 and b be a continuous function on I. Assume that u satisfies the following n,q-differential inequality*

$$D_{n,q}\, u(t) \le a(t) + b(t)\, u(t), \quad t \in I. \tag{3.1}$$

Then

$$u(t) \le e_{n,q}(b\,;t)\Big[u(0) + \int_0^t a(\tau)\, E_{n,q}(-b\,;q\tau^n)\, d_{n,q}\tau\Big], \quad t \in I. \tag{3.2}$$

Proof. Inequality (3.1) implies that

$$\begin{aligned} D_{n,q}[u(t)\, E_{n,q}(-b\,;t)] &= D_{n,q}\, u(t)\, E_{n,q}(-b\,;qt^n) - u(t)\, b(t)\, E_{n,q}(-b\,;qt^n) \\ &= \big[D_{n,q}\, u(t) - b(t)\, u(t)\big] E_{n,q}(-b\,;qt^n) \\ &\le a(t)\, E_{n,q}(-b\,;qt^n). \end{aligned}$$

Integrating both sides from 0 to t, we get

$$u(t)\, E_{n,q}(-b\,;t) - u(0)\, E_{n,q}(-b\,;0) \le \int_0^t a(\tau)\, E_{n,q}(-b\,;q\tau^n)\, d_{n,q}\tau.$$

In view of $E_{n,q}(-b\,;0) = 1$, we get

$$u(t)\, E_{n,q}(-b\,;t) - u(0) \le \int_0^t a(\tau)\, E_{n,q}(-b\,;q\tau^n)\, d_{n,q}\tau, \quad t \in I. \tag{3.3}$$

Multiplying both sides of (3.3) by $e_{n,q}(b\,;t)$, we conclude the desired inequality (3.2). ☐

Corollary 3.2. *Let b be a continuous function on I. Assume that u satisfies the following n,q-differential inequality*

$$D_{n,q}\,u(t) \le b(t)\,u(t), \quad t \in I. \tag{3.4}$$

Then

$$u(t) \le u(0)\,e_{n,q}(b\,;t), \quad t \in I. \tag{3.5}$$

Theorem 3.3. *Let a be a continuous function at 0, $f(t,s), h(t,s) : I \times I \to \mathbb{R}_+$ be non-decreasing functions in $t \in I$ such that f is continuous on $I \times I$ and $h(t,s)$ is continuous at $s = 0$. Assume that*

$$p(t) = \int_0^\sigma h(t,s)\,e_{n,q}(f(t,\cdot)\,;s)\,d_{n,q}s < 1, \quad t \in I.$$

If

$$u(t) \le a(t) + \int_0^t f(t,s)u(s)\,d_{n,q}s + \int_0^\sigma h(t,s)\,u(s)\,d_{n,q}s, \quad t \in I, \tag{3.6}$$

then

$$u(t) \le a(t) + e_{n,q}(f(t,\cdot)\,;t)\left[M(t) + \int_0^t a(\tau)\,f(t,\tau)E_{n,q}(-f(t,\cdot)\,;q\tau^n)\,d_{n,q}\tau\right], \quad t \in I. \tag{3.7}$$

where

$$M(t) = \frac{1}{1-p(t)}\int_0^\sigma h(t,s)\left[a(s) + e_{n,q}(f(t,\cdot)\,;s)\int_0^s a(\tau)\,f(t,\tau)E_{n,q}(-f(t,\cdot)\,;q\tau^n)\,d_{n,q}\tau\right]d_{n,q}s$$

Proof. Fix any $T \in I$. Then for $t \in [-T,T]$, from (3.6), we have

$$u(t) \le a(t) + \int_0^t f(T,s)\,u(s)\,d_{n,q}s + \int_0^\sigma h(T,s)\,u(s)\,d_{n,q}s. \tag{3.8}$$

Define a function $z(T,t), t \in [-T,T]$ by

$$z(T,t) = \int_0^t f(T,s)\,u(s)\,d_{n,q}s + \int_0^\sigma h(T,s)\,u(s)\,d_{n,q}s.$$

For $t \in [-T,T]$, we see that $u(t) \le a(t) + z(T,t)$,

$$z(T,0) = \int_0^\sigma h(T,s)\,u(s)\,d_{n,q}s, \tag{3.9}$$

and

$$D_{n,q}\,z(T,t) = f(T,t)\,u(t) \le f(T,t)(a(t) + z(T,t)).$$

which implies by Lemma 3.1,

$$z(T,t) \le e_{n,q}(f(T,\cdot)\,;t)\big[z(T,0) + \int_0^t a(\tau)f(T,\tau)\,E_{n,q}(-f(T,\cdot)\,;q\tau^n)\,d_{n,q}\tau\big].$$

where $t \in [-T,T]$. In view of $u(t) \le a(t) + z(T,t)$, we get

$$u(t) \le a(t) + e_{n,q}(f(T,\cdot)\,;t)\big[z(T,0) + \int_0^t a(\tau)f(T,\tau)\,E_{n,q}(-f(T,\cdot)\,;q\tau^n)\,d_{n,q}\tau\big]. \tag{3.10}$$

Using (3.9) and (3.10), we conclude that

$$z(T,0) \le M(T). \tag{3.11}$$

Inequalities (3.10) and (3.11) imply that

$$u(t) \leq a(t) + e_{n,q}(f(T,\cdot)\,;t)\left[M(T) + \int_0^t a(\tau)f(T,\tau)\,E_{n,q}(-f(T,\cdot)\,;q\tau^n)\,d_{n,q}\tau\right] \tag{3.12}$$

Since T is arbitrary, replacing T by t in (3.12), we get the desired inequality (3.7). □

Remark. When $a(t) = k$ (a real constant), inequality (3.7) yields

$$u(t) \leq \frac{k}{1-p(t)} e_{n,q}(f(t,\cdot)\,;t), \qquad t \in I. \tag{3.13}$$

If we take $h(t,s) = 0$, then $M(t) = 0$ and we deduce the following result

Corollary 3.4. *Let a be a continuous function at 0, $f(t,s) : I \times I \to \mathbb{R}_+$ be non-decreasing in $t \in I$ and continuous on $I \times I$. Assume that*

$$u(t) \leq a(t) + \int_0^t f(t,s)\,u(s)\,d_{n,q}s \quad t \in I, \tag{3.14}$$

then, the following statments are true

(i) $$u(t) \leq a(t) + e_{n,q}(f(t,\cdot)\,;t)\int_0^t a(\tau)f(t,\tau)E_{n,q}(-f(t,\cdot); q\tau^n)\,d_{n,q}\tau, \qquad t \in I. \tag{3.15}$$

(ii) *If a is non-decreasing on I, then*

$$u(t) \leq a(t)\,e_{n,q}(f(t,\cdot)\,;t), \quad t \in I. \tag{3.16}$$

The following corollary comes from Theorem 3.3 by taking $f(t,s) = b(s)$ and $h(t,s) = c(s)$, $t,s \in I$.

Corollary 3.5. *Let a, c be continuous at 0, b be continuous on I and $b(t), c(t) \geq 0$ for all $t \in I$. Assume that*

$$p = \int_0^\sigma c(s)\,e_{n,q}(b\,;s)\,d_{n,q}s < 1.$$

If

$$u(t) \leq a(t) + \int_0^t b(s)\,u(s)\,d_{n,q}s + \int_0^\sigma c(s)\,u(s)\,d_{n,q}s, \quad t \in I, \tag{3.17}$$

then

$$u(t) \leq a(t) + e_{n,q}(b\,;t)\left[M + \int_0^t a(\tau)b(\tau)E_{n,q}(-b; q\tau^n)\,d_{n,q}\tau\right], \qquad t \in I. \tag{3.18}$$

where

$$M = \frac{1}{1-p}\int_0^\sigma c(s)\left[a(s) + e_{n,q}(b; s)\int_0^s a(\tau)b(\tau)E_{n,q}(-b; q\tau^n)\,d_{n,q}\tau\right]d_{n,q}s$$

If $c(s) = 0$, then the inequality in Corollary 3.5 reduces to the following Gronwall's inequality of integral type which is a special version of the inequality given in Corollary 3.4

Corollary 3.6 (Gronwall's inequality of integral type)**.** *Let a be continuous at 0 and b be non-negative and continuous on I. Suppose that u satisfies the n,q-integral inequality*

$$u(t) \leq a(t) + \int_0^t b(s)\, u(s)\, d_{n,q}s, \quad t \in I. \tag{3.19}$$

Then

$$u(t) \leq a(t) + e_{n,q}(b\,;t) \int_0^t a(\tau) b(\tau) E_{n,q}(-b\,; q\tau^n)\, d_{n,q}\tau, \quad t \in I. \tag{3.20}$$

If in addition a is non-decreasing on I, then

$$u(t) \leq a(t)\, e_{n,q}(b\,;t), \quad t \in I. \tag{3.21}$$

Theorem 3.7 (Bernoulli's inequality)**.** *For $r \in \mathbb{R}$, the following inequality holds: $e_{n,q}(r\,;t) \geq 1 + rt, \quad t \in [0, \theta)$.*

Proof. Let $u(t) = rt$. Then, $ru(t) + r = r^2 t + r \geq r = D_{n,q}\, u(t)$. Since $u(0) = 0$, we have by Lemma 3.1

$$\begin{aligned} u(t) &\leq e_{n,q}(r\,;t) \left[u(0) + \int_0^t r\, E_{n,q}(-r\,; q\tau^n) d_{n,q}\tau \right] \\ &= e_{n,q}(r\,;t) \left[-\int_0^t D_{n,q} E_{n,q}(-r\,;\tau) d_{n,q}\tau \right] \\ &= e_{n,q}(r\,;t) \big[E_{n,q}(-r\,;0) - E_{n,q}(-r\,;t) \big] \\ &= e_{n,q}(r\,;t) - 1. \end{aligned}$$

Therefore, $e_{n,q}(r\,;t) \geq 1 + u(t) = 1 + rt$. □

Theorem 3.8. *Let a, c and g be continuous functions at 0, b and f be continuous functions on I, Moreover, $b(t), c(t), f(t) \geq 0$ for all $t \in I$. Suppose that*

$$p = \int_0^\sigma c(s)\, K_2(s)\, d_{n,q}s < 1.$$

If u satisfies the n,q-integral inequality

$$u(t) \leq a(t) + f(t) \int_0^t b(s)\, u(s)\, d_{n,q}s + g(t) \int_0^\sigma c(s)\, u(s)\, d_{n,q}s, \quad t \in I, \tag{3.22}$$

then

$$u(t) \leq K_1(t) + M K_2(t), \quad t \in I, \tag{3.23}$$

where

$$K_1(t) = a(t) + f(t)\, e_{n,q}(b\ f\,;t) \int_0^t a(\tau)\, b(\tau)\, E_{n,q}(-b\ f\,; q\tau^n)\, d_{n,q}\tau,$$

$$K_2(t) = g(t) + f(t)\, e_{n,q}(b\ f\,;t) \int_0^t g(\tau)\, b(\tau)\, E_{n,q}(-b\ f\,; q\tau^n)\, d_{n,q}\tau,$$

and

$$M = \frac{1}{1-p} \int_0^\sigma c(s)\, K_1(s)\, d_{n,q}s.$$

Proof. Set

$$z(t) = \int_0^t b(s)\, u(s)\, d_{n,q}s, \tag{3.24}$$

and

$$\lambda = \int_0^\sigma c(s)\, u(s)\, d_{n,q}s. \tag{3.25}$$

Then $z(0) = 0$, Inequality (3.22) can be written on the form

$$u(t) \le a(t) + f(t)\, z(t) + \lambda\, g(t). \tag{3.26}$$

We have

$$D_{n,q}\, z(t) = b(t)\, u(t). \tag{3.27}$$

In view of (3.26) and (3.27), we see that

$$D_{n,q}\, z(t) \le b(t)\big[a(t) + \lambda\, g(t)\big] + b(t)\, f(t)\, z(t).$$

This implies by Lemma 3.1 that

$$z(t) \le e_{n,q}(b\ f\,; t) \int_0^t b(\tau)\big[a(\tau) + \lambda\, g(\tau)\big]\, E_{n,q}(-b\ f\,; q\tau^n)\, d_{n,q}\tau. \tag{3.28}$$

Using (3.28) in (3.26) we get

$$\begin{aligned} u(t) &\le a(t) + \lambda\, g(t) \\ &\quad + f(t)\, e_{n,q}(b\ f\,; t) \int_0^t b(\tau)\big[a(\tau) + \lambda\, g(\tau)\big] E_{n,q}(-b\ f\,; q\tau^n)\, d_{n,q}\tau \\ &= K_1(t) + \lambda\, K_2(t). \end{aligned} \tag{3.29}$$

From (3.25) and (3.29), it is easy to observe that

$$\lambda \le M. \tag{3.30}$$

Using (3.30) in (3.29) we get the desired inequality (3.23). □

The following result shows that if, in Theorem 3.8, $g(t) = 0$, then $K_2(t) = 0$.

Corollary 3.9. *Let a be continuous at 0 and $b, f : I \to \mathbb{R}_+$ be continuous on I. Suppose that*

$$u(t) \le a(t) + f(t) \int_0^t b(s)\, u(s)\, d_{n,q}s, \quad t \in I, \tag{3.31}$$

then

$$u(t) \le a(t) + f(t)\, e_{n,q}(b\ f\,; t) \int_0^t a(\tau)\, b(\tau)\, E_{n,q}(-b\ f\,; q\tau^n)\, d_{n,q}\tau, \quad t \in I. \tag{3.32}$$

Theorem 3.10. *Let $u, a, b, c : I \to \mathbb{R}+$ be such that a, b are continuous on I and c is continuous at 0. Assume that*

$$p = \int_0^\sigma c(\tau)\, e_{n,q}(a + b\,; \tau)\, d_{n,q}\tau < 1.$$

If

$$u(t) \le k + \int_0^t a(s)\left[u(s) + \int_0^s b(\tau)\, u(\tau)\, d_{n,q}\tau + \int_0^\sigma c(\tau)\, u(\tau)\, d_{n,q}\tau\right] d_{n,q}s, \quad t \in I, \tag{3.33}$$

then

$$u(t) \le \frac{k}{1-p}\, e_{n,q}(a + b\,; t), \quad t \in I. \tag{3.34}$$

Proof. Define a function $z(t)$ by the right hand side of (3.33). Then we have $z(t) \geq 0$, $u(t) \leq z(t)$, $z(0) = k$ and

$$\begin{aligned} D_{n,q}\, z(t) &= a(t)\left[u(t) + \int_0^t b(\tau)\, u(\tau)\, d_{n,q}\tau + \int_0^\sigma c(\tau)\, u(\tau)\, d_{n,q}\tau\right] \\ &\leq a(t)\left[z(t) + \int_0^t b(\tau)\, z(\tau)\, d_{n,q}\tau + \int_0^\sigma c(\tau)\, z(\tau)\, d_{n,q}\tau\right], \quad t \in I. \end{aligned} \tag{3.35}$$

Define a function $v(t)$ by

$$v(t) = z(t) + \int_0^t b(\tau)\, z(\tau)\, d_{n,q}\tau + \int_0^\sigma c(\tau)\, z(\tau)\, d_{n,q}\tau. \tag{3.36}$$

Hence $z(t) \leq v(t)$, $D_{n,q}z(t) \leq a(t)\, v(t)$,

$$v(0) = z(0) + \int_0^\sigma c(\tau)\, z(\tau)\, d_{n,q}\tau = k + \int_0^\sigma c(\tau)\, z(\tau)\, d_{n,q}\tau, \tag{3.37}$$

and

$$D_{n,q}\, v(t) = D_{n,q}\, z(t) + b(t)\, z(t) \leq a(t)\, v(t) + b(t)\, z(t) = (a(t) + b(t))v(t),$$

which implies by Corollary 3.2

$$v(t) \leq v(0)\, e_{n,q}(a + b\,; t), \quad t \in I. \tag{3.38}$$

Using (3.38) and the inequality $z(t) \leq v(t)$ we obtain

$$z(t) \leq v(0)\, e_{n,q}(a + b\,; t), \quad t \in I. \tag{3.39}$$

Using (3.39) on the right hand side of (3.37), it is easy to observe that

$$v(0) \leq \frac{k}{1 - p}. \tag{3.40}$$

Using (3.39) in (3.40) and inequality $u(t) \leq z(t)$ we get the desired inequality in (3.34). □

Conclusion and future directions. This article is devoted to prove many Pachpatte's inequalities associated with power quantum difference operators. In the near future our aim will be investigating the existence and uniqueness of solutions of power quantum difference equations. Thereafter, we will study in more details the theory of power quantum difference equations associated with these operators.

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