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APPLICATIONS OF $\lambda-$STATISTICAL CONVERGENCE IN INTUITIONISTIC FUZZY n-NORMED SPACES

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Abstract. In the present paper we shall introduce and study the new notion namely, $\mathcal{I}-\lambda-$statistical convergence by using ideal with respect to the intuitionistic fuzzy norm $(\mu,\nu)_n$. We also study the relation between $\mathcal{I}-\lambda-$statistical convergence and $\mathcal{I}$-statistical convergence.

1. Introduction and Preliminaries

Fuzzy set theory as compared to other mathematical theories is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [24] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming, population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operator, etc. Recently, fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. In [12], Park introduced the concept of intuitionistic fuzzy metric space and later on Saadati and Park [17] introduced the concept of intuitionistic fuzzy normed space. Recently Mursaleen and Lohani [8] defined the concept of intuitionistic fuzzy 2-normed space which is generalization of the notion of intuitionistic fuzzy.
The notion of statistical convergence was introduced by Fast [1] and Schoenberg [23] independently. A lot of developments have been made in this areas after the works of Šalát [18], and Fridy [2]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Recently, Savas [19] introduced the concept of $\lambda-$statistical

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convergence with respect to the intuitionistic fuzzy 2-normed space. More investigations in this direction and more applications can be found in ([9], [10], [20], [21], [13], [14], [15], [16]) where many important references can be found.
In this paper, we shall study $\mathcal{I}-[V,\lambda]-$summable and $\mathcal{I}-\lambda-$ statistical convergence on the intuitionistic fuzzy $n-$normed space $(\mu,\nu)_n$. We mainly examine the relation between these two new methods in intuitionistic fuzzy normed space $(\mu,\nu)_n$

Definition 1.1. [22] *A binary operation* $*:[0,1]\times[0,1]\to[0,1]$ *is said to be a continuous* $t-$*norm if it satisfies the following conditions:*
(a) $*$ *is associative and commutative,*
(b) $*$ *is continuous,*
(c) $a*1=a$ *for all* $a\in[0,1]$
(d) $a*b\leq c*d$ *whenever* $a\leq c$ *and* $b\leq d$ *for each* $a,b,c,d\in[0,1]$.

Definition 1.2. [22] *A binary operation* $\Diamond:[0,1]\times[0,1]\to[0,1]$ *is said to be a continuous* $t-$*conorm if it satisfies the following conditions:*
(a) $\Diamond$ *is associative and commutative,*
(b) $\Diamond$ *is continuous,*
(c) $a\Diamond 0=a$ *for all* $a\in[0,1]$
(d) $a\Diamond b\leq c\Diamond d$ *whenever* $a\leq c$ *and* $b\leq d$ *for each* $a,b,c,d\in[0,1]$.

Using the notations of continuous $t-$norm and $t-$conorm, Saadati and Park [17] have recently introduced the concept of intuitionistic fuzzy norm space as follows:

Definition 1.3. *The five-tuple* $(X,\mu,\nu,*,\Diamond)$ *is said to be an intuitionistic fuzzy norm space (for short,* ***IFNS****) if* X *is a vector space,* $*$ *is continuous* $t-$*norm,* $\Diamond$ *is continuous* $t-$*conorm, and* μ,ν *are fuzzy sets on* $X\times(0,\infty)$ *satisfying the following conditions. For every* $x,y\in X$ *and* $s,t>0$.
(a) $\mu(x,t)+\nu(x,t)\leq 1$,
(b) $\mu(x,t)>0$,
(c) $\mu(x,t)=1$ *if and only if* $x=0$,
(d) $\mu(\alpha x,t)=\mu(x,\frac{t}{|\alpha|})$ *for each* $\alpha\neq 0$,
(e) $\mu(x,t)*\mu(y,s)\leq\mu(x+y,t+s)$,
(f) $\mu(x,.):(0,\infty)\to[0,1]$ *is continuous,*
(g) $\lim_{t\to\infty}\mu(x,t)=1$ *and* $\lim_{t\to 0}\mu(x,t)=0$,
(h) $\nu(x,t)<1$,
(i) $\nu(x,t)=1$ *if and only if* $x=0$,
(j) $\nu(\alpha x,t)=\nu(x,\frac{t}{|\alpha|})$ *for each* $\alpha\neq 0$,
(k) $\nu(x,t)\Diamond\nu(y,s)\geq\nu(x+y,t+s)$,
(l) $\nu(x,.):(0,\infty)\to[0,1]$ *is continuous,*
(m) $\lim_{t\to\infty}\nu(x,t)=0$ *and* $\lim_{t\to 0}\nu(x,t)=1$.
In this case (μ,ν) *is called an intuitionistic fuzzy norm.*

The concept of 2-normed spaces was initially developed by Gähler [3] in the mid of 1960's, while that for n-normed spaces one can see in Misiak [11]. Since then, many others have studied this concept and obtained various results, see Gunawan ([4], [5]) and Gunawan and Mashadi [6].

Definition 1.4. *Let* $n\in\mathbb{N}$ *and* X *be a linear space over the field* $\mathbb{R}$ *of reals of dimension* d*, where* $d\geq n\geq 2$. *A real valued function* $||\cdot,\cdots,\cdot||$ *on* X^n *satisfying the following four conditions:*

(a) $||x_1, x_2, \cdots, x_n|| = 0$ *if and only if* $x_1, x_2, \cdots, x_n$ *are linearly dependent in* X,
(b) $||x_1, x_2, \cdots, x_n||$ *is invariant under permutation,*
(c) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ *for any* $\alpha \in \mathbb{R}$, *and*
(d) $||x_1, x_2, \cdots, x_{n-1}, y + z|| \leq ||x_1, x_2, \cdots, x_{n-1}, y|| + ||x_1, x_2, \cdots, x_{n-1}, z||$
is called an n*-norm on* X, *and the pair* $(X, ||\cdot, \cdots, \cdot||)$ *is called a* n*-normed space over the field* $\mathbb{R}$.
For example, we may take $X = \mathbb{R}^n$ *being equipped with the* n*-norm* $||x_1, x_2, \cdots, x_n||_E$ = *the volume of the* n*-dimensional parallelopiped spanned by the vectors* $x_1, x_2, \cdots, x_n$ *which may be given explicitly by the formula*

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ *for each* $i = 1, 2, \cdots, n$.

Definition 1.5. *The five-tuple* $(X, \mu, \nu, *, \lozenge)$ *is said to be an intuitionistic fuzzy* n*-norm space (for short,* ***IFnNLS****) if* X *is a linear space over field* F, $*$ *is continuous* $t-$*norm,* $\lozenge$ *is continuous* $t-$*conorm, and* μ, ν *are fuzzy sets on* $X^n \times (0, \infty)$, μ *denotes the degree of membership and* ν *denotes the degree of non-membership of* $(x_1, x_2, \cdots, x_n, t) \in X^n \times (0, \infty)$ *satisfying the following conditions for every* $(x_1, x_2, \cdots, x_n) \in X^n$ *and* $s, t > 0$*:*
(a) $\mu(x_1, x_2, \cdots, x_n, t) + \nu(x_1, x_2, \cdots, x_n, t) \leq 1$,
(b) $\mu(x_1, x_2, \cdots, x_n, t) > 0$,
(c) $\mu(x_1, x_2, \cdots, x_n, t) = 1$ *if and only if* $x_1, x_2, \cdots, x_n$ *are linearly dependent,*
(d) $\mu(x_1, x_2, \cdots, x_n, t)$ *is invariant under any permutation of* $x_1, x_2, \cdots, x_n$,
(e) $\mu(x_1, x_2, \cdots, \alpha x_n, t) = \mu(x_1, x_2, \cdots, x_n, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,
(f) $\mu(x_1, x_2, \cdots, x_n, s) * \mu(x_1, x_2, \cdots, x_n^{'}, t) \leq \mu(x_1, x_2, \cdots, x_n + x_n^{'}, s + t)$,
(g) $\mu(x_1, x_2, \cdots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ *is continuous in* t,
(h) $\lim_{t\to\infty} \mu(x_1, x_2, \cdots, x_n, t) = 1$ *and* $\lim_{t\to 0} \mu(x_1, x_2, \cdots, x_n, t) = 0$,
(i) $\nu(x_1, x_2, \cdots, x_n, t) < 1$,
(j) $\nu(x_1, x_2, \cdots, x_n, t) = 0$ *if and only if* $x_1, x_2, \cdots, x_n$ *are linearly dependent,*
(k) $\nu(x_1, x_2, \cdots, x_n, t)$ *is invariant under any permutation of* $x_1, x_2, \cdots, x_n$,
(l) $\nu(x_1, x_2, \cdots, \alpha x_n, t) = \nu(x_1, x_2, \cdots, x_n, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,
(m) $\nu(x_1, x_2, \cdots, x_n, s) \lozenge \nu(x_1, x_2, \cdots, x_n^{'}, t) \geq \nu(x_1, x_2, \cdots, x_n + x_n^{'}, s + t)$,
(m) $\nu(x_1, x_2, \cdots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ *is continuous in* t,
(o) $\lim_{t\to\infty} \nu(x_1, x_2, \cdots, x_n, t) = 0$ *and* $\lim_{t\to 0} \nu(x_1, x_2, \cdots, x_n, t) = 1$,
In this case (μ, ν) *is called an intuitionistic fuzzy* n*-norm on* X, *and we denote it by* $(\mu, \nu)_n$.

Example 1.6. *Let* $(X, \|.,\cdots,.\|)$ *be an* $n-$*normed space. Also let* $a * b = ab$ *and* $a \circ b = \min\{a + b, 1\}$ *for all* $a, b \in [0, 1]$,

$$\mu(x_1, x_2, \cdots, x_n, t) = \frac{t}{t + \|x_1, x_2, \cdots, x_n\|} \quad \text{and} \quad \nu(x_1, x_2, \cdots, x_n, t) = \frac{\|x_1, x_2, \cdots, x_n\|}{t + \|x_1, x_2, \cdots, x_n\|}.$$

Then $(X, \mu, \nu, *, \lozenge)$ *is an IFnNLS.*

Definition 1.7. *Let* $(X, \mu, \nu, *, \lozenge)$ *be an intuitionistic fuzzy* n*-normed space. A sequence* $x = (x_k)$ *is said to be convergent to* $L \in X$ *with respect to* $(\mu, \nu)_n$ *if, for every* $\varepsilon > 0$ *and* $t > 0$, *there exists* $k_0 \in \mathbb{N}$ *such that* $\mu(x_k - L, z_1, \cdots, z_{n-1}; t) > 1 - \varepsilon$ *and* $\nu(x_k - L, z_1, \cdots, z_{n-1}; t) < \varepsilon$ *for all* $k \geq k_0$ *and for all* $z_1, \cdots, z_{n-1} \in X$. *In this case we write* $(\mu, \nu)_n - \lim x = L$ *or* $x_k \xrightarrow{(\mu,\nu)_n} L$ *as* $k \to \infty$.

Ideal convergence is a generalization of statistical convergence and any concept involving ideal convergence plays a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially information theory, computer science, biological science, dynamical systems, geographic information systems, and motion planning in robotics. Kostyrko et al. [7] was initially introduced the notion of $\mathcal{I}-$convergence based on the structure of admissible ideal $\mathcal{I}$ of subset of natural number $\mathbb{N}$.
Let Y be a non empty set. Then a family of sets $\mathcal{I} \subset 2^Y$ (Power set of Y) is said to be an *ideal* in Y if $\emptyset \notin \mathcal{I}$, $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$ and $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$, while an *admissible ideal* $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is an ideal in Y then the collection $F(\mathcal{I}) = \{M \subset Y : M^c \in \mathcal{I}\}$ forms a filter in Y which is called the *filter* associated with I.

Definition 1.8. [7] *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}-$convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.*

Definition 1.9. *Let $(X, \mu, \nu, *, \Diamond)$ be an intuitionistic fuzzy n-normed space. Then a sequence $x = (x_k)$ is said to be $\mathcal{I}-$statistically convergent to $L \in X$ with respect to $(\mu, \nu)_n$, if, for every $\varepsilon > 0$ and $t > 0$, and for nonzero $z_1, \cdots, z_{n-1} \in X$ such that*

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \mu(x_k - L, z_1, \cdots, z_{n-1}; t) \leq 1-\varepsilon \text{ or } \nu(x_k - L, z_1, \cdots, z_{n-1}; t) \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}.$$

In this case we write $x_k \xrightarrow{(\mu,\nu)} L\big(S^{(\mu,\nu)_n}(\mathcal{I})\big)$.

2. $\lambda-$Statistical convergence in IFnNLS

In this section, we study the concept of $\mathcal{I} - \lambda-$statistically convergence in the intuitionistic fuzzy n-normed space $(\mu, \nu)_n$.

Definition 2.1. *Let K be subset of $\mathbb{N}$, the set of natural numbers. Then the asymptotic density of K denoted by $\delta(K)$, is defined as*

$$\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set

Definition 2.2. *A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - L| > \varepsilon\}$ has asymptotic density zero, i.e.*

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim x = L$ (see [1], [2]*)*

Note that every convergent sequence is statistically convergent to the same limit, but converse need not be true.

Definition 2.3. *Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The collection of such sequences λ will be denoted by Δ.*
The generalized de la Vallée Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

Definition 2.4. *Let* $(X, \mu, \nu, *, \Diamond)$ *be an intuitionistic fuzzy n-normed space. A sequence* $x = (x_k)$ *is said to be* $\mathcal{I} - [V, \lambda]$*-summable to* $L \in X$ *with respect to* $(\mu, \nu)_n$*, if, for every* $\delta > 0$, $t > 0$, *and for nonzero* $z_1, \cdots, z_{n-1} \in X$ *such that*

$$\{n \in \mathbb{N} : \mu(t_n(x) - L, z_1, \cdots, z_{n-1}; t) \leq 1 - \delta \text{ or } \nu(t_n(x) - L, z_1, \cdots, z_{n-1}; t) \geq \delta\} \in \mathcal{I}.$$

Now we define $\mathcal{I} - \lambda-$statistical convergence with respect to intuitionistic fuzzy n-normed space.

Definition 2.5. *Let* $(X, \mu, \nu, *, \Diamond)$ *be an intuitionistic fuzzy n-normed space. A sequence* $x = (x_k)$ *is said to be* $\mathcal{I} - \lambda-$*statistically convergent or* $\mathcal{I} - S_\lambda$ *convergent to L with respect to* $(\mu, \nu)_n$*, if, for every* $\varepsilon > 0$ *and* $\delta > 0$, *and* $t > 0$, *for nonzero* $z_1, \cdots, z_{n-1} \in X$ *such that*

$$\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : \mu(x_k - L, z_1, \cdots, z_{n-1}; t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, z_1, \cdots, z_{n-1}; t) \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}.$$

In this case we write $\mathcal{I} - S_\lambda^{(\mu,\nu)_n} - \lim x = L$ *or* $x_k \to L\big(\mathcal{I} - S_\lambda^{(\mu,\nu)_n}\big)$.

We shall denote by $S^{(\mu,\nu)_n}(\mathcal{I}), S_\lambda^{(\mu,\nu)_n}(\mathcal{I})$ and $[V, \lambda]^{(\mu,\nu)_n}(\mathcal{I})$ the collections of all $\mathcal{I}-$statistically convergent, $\mathcal{I} - S_\lambda^{(\mu,\nu)_n}$ convergent and $\mathcal{I} - [V, \lambda]^{(\mu,\nu)_n}-$convergent sequences respectively.

Theorem 2.6. *Let* $(X, \mu, \nu, *, \Diamond)$ *be an intuitionistic fuzzy n-normed space. If a sequence* $x = (x_k)$ *in X is* $\mathcal{I} - \lambda$ *statistically convergent sequence with respect to* $(\mu, \nu)_n$*, then limit is unique.*

Proof. Suppose that $S_\lambda^{(\mu,\nu)_n}(\mathcal{I}) - \lim x = L_1$, $S_\lambda^{(\mu,\nu)_n}(\mathcal{I}) - \lim x = L_2$, ..., $S_\lambda^{(\mu,\nu)_n}(\mathcal{I}) - \lim x = L_n$.
For a given $\varepsilon > 0$, choose $r > 0$ such that $(1 - r) * (1 - r) * \cdots * (1 - r) > 1 - \varepsilon$ and $r \Diamond r \Diamond \cdots \Diamond r < \varepsilon$. Then for any $t > 0$, define the following sets as:
$K_{\mu,1}(r, t) = \{k \in I_n : \mu(x_k - L_1, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2}) \leq 1 - r)\}$,
$K_{\mu,2}(r, t) = \{k \in I_n : \mu(x_k - L_2, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2}) \leq 1 - r)\}$ and
$K_{\mu,n}(r, t) = \{k \in I_n : \mu(x_k - L_n, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2}) \leq 1 - r)\}$
Also,
$K_{\nu,1}(r, t) = \{k \in I_n : \mu(x_k - L_1, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2}) \geq r)\}$,
$K_{\nu,2}(r, t) = \{k \in I_n : \mu(x_k - L_2, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2}) \geq r)\}$and
$K_{\nu,n}(r, t) = \{k \in I_n : \mu(x_k - L_n, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2}) \geq r)\}$
Since $S_\lambda^{(\mu,\nu)_n}(\mathcal{I}) - \lim x = L_1$, we have
$\{\frac{1}{\lambda_n}\Big(K_{\mu,1}(r, t)\Big)$ and $\frac{1}{\lambda_n}\Big(K_{\nu,1}(r, t)\Big) \geq \delta\} \in \mathcal{I}$ for all $t > 0$,
Furthermore, using $S_\lambda^{(\mu,\nu)_n}(\mathcal{I}) - \lim x = L_2$, we get
$\{\frac{1}{\lambda_n}\Big(K_{\mu,2}(r, t)\Big)$ and $\frac{1}{\lambda_n}\Big(K_{\nu,2}(r, t)\Big) \geq \delta\} \in \mathcal{I}$ for all $t > 0$,
Also, $S_\lambda^{(\mu,\nu)_n}(\mathcal{I}) - \lim x = L_n$, we have
$\{\frac{1}{\lambda_n}\Big(K_{\mu,n}(r, t)\Big)$ and $\frac{1}{\lambda_n}\Big(K_{\nu,n}(r, t)\Big) \geq \delta\} \in \mathcal{I}$ for all $t > 0$.
Now let $K_{\mu,\nu}(r, t) = \Big(K_{\mu,1}(r, t) \cup K_{\mu,2}(r, t) \cdots \cup K_{\mu,n}(r, t)\Big) \cap \Big(K_{\nu,1}(r, t) \cup K_{\nu,2}(r, t) \cdots \cup$

$K_{\nu,n}(r,t)\Big) \in \mathcal{I}$. Then we see that $\{\frac{1}{\lambda_n}\Big(K_{\mu,\nu}(r,t)\Big)\} \in \mathcal{I}$. This implies that its compliment $\frac{1}{\lambda_n}\Big(K^c_{\mu,\nu}(r,t)\Big)$ is a non empty set in $F(\mathcal{I})$. If $k \in \frac{1}{\lambda_n}\Big(K^c_{\mu,\nu}(r,t)\Big)$, then we have two possible cases. That is,
$k \in \frac{1}{\lambda_n}\Big(K^c_{\mu,1}(r,t) \cap K^c_{\mu,2}(r,t) \cap \cdots \cap K^c_{\mu,n}(r,t)\Big)$ or $k \in \frac{1}{\lambda_n}\Big(K^c_{\nu,1}(r,t) \cap K^c_{\nu,2}(r,t) \cap \cdots \cap K^c_{\nu,n}(r,t)\Big)$.
We first consider that $k \in \frac{1}{\lambda_n}\Big(K^c_{\mu,1}(r,t) \cap K^c_{\mu,2}(r,t) \cap \cdots \cap K^c_{\mu,n}(r,t)\Big)$. Then we have

$$\begin{aligned}
&\mu(L_1 - L_2 - \cdots - L_n, z_1, z_2, \cdots, z_{n-1}; t)\\
&\geq \frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k - L_1, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2}) * \frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k - L_2, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2})\\
&\qquad * \cdots * \frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k - L_n, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2})\\
&\qquad > (1-r) * (1-r) * \cdots * (1-r) > 1 - \varepsilon.
\end{aligned} \tag{2.1}$$

Using (2.1) and Since $\varepsilon > 0$ was arbitrary, we get
$\mu(L_1 - L_2 - \cdots - L_n, z_1, z_2, \cdots, z_{n-1}; t) = 1$ for all $t > 0$ which yields $L_1 = L_2 = \cdots = L_n$. On the other hand, if $k \in \frac{1}{\lambda_n}\Big(K^c_{\nu,1}(r,t) \cap K^c_{\nu,2}(r,t) \cap \cdots \cap K^c_{\nu,n}(r,t)\Big)$, then we may write

$$\begin{aligned}
&\nu(L_1 - L_2 - \cdots - L_n, z_1, z_2, \cdots, z_{n-1}; t)\\
\leq\ & \frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k - L_1, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2}) \Diamond \frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k - L_2, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2})\\
& \Diamond \cdots \Diamond \frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k - L_n, z_1, z_2, \cdots, z_{n-1}; \frac{t}{2})\\
<\ & r \Diamond r \Diamond \cdots \Diamond r < \varepsilon.
\end{aligned}$$

Therefore, we have $\nu(L_1 - L_2 - \cdots - L_n, z_1, z_2, \cdots, z_{n-1}; t) = 0$ for all $t > 0$, which implies that $L_1 = L_2 = \cdots = L_n$. Therefore, in all cases we conclude that $S_\lambda^{(\mu,\nu)_n}(\mathcal{I})-$ limit is unique.
This completes the proof of the theorem. □

Theorem 2.7. *Let $(X, \mu, \nu, *, \Diamond)$ be an intuitionistic fuzzy n-normed space. Let $\lambda = (\lambda_n) \in \Delta$. Then*
(i) $x_k \to L[V,\lambda]^{(\mu,\nu)_n}(\mathcal{I}) \Rightarrow x_k \to L\big(S_\lambda^{(\mu,\nu)_n}(\mathcal{I})\big)$ and the inclusion $[V,\lambda]^{(\mu,\nu)_n}(\mathcal{I}) \subset S_\lambda^{(\mu,\nu)_n}(\mathcal{I})$ is proper for every ideal $\mathcal{I}$.
(ii) If $x \in m(X)$, the space of all bounded sequences of X and $x_k \to L\big(S_\lambda^{(\mu,\nu)_n}(\mathcal{I})\big)$, then $x_k \to L[V,\lambda]^{(\mu,\nu)_n}(\mathcal{I})$.
(iii) $S_\lambda^{(\mu,\nu)_n}(\mathcal{I}) \cap m(X) = [V,\lambda]^{(\mu,\nu)_n}(\mathcal{I}) \cap m(X)$.

Proof. Let $\varepsilon > 0$ and $x_k \to L[V,\lambda]^{(\mu,\nu)_n}(\mathcal{I})$. Then we have

$$\sum_{k\in I_n}\big(\mu(x_k-L,z_1,\cdots,z_{n-1};t)\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\big)$$
$$\geq \sum_{\substack{k\in I_n\&\mu(x_k-L,z_1,\cdots,z_{n-1};t<1-\varepsilon\\ \nu(x_k-L,z_1,\cdots,z_{n-1};t)>\varepsilon}}\big(\mu(x_k-L,z_1,\cdots,z_{n-1};t)\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\big)$$
$$\geq \varepsilon|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|$$

So for a given $\delta>0$,

$$\tfrac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta$$
$$\Rightarrow\frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq(1-\varepsilon)\delta\text{ or }\frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\delta.$$

Thus, we have

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta\Big\}$$
$$\subset\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\Big\{\sum_{k\in I_n}\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\sum_{k\in I_n}\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\Big\}\varepsilon\delta\Big\}.$$

Since $x_k\to L[V,\lambda]^{(\mu,\nu)_n}(\mathcal{I})$, so the set on the right hand side belongs to $\mathcal{I}$ and so it follows that $x_k\to L\big(S_\lambda^{(\mu,\nu)_n}\big)(\mathcal{I})$.
To show that $S_\lambda^{(\mu,\nu)_n}(\mathcal{I})\subsetneq[V,\lambda]^{(\mu,\nu)_n}(\mathcal{I})$, take a fixed $A\in\mathcal{I}$.

Consider $X=\mathbb{R}^n$ with

$$\|x_1,x_2\cdots,x_n\|=\ abs\left(\begin{vmatrix}x_{11}&\ldots&x_{1n}\\ \vdots&\ddots&\vdots\\ x_{n1}&\ldots&x_{nn}\end{vmatrix}\right).$$

where $x_i=(x_{i1},x_{i2}\cdots,x_{in})\in\mathbb{R}^n$ for each $i=1,2,...,n$ and let $a*b=ab$, $a\circ b=\min\{a+b,1\}$ for all $a,b\in[0,1]$.
Now for all $(x,z_1,z_2\cdots,z_{n-1})\in\mathbb{R}^n$ and $t>0$, $\mu(x,z_1,\cdots,z_{n-1};t)=\frac{t}{t+\|x,z_1,\cdots,z_{n-1}\|}$ and $\nu(x,z_1,\cdots,z_{n-1};t)=\frac{\|x,z_1,\cdots,z_{n-1}\|}{t+\|x,z_1,\cdots,z_{n-1}\|}$.
Then $(\mathbb{R}^n,\mu,\nu,*,\Diamond)$ be an intuitionistic fuzzy n-normed space. Now we define a sequence $x=(x_k)$ by

$$x_k=\begin{cases}(k,0,...,0), & \text{for } n-[\sqrt{\lambda_n}]+1\leq k\leq n, n\notin A;\\ (k,0,...,0), & n-\lambda_n+1\leq k\leq n, n\in A;\\ \theta & \text{otherwise.}\end{cases}$$

Then $x\notin m(X)$ and for every $\varepsilon>0$ $(0<\varepsilon<1)$ since

$$\frac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-0,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-0,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|=\frac{[\sqrt{\lambda_n}]}{\lambda_n}\to 0$$

as $n\to\infty$ and $n\notin A$, so for every $\delta>0$,

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-0,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-0,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta\Big\}$$

$\subset A\cup\{1,2,...,m\}$ for some $m\in\mathbb{N}$.
Since $\mathcal{I}$ is admissible, it follows that $x_k\to\theta\big(S_\lambda^{(\mu,\nu)_n}(\mathcal{I})\big)$. Obviously

$\frac{1}{\lambda_n}\sum_{k\in I_n}\Big(\mu(x_k-\theta,z_1,\cdots,z_{n-1};t)$ or $\sum_{k\in I_n}\nu(x_k-\theta,z_1,\cdots,z_{n-1};t)\Big)\to\infty$

i.e. $x_k\nrightarrow\theta[V,\lambda]^{(\mu,\nu)_n}(\mathcal{I})$. Note that if $A\in\mathcal{I}$ is finite then $x_k\nrightarrow\theta\big(S_\lambda^{(\mu,\nu)_n}\big)$. This example also shows that $\mathcal{I}-S_\lambda^{(\mu,\nu)_n}-$convergence is more general than $S_\lambda^{(\mu,\nu)_n}-$convergence.

(ii) Suppose that $x_k\to L\big(S_\lambda^{(\mu,\nu)_n}(\mathcal{I})\big)$ and $x\in m(X)$. Let $\mu(x_k-L,z_1,\cdots,z_{n-1};t)\geq 1-M$ or $\nu(x_k-L,z_1,\cdots,z_{n-1};t)\leq M\ \ \forall\ k$. Let ε be given. Now

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\big(\mu(x_k-L,z_1,\cdots,z_{n-1};t)\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\big)$$

$$\begin{aligned}
&= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\&\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\\ \nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon}}\big(\mu(x_k-L,z_1,\cdots,z_{n-1};t)\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\big)\\
&+ \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\&\mu(x_k-L,z_1,\cdots,z_{n-1};t>1-\varepsilon\\ \nu(x_k-L,z_1,\cdots,z_{n-1};t)<\varepsilon}}\big(\mu(x_k-L,z_1,\cdots,z_{n-1};t)\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\big)\\
&\leq \frac{M}{\lambda_n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|+\varepsilon.
\end{aligned}$$

Note that

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\frac{\varepsilon}{M}\Big\}$$

$=A(\varepsilon)\in\mathcal{I}$. If $n\in(A(\varepsilon))^c$ then
$\frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k-L,z_1,\cdots,z_{n-1};t)>1-2\varepsilon$ or $\frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k-L,z_1,\cdots,z_{n-1};t)<2\varepsilon.$

Hence

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-2\varepsilon\text{ or }\frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq 2\varepsilon\Big\}\subset A(\varepsilon)$$

and so belongs to $\mathcal{I}$. This shows that $x_k\to L[V,\lambda]^{(\mu,\nu)_n}(\mathcal{I})$.

(iii) This readily follows from (i) and (ii). □

Theorem 2.8. *Let $(X,\mu,\nu,*,\lozenge)$ be an intuitionistic fuzzy n-normed space. If $\inf_{n\to\infty}\frac{\lambda_n}{n}>0$, then $S^{(\mu,\nu)}(\mathcal{I})\subset S_\lambda^{(\mu,\nu)}(\mathcal{I})$.*

Proof. For given $\varepsilon>0$, we have

$$\frac{1}{n}|\{k\leq n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|$$

$$\begin{aligned}
&\geq \frac{1}{n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\\
&\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\text{ or }\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|.
\end{aligned}$$

If $\lim\inf_{n\to\infty}\dfrac{\lambda_n}{n}=\alpha$, then from definition $\{n\in\mathbb{N}:\frac{\lambda_n}{n}<\frac{\alpha}{2}\}$ is finite. For $\delta>0$, the set

$$\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon \text{ or } \nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta\}$$

$$\subset \quad \{n\in\mathbb{N}:\frac{1}{n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon \text{ or } \nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\frac{\alpha}{2}\delta\}$$
$$\cup \quad \{n\in\mathbb{N}:\frac{\lambda_n}{n}<\frac{\alpha}{2}\}.$$

Since $\mathcal{I}$ is admissible, the set on the right-hand side belongs to $\mathcal{I}$ and this completes the proof. □

Theorem 2.9. *Let $(X,\mu,\nu,*,\diamondsuit)$ be an intuitionistic fuzzy n-normed space and let $x=(x_k)$ in X. If $S_\lambda^{(\mu,\nu)_n}(\mathcal{I})-\lim x=L$, then there exists a subsequence (x_{mk}) of $x=(x_k)$ such that $S^{(\mu,\nu)_n}(\mathcal{I})-\lim x_{mk}=L$.*

Proof. Let $S_\lambda^{(\mu,\nu)_n}(\mathcal{I})-\lim x=L$. Then, for every $t>0$ and given $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon \text{ or } \nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta\}\in\mathcal{I}$$

for all $n\geq n_0$. Clearly, for each $n\geq n_0$, we can select an $m_k\in I_n$ such that

$$\mu(x_{mk}-L,z_1,z_2,\cdots,z_{n-1};t)\leq\frac{1}{\lambda_n}|\{\sum_{k\in I_n}\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\}|\geq\delta\}\in\mathcal{I}$$

and

$$\nu(x_{mk}-L,z_1,z_2,\cdots,z_{n-1};t)\geq\tfrac{1}{\lambda_n}|\{\sum_{k\in I_n}\nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta\}\in\mathcal{I}.$$

It follows that $S^{(\mu,\nu)_n}(\mathcal{I})-\lim x_{mk}=L$. □

Theorem 2.10. *Let $(X,\mu,\nu,*,\diamondsuit)$ be an intuitionistic fuzzy n-normed space and $x=(x_k)$ be a sequence in X. Let $\mathcal{I}$ be a non-trivial ideal in $\mathbb{N}$. If there is a $S_\lambda^{(\mu,\nu)_n}(\mathcal{I})-$convergent sequence $y=(y_k)$ in X such that $\{k\in\mathbb{N}:y_k\neq x_k\}\in\mathcal{I}$, then x is also $S_\lambda^{(\mu,\nu)_n}(\mathcal{I})-$convergent.*

Proof. Suppose that $\{k\in\mathbb{N}:y_k\neq x_k\}\in\mathcal{I}$ and $S_\lambda^{(\mu,\nu)_n}(\mathcal{I})-y=L$. Then for every $\varepsilon>0$, $t>0$, the set

$$\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:\mu(y_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon \text{ or } \nu(y_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta\}\in\mathcal{I}.$$

For every $0<\varepsilon<1$ and $t>0$, we have

$$\begin{aligned}&\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:\mu(x_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon \text{ or } \nu(x_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta\}\\ &\subseteq\{k\in\mathbb{N}:y_k\neq x_k\}\cup\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:\mu(y_k-L,z_1,\cdots,z_{n-1};t)\leq 1-\varepsilon\\ &\qquad \text{or } \nu(y_k-L,z_1,\cdots,z_{n-1};t)\geq\varepsilon\}|\geq\delta\}.\end{aligned}\tag{2.2}$$

As both the sets of right-hand side of (2.2) are in $\mathcal{I}$, we have

$$\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \mu(x_k - L, z_1, \cdots, z_{n-1}; t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, z_1, \cdots, z_{n-1}; t) \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}.$$

This completes the proof of the theorem. □

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