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PRESENTATION OF YOUNG'S INEQUALITY

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Abstract. The paper presents different forms of Young's inequality. Main results include generalizations of the discrete and integral form. Issues on inequalities are studied using the geometric-arithmetic mean inequality, integral method and Jensen's inequality. A functional approach to Young's inequality is also considered.

1. Introduction

Dealing with inequalities we use means. In this paper, we mainly apply the two basic means, arithmetic and geometric. Let $a_1, \dots, a_n \in \mathbb{R}$ be points on the real axis, and let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be non-negative coefficients satisfying $\sum_{i=1}^n \alpha_i = 1$. The sum

$$c_{AM} = \sum_{i=1}^{n} \alpha_i a_i \tag{1.1}$$

is called convex combination (generalized or weighted arithmetic mean), and its center c_{AM} belongs to the closed interval $[a_{\min}, a_{\max}]$. Using the notion of convex hull, we usually write $c_{AM} \in \operatorname{conv}\{a_1, \dots, a_n\}$. Assuming that all $a_i \geq 0$ (respectively $a_i > 0$), and that all $\alpha_i > 0$ (respectively $\alpha_i \geq 0$), the product

$$c_{GM} = \prod_{i=1}^{n} a_i^{\alpha_i} \tag{1.2}$$

is called geometric combination (generalized or weighted geometric mean). By applying the logarithmic and exponential function to the equation in (1.2), it follows that $c_{GM} \in \operatorname{conv}\{a_1, \dots, a_n\}$. According to the theory of quasi-arithmetic means, the geometric center is less than or equal to the arithmetic center, $c_{GM} \leq c_{AM}$.

2. Young's inequality

The section presents an overview of the most important results on Young's inequality. Moreover, there is the new result, Theorem 2.1.

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The discrete form of Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \tag{2.1}$$

where points $a, b \geq 0$ and coefficients $p, q > 1$ satisfy $1/p + 1/q = 1$, is a special case of the generalized geometric-arithmetic mean inequality $c_{GM} \leq c_{AM}$.

More important is the integral form of Young's inequality. Let $g : [0, \infty) \to [0, \infty)$ be a bijective continuous function (such a function is strictly increasing satisfying $g(0) = 0$ and $\lim_{x\to\infty} g(x) = \infty$), and let g^{-1} be its inverse function.

Let a be a non-negative number. Given the integer n, we take the points $x_{ni} = (a/n)i$ assuming that $x_{n0} = 0$, and so we have $x_{ni} - x_{n\,i-1} = a/n$. As the following calculation demonstrates,

$$\begin{aligned} ag(a) &= \sum_{i=1}^{n} \frac{a}{n}\left[ig(x_{ni}) - (i-1)g(x_{n\,i-1})\right] \\ &= \sum_{i=1}^{n} \frac{a}{n} g(x_{n\,i-1}) + \sum_{i=1}^{n}\left[g(x_{ni}) - g(x_{n\,i-1})\right]\frac{a}{n}i \\ &= \sum_{i=1}^{n}\left[x_{ni} - x_{n\,i-1}\right] g(x_{n\,i-1}) + \sum_{i=1}^{n}\left[g(x_{ni}) - g(x_{n\,i-1})\right] g^{-1}\big(g(x_{ni})\big), \end{aligned} \tag{2.2}$$

the product $ag(a)$ is presented with the integral sums of the functions g and g^{-1}. Letting n to infinity, we obtain the necessary equality

$$ag(a) = \int_0^a g(x)\,dx + \int_0^{g(a)} g^{-1}(x)\,dx. \tag{2.3}$$

Take another non-negative number b. Using the above and related equality

$$bg^{-1}(b) = \int_0^b g^{-1}(x)\,dx + \int_0^{g^{-1}(b)} g(x)\,dx, \tag{2.4}$$

we can prove that

$$\int_a^{g^{-1}(b)} \big(b - g(x)\big)\,dx = \int_{g(a)}^{b} \big(g^{-1}(x) - a\big)\,dx. \tag{2.5}$$

A similar proof for the equality in equation (2.3) is given in [8].

The following theorem extends the equality in equation (2.3) to any two non-negative numbers a and b.

Theorem 2.1. *Let $g : [0, \infty) \to [0, \infty)$ be a bijective continuous function, and let a and b be non-negative real numbers.*

Then

$$ab = \int_0^a g(x)\,dx + \int_0^b g^{-1}(x)\,dx - \int_a^{g^{-1}(b)} \big(b - g(x)\big)\,dx. \tag{2.6}$$

Proof. Using the basic properties of the integral calculus, and the equality in equation (2.4), it follows that

$$\begin{aligned} ab+\int_a^{g^{-1}(b)}\big(b-g(x)\big)\,dx &= ab+b\int_a^{g^{-1}(b)}dx-\int_a^{g^{-1}(b)}g(x)\,dx \\ &= bg^{-1}(b)-\int_0^{g^{-1}(b)}g(x)\,dx+\int_0^a g(x)\,dx \\ &= \int_0^a g(x)\,dx+\int_0^b g^{-1}(x)\,dx \end{aligned}$$

proving the equality in equation (2.6). □

The equalities in equations (2.3), (2.5) and (2.6) are equivalent.

The integral value $\int_a^{g^{-1}(b)}\big(b-g(x)\big)\,dx$ is non-negative, and it is equal to zero if, and only if, $a=g^{-1}(b)$, or equivalently $g(a)=b$. Applying the variable values $x=g^{-1}(b)$ and $x=a$ to the function $g(x)$, we get the estimation

$$0\le\int_a^{g^{-1}(b)}\big(b-g(x)\big)\,dx\le\big(b-g(a)\big)\big(g^{-1}(b)-a\big). \tag{2.7}$$

A similar estimation was obtained in [1]. Combining the above inequality and the equality in equation (2.6), we get the Young inequality extended to the right side, as follows.

Corollary 2.2. *Let $g:[0,\infty)\to[0,\infty)$ be a bijective continuous function, and let a and b be non-negative real numbers.*

Then

$$ab\le\int_0^a g(x)\,dx+\int_0^b g^{-1}(x)\,dx\le ab+\big(b-g(a)\big)\big(g^{-1}(b)-a\big). \tag{2.8}$$

Theorem 2.1 also provides a clear picture of the equality cases. The equalities in equation (2.8) appear if, and only if, $g(a)=b$. In fact, the inequality in equation (2.8) is equivalent with the equalities in equations (2.3), (2.5) and (2.6).

The equality in equation (2.3) has been proved in [8] by using the lower and upper integral sums within ε-notation. The left inequality of equation (2.8) is the well-known integral form of Young's inequality. He proved the inequality using the assumption that the function g is differentiable, see [9]. Convenient proof of this inequality was obtained in [8] by using the convexity of the antiderivative function $G(x)=\int_0^x g(t)\,dt$. The whole double inequality in equation (2.8) was also proved in [8] applying the mean value theorem of the integral calculus.

3. Main results

The first objective of this section are generalizations of the discrete and integral form of Young's inequality. The second objective is the application of a non-decreasing convex function to the extended Young's inequality.

3.1. **Generalizations of the discrete form.** The aim of this subsection is to extend the discrete form of Young's inequality, and then apply a convex function to this extension.

Lemma 3.1. *Let $a_1,\dots,a_n$ be non-negative points on the real axis, and let $\alpha_1,\dots,\alpha_n$ be positive coefficients satisfying $\sum_{i=1}^n \alpha_i = 1$.*

Then

$$\prod_{i=1}^n a_i^{\alpha_i} \le \sum_{i=1}^n \alpha_i a_i \le \sum_{i=1}^n a_i - (n-1)\prod_{i=1}^n a_i^{\frac{1-\alpha_i}{n-1}}. \tag{3.1}$$

Proof. We prove the right inequality in equation (3.1). Applying the geometric-arithmetic mean inequality to the geometric mean $\prod_{i=1}^n a_i^{(1-\alpha_i)/(n-1)}$, we obtain the inequality

$$\prod_{i=1}^n a_i^{\frac{1-\alpha_i}{n-1}} \le \sum_{i=1}^n \frac{1-\alpha_i}{n-1} a_i = \frac{1}{n-1}\left(\sum_{i=1}^n a_i - \sum_{i=1}^n \alpha_i a_i\right), \tag{3.2}$$

which after rearrangement coincides with the right inequality in equation (3.1). □

Remark. *The left term of the inequality in equation* (3.1) *is geometric mean, and the middle term is arithmetic mean. Using the concrete examples it can be shown that the right term does not necessarily belong to* $\operatorname{conv}\{a_1,\dots,a_n\}$, *except in the initial case $n=2$.*

If $n=2$, then the geometric mean $a_1^{1-\alpha_1}a_2^{1-\alpha_2}$ belongs to the interval $\operatorname{conv}\{a_1,a_2\}$ *and thus it is equal to some binomial convex combination $\beta_1 a_1 + \beta_2 a_2$. Therefore, the whole right-hand side takes the convex combination form*

$$\begin{aligned} a_1 + a_2 - a_1^{1-\alpha_1}a_2^{1-\alpha_2} &= a_1 + a_2 - \beta_1 a_1 - \beta_2 a_2 \\ &= (1-\beta_1)a_1 + (1-\beta_2)a_2 \end{aligned} \tag{3.3}$$

belonging to the interval $\operatorname{conv}\{a_1,a_2\}$.

Substituting $a = a_1^{\alpha_1}$, $b = a_2^{\alpha_2}$, $\alpha_1 = 1/p$ and $\alpha_2 = 1/q$ in equation (3.1) for $n=2$, and using the fact that

$$a^p + b^q - a^{p-1}b^{q-1} = ab + \big(b - a^{p-1}\big)\big(b^{q-1} - a\big), \tag{3.4}$$

we get the following extension of the discrete form of Young's inequality.

Corollary 3.2. *Let $a,b\ge 0$ be points on the real axis, and let $p,q>1$ be coefficients satisfying $1/p+1/q=1$.*

Then

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q \le ab + \big(b - a^{p-1}\big)\big(b^{q-1} - a\big). \tag{3.5}$$

Similarly to the integral case, the equalities are valid in equation (3.5) if, and only if, $a^p = b^q$. The equation (3.5) can be obtained by applying the substitutions $g(x) = x^{p-1}$ and $g^{-1}(x) = x^{q-1}$ to the integral inequality in equation (2.8).

As an adventitious consequence of Lemma 3.1, we expose the extensions of Bernoulli's inequalities in the following manner.

Corollary 3.3. *Let a be a non-negative real number, and let r be a real number.*

Then the following inequalities are valid and equivalent:

$$a - 1 \le \frac{a^{1-r}-1}{1-r} \le a^{-r}(a-1),\ \ r<0 \tag{3.6}$$

$$a^r - 1 \leq r(a-1) \leq a^{1-r}(a^r - 1),\ \ 0 < r < 1 \tag{3.7}$$

$$a - 1 \leq \frac{a^r - 1}{r} \leq a^{r-1}(a-1),\ \ r > 1 \tag{3.8}$$

Proof. For example, the inequality in equation (3.7) follows by applying the substitutions $a_1 = 1$, $a_2 = a$, $\alpha_1 = 1 - r$ and $\alpha_2 = r$ to equation (3.1) with $n = 2$. □

We finish the subsection generalizing Lemma 3.1 by involving a convex function and applying the famous Jensen's inequality (see [2]).

Theorem 3.4. *Let $a_1, \ldots, a_n$ be non-negative points, and let $\alpha_1, \ldots, \alpha_n$ be positive coefficients satisfying $\sum_{i=1}^n \alpha_i = 1$. Let $f : \operatorname{conv}\{a_1, \ldots, a_n\} \to \mathbb{R}$ be a non-decreasing convex function.*

Then

$$f\left(\prod_{i=1}^n a_i^{\alpha_i}\right) \leq \sum_{i=1}^n \alpha_i f(a_i) \leq \sum_{i=1}^n f(a_i) - (n-1) f\left(\prod_{i=1}^n a_i^{\frac{1-\alpha_i}{n-1}}\right). \tag{3.9}$$

Proof. Prove the right inequality in equation (3.9). Acting with the non-decreasing and convex function f to the geometric-arithmetic mean inequality

$$\prod_{i=1}^n a_i^{\frac{1-\alpha_i}{n-1}} \leq \sum_{i=1}^n \frac{1-\alpha_i}{n-1} a_i \tag{3.10}$$

and applying Jensen's inequality, we get

$$\begin{aligned} f\left(\prod_{i=1}^n a_i^{\frac{1-\alpha_i}{n-1}}\right) &\leq f\left(\sum_{i=1}^n \frac{1-\alpha_i}{n-1} a_i\right) \leq \sum_{i=1}^n \frac{1-\alpha_i}{n-1} f(a_i) \\ &= \frac{1}{n-1}\left(\sum_{i=1}^n f(a_i) - \sum_{i=1}^n \alpha_i f(a_i)\right), \end{aligned} \tag{3.11}$$

which can be easily transformed to the required inequality. □

3.2. **Generalizations of the integral form.** First, we want to apply a non-decreasing convex function to the extended Young's inequality in equation (2.8). For this purpose we put

$$c = \int_0^a g(x)\,dx + \int_0^b g^{-1}(x)\,dx. \tag{3.12}$$

The numbers ab, c, and $g(a)g^{-1}(b)$ belong to the interval $\operatorname{conv}\{ag(a), bg^{-1}(b)\}$. Replacing the middle term in equation (2.8) with the convex combination $c = \alpha\, ag(a) + \beta\, bg^{-1}(b)$, and using the fact that $ab + \big(b - g(a)\big)\big(g^{-1}(b) - a\big) = ag(a) + bg^{-1}(b) - g(a)g^{-1}(b)$, we get the inequality

$$ab \leq \alpha\, ag(a) + \beta\, bg^{-1}(b) \leq ag(a) + bg^{-1}(b) - g(a)g^{-1}(b), \tag{3.13}$$

where

$$\alpha = \frac{bg^{-1}(b) - c}{bg^{-1}(b) - ag(a)}\,,\ \ \beta = \frac{c - ag(a)}{bg^{-1}(b) - ag(a)}. \tag{3.14}$$

Theorem 3.5. *Let $g:[0,\infty)\to[0,\infty)$ be a bijective continuous function, and let a and b be different non-negative real numbers. Let $f:\operatorname{conv}\{ag(a),bg^{-1}(b)\}\to\mathbb{R}$ be a non-decreasing convex function.*

Then

$$f(ab)\le \alpha f\big(ag(a)\big)+\beta f\big(bg^{-1}(b)\big)\le f\big(ag(a)\big)+f\big(bg^{-1}(b)\big)-f\big(g(a)g^{-1}(b)\big). \tag{3.15}$$

Proof. The left inequality in equation (3.15) is a direct consequence of the monotonicity and convexity of the function f.

To prove the right inequality, we use the function $f^{\text{line}}_{\{A,B\}}$ of the chord line passing through the points $A\big(ag(a),f(ag(a))\big)$ and $B\big(bg^{-1}(b),f(bg^{-1}(b))\big)$ of the graph of f. By applying the convexity of f, as well as the monotonicity and affinity of $f^{\text{line}}_{\{A,B\}}$, it follows that

$$\begin{aligned}
\alpha f\big(ag(a)\big)+\beta f\big(bg^{-1}(b)\big) &= f^{\text{line}}_{\{A,B\}}\left(\alpha\, ag(a)+\beta\, bg^{-1}(b)\right)=f^{\text{line}}_{\{A,B\}}(c)\\
&\le f^{\text{line}}_{\{A,B\}}\left(ag(a)+bg^{-1}(b)-g(a)g^{-1}(b)\right)\\
&= f\left(ag(a)\right)+f\left(bg^{-1}(b)\right)-f^{\text{line}}_{\{A,B\}}\left(g(a)g^{-1}(b)\right)\\
&\le f\left(ag(a)\right)+f\left(bg^{-1}(b)\right)-f\left(g(a)g^{-1}(b)\right),
\end{aligned}$$

and the inequality proof is done. □

To generalize the extended Young's inequality in equation (2.8) we interpolate new points, and thus achieve the main result.

Theorem 3.6. *Let $g:[0,\infty)\to[0,\infty)$ be a bijective continuous function, and let a and b be non-negative real numbers. Let $n\ge 2$ be an integer, and let $a=x_1,\dots,x_n=g^{-1}(b)$ be either the non-decreasing or non-increasing n-tuple.*

Then

$$\begin{aligned}
ag(a)+\sum_{i=1}^{n-1}[g(x_{i+1})-g(x_i)]x_i &\le \int_0^a g(x)\,dx+\int_0^b g^{-1}(x)\,dx\\
&\le bg^{-1}(b)+\sum_{i=1}^{n-1}[x_i-x_{i+1}]g(x_i).
\end{aligned} \tag{3.16}$$

Proof. Let us prove the inequality in (3.16) using the induction on the integer $n\ge 2$.

The base of induction. Taking $n=2$ the inequality in (3.16) reduces to the inequality in (2.8).

The step of induction. Suppose that the inequality in equation (3.16) is true for all corresponding n-tuples of points x_i. Using $n+1$ points x_i where $x_{n+1}=g^{-1}(b)$, and applying the induction premise to the left inequality, we get

$$\begin{aligned}
ag(a)+\sum_{i=1}^{n}[g(x_{i+1})-g(x_i)]x_i &\le \int_0^a g(x)\,dx+\int_0^{g(x_n)} g^{-1}(x)\,dx+\int_{g(x_n)}^b x_n\,dx\\
&\le \int_0^a g(x)\,dx+\int_0^{g(x_n)} g^{-1}(x)\,dx+\int_{g(x_n)}^b g^{-1}(x)\,dx\\
&= \int_0^a g(x)\,dx+\int_0^b g^{-1}(x)\,dx,
\end{aligned}$$

proving the left inequality for $n+1$ points. To prove the right inequality we proceed in a similar way. □

The geometric image of the inequality in equation (3.16), the case $a \leq g^{-1}(b)$, can be seen in Figure 1. Expressed by the numeric values, it is as follows.

The numeric value of the middle term is equal to "area of the curvilinear triangle with vertices $O(0,0)$, $A(a,0)$ and $A'(a,g(a))$" plus "area of the curvilinear triangle with vertices $O(0,0)$, $B(0,b)$ and $B'(g^{-1}(b),b)$". The value of the left term is equal to "area of the rectangle with sides a and $g(a)$" plus "areas of the rectangles with sides x_i and $g(x_{i+1}) - g(x_i)$". The value of the right term is equal to "area of the rectangle with sides $g^{-1}(b)$ and b" minus "areas of the rectangles with sides $x_{i+1} - x_i$ and $g(x_i)$".

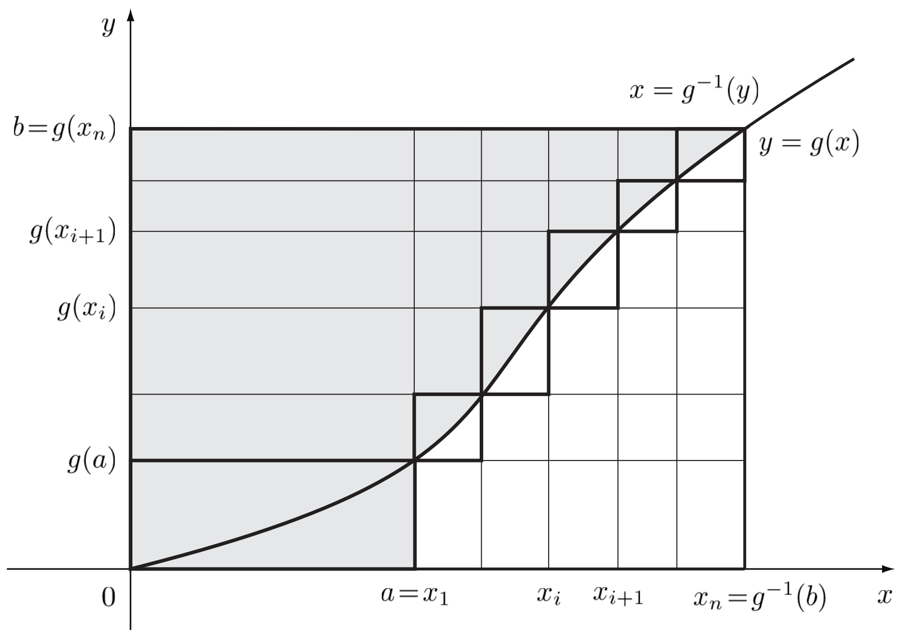

FIGURE 1. The geometric presentation of the inequality in (3.16)

Remark. *If, for the given integer n, every interval $[x_{ni}, x_{n\,i+1}]$ of the n-tuple of Theorem 3.6 contracts to the point as n approaches infinity, then both the left and right term of equation* (3.16) *approaches the middle term. It actually says that the left term approaches*

$$ag(a) + \int_{g(a)}^{b} g^{-1}(x)\,dx, \tag{3.17}$$

and the right term approaches

$$bg^{-1}(b) + \int_{g^{-1}(b)}^{a} g(x)\,dx. \tag{3.18}$$

The bounds of Young's inequality were discussed in [1] and [4]. Review of the earlier work on Young's inequality and its consequences can be found in books [7, pages 239-246] and [5, pages 14-19].

4. A functional approach

Let $\mathcal{X}$ be a non-empty set. Let $\mathbb{X}$ be a subspace of the linear space $\mathbb{R}^{\mathcal{X}}$ of all real functions on the domain $\mathcal{X}$ containing the unit function e_0, defined by $e_0(x) = 1$ for every $x \in \mathcal{X}$. A linear functional $L : \mathbb{X} \to \mathbb{R}$ is said to be positive (non-negative) if $L(g) \geq 0$ for every non-negative function $g \in \mathbb{X}$. The positive functional L is called unital or normalized if $L(e_0) = 1$. For any function $g \in \mathbb{X}$, the number $L(g)$ is located in the closed convex hull of the set $\{g(x) : x \in \mathcal{X}\}$, that is, in the closed interval of real numbers which contains the image of g.

Jessen (see [3]) affirmed the functional form of Jensen's inequality. Suitable adaptation of Jessen's result is as follows.

Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval, and let $g \in \mathbb{X}$ be a function such that $g(x) \in \mathcal{I}$ for every $x \in \mathcal{X}$. Let $f : \mathcal{I} \to \mathbb{R}$ be a convex continuous function such that $f(g) \in \mathbb{X}$. Let $L : \mathbb{X} \to \mathbb{R}$ be a unital positive linear functional.

Then

$$L(g) \in \mathcal{I} \tag{4.1}$$

and

$$f(L(g)) \leq L(f(g). \tag{4.2}$$

It should be noted that the inclusion in (4.1) is not generally true if the interval $\mathcal{I}$ is not closed, and the inequality in (4.2) is not generally valid if the function f is not continuous. Functional variants of Jensen's inequality and its applications have been considered in [6].

We take $\mathcal{X} = [0, \infty)$, and assume that $\mathbb{X}$ is the linear space of all continuous real functions on the domain $[0, \infty)$. For every positive number a we can define the unital positive linear functional $L_a : \mathbb{X} \to \mathbb{R}$ by

$$L_a(g) = \frac{1}{a} \int_0^a g(x)\, dx. \tag{4.3}$$

According to the left inequality of equation (2.8), it follows that the inequality

$$ab \leq aL_a\big(g\big) + bL_b\big(g^{-1}\big) \tag{4.4}$$

holds for all pairs of numbers $a, b > 0$ and mutually inverse functions $g, g^{-1} \in \mathbb{X}$.

In what follows, we will use functionals L_a and L_b satisfying the inequality in (4.4) independently on equation (4.3). Applying the substitution $b = a$ to the inequality in equation (4.4), and using the function $h = g + g^{-1}$, we get

$$a \leq L_a(h). \tag{4.5}$$

Summarizing the inequality in equation (4.4), and the related inequality where g and g^{-1} are replaced, we obtain

$$2ab \leq aL_a(h) + bL_b(h). \tag{4.6}$$

To generalize the above formula, we include three positive numbers.

Lemma 4.1. *Let a_1, a_2, a_3 be positive real numbers, and let $L_{a_1}, L_{a_2}, L_{a_3}$ be unital positive linear functionals on $\mathbb{X}$ such that every pair satisfies the inequality in equation* (4.4)*. Let $h = g + g^{-1}$, and let $L_{a_k}(h) = L_k$ be abbreviations for $k = 1, 2, 3$.*

Then

$$12a_1a_2a_3 \leq \sum_{j\leq i=1}^{3} (a_i + a_j)L_iL_j. \tag{4.7}$$

Proof. Applying the inequality in equation (4.6) to the products $2a_1a_2$, $2a_1a_3$ and $2a_2a_3$, we get

$$\begin{aligned} 2(2a_1a_2)a_3 &\leq 2(a_1L_1 + a_2L_2)a_3 \\ &= 2a_1a_3L_1 + 2a_2a_3L_2 \\ &\leq a_1L_1L_1 + a_3L_1L_3 + \\ &\quad a_2L_2L_2 + a_3L_2L_3. \end{aligned} \tag{4.8}$$

In the same way, or using the above inequality, we obtain

$$\begin{aligned} 2(2a_1a_3)a_2 &\leq a_1L_1L_1 + a_2L_1L_2 + \\ &\quad a_2L_2L_3 + a_3L_3L_3 \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} 2(2a_2a_3)a_1 &\leq a_1L_1L_2 + a_1L_1L_3 + \\ &\quad a_2L_2L_2 + a_3L_3L_3. \end{aligned} \tag{4.10}$$

The required inequality in equation (4.7) follows by summarizing the above three inequalities. □

Relying on mathematical induction, we can obtain the following generalization.

Theorem 4.2. *Let $n \geq 2$ be a positive integer, let $a_1, \ldots, a_n$ be positive real numbers, and let $L_{a_1}, \ldots, L_{a_n}$ be unital positive linear functionals on $\mathbb{X}$ such that every pair satisfies the inequality in equation (4.4). Let $h = g + g^{-1}$, and let $L_{a_k}(h) = L_k$ be abbreviations for $k = 1, \ldots, n$.*
Then

$$n!2^{n-2}a_1 \ldots a_n \leq \sum_{i_{n-1}\leq\ldots\leq i_1=1}^{n} \big(a_{i_1} + \ldots + a_{i_{n-1}}\big)L_{i_1} \ldots L_{i_{n-1}}. \tag{4.11}$$

Proof. To prove the formula in equation (4.11), we use mathematical induction depending on the number of points a_k. Taking $n = 2$, we have the basic inequality in equation (4.6).

Assuming that the inequality in equation (4.11) is valid for all positive integers which are less than or equal to $n-1\geq 2$, we get the series of inequalities

$$
\begin{aligned}
&n!2^{n-2}a_1\dots a_n=\sum_{k=1}^{n}2a_k\big[(n-1)!2^{(n-1)-2}a_1\dots a_{k-1}a_{k+1}\dots a_n\big]\\
&\leq\sum_{k=1}^{n}2a_k\sum_{\substack{i_{n-2}\leq\ldots\leq i_1=1\\ i_{n-2},\ldots,i_1\neq k}}^{n}\big(a_{i_1}+\ldots+a_{i_{n-2}}\big)L_{i_1}\dots L_{i_{n-2}}\\
&=\sum_{k=1}^{n}\sum_{\substack{i_{n-2}\leq\ldots\leq i_1=1\\ i_{n-2},\ldots,i_1\neq k}}^{n}\big(2a_ka_{i_1}+\ldots+2a_ka_{i_{n-2}}\big)L_{i_1}\dots L_{i_{n-2}}\\
&\leq\sum_{k=1}^{n}\sum_{\substack{i_{n-2}\leq\ldots\leq i_1=1\\ i_{n-2},\ldots,i_1\neq k}}^{n}\big(a_kL_k+a_{i_1}L_{i_1}+\ldots+a_kL_k+a_{i_{n-2}}L_{i_{n-2}}\big)L_{i_1}\dots L_{i_{n-2}}\\
&=\sum_{i_{n-1}\leq\ldots\leq i_1=1}^{n}\big(a_{i_1}+\ldots+a_{i_{n-1}}\big)L_{i_1}\dots L_{i_{n-1}},
\end{aligned}
\tag{4.12}
$$

whose the first and last term represent the formula in equation (4.11). □

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