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DERIVATIVES OF APPELL FUNCTIONS WITH RESPECT TO PARAMETERS

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Abstract. We give nth-order derivatives of the four Appell functions with respect to their parameters. Certain related results are also presented.

1. Introduction

Ancarani and Gasaneo [1, 2, 3] obtained nth-order derivatives of the hypergeometric functions ${}_1F_1$, ${}_2F_1$ and ${}_pF_q$ with respect to parameters. Various applications of these derivatives were also given. In the present paper, we obtain nth-order derivatives of Appell functions with respect to the parameters. It is expected that these results are useful to workers in special functions, mathematical physics and engineering, to name a few. Sectionwise treatment is as follows.

In Section 2, we list certain preliminary results and definitions that are needed in the sequel. In Section 3, we obtain first order derivatives of Appell functions with respect to the parameters. These derivatives are expressible in terms of Srivastava's triple hypergeometric function. In Section 4, we find nth-order derivatives of Appell functions with respect to the parameters in terms of $(n+2)$–variable hypergeometric functions. Finally, in Section 5, we discuss some particular cases and related results involving contiguous relations.

2. Preliminaries

The generalized hypergeometric function ${}_pF_q$ is defined as [4, 8]

$$ {}_pF_q = {}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\middle|; z\right) = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_m}{\prod_{i=1}^{q}(b_i)_m}\frac{z^m}{m!} \tag{2.1} $$

where Pochhammer symbol $(a)_m$ is given by

$$ (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}, \quad m = 0, 1, 2, \dots. \tag{2.2} $$

Psi function $\psi(z)$ is defined as [8]

$$ \psi(z) = \frac{d}{dz}\ln\Gamma(z) $$

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and satisfies the following recurrence relation

$$\psi(z+n)-\psi(z)=\sum_{l=1}^{n}\frac{1}{z+l-1},\quad n=1,2,\dots. \tag{2.3}$$

The four Appell functions of two variables are defined as [7, 8]:

$$F_1(a,b,b';c;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1+m_2}(b)_{m_1}(b')_{m_2}}{(c)_{m_1+m_2}}\frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!},$$
$$\max\{|x|,|y|\}\le 1; \tag{2.4}$$

$$F_2(a,b,b';c,c';x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1+m_2}(b)_{m_1}(b')_{m_2}}{(c)_{m_1}(c')_{m_2}}\frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!},$$
$$|x|+|y|\le 1; \tag{2.5}$$

$$F_3(a,a',b,b';c;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1}(a')_{m_2}(b)_{m_1}(b')_{m_2}}{(c)_{m_1+m_2}}\frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!},$$
$$\max\{|x|,|y|\}\le 1; \tag{2.6}$$

$$F_4(a,b;c,c';x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1+m_2}(b)_{m_1+m_2}}{(c)_{m_1}(c')_{m_2}}\frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!},$$
$$\sqrt{|x|}+\sqrt{|y|}\le 1. \tag{2.7}$$

The Srivastava's triple hypergeometric function [7] is defined by

$$\begin{aligned}F^{(3)}[x,y,z] &= F^{(3)}\left[\begin{smallmatrix}(a)\,::\,(b);\,(b')\,;\,(b'')\,:\,(c)\,;\,(c')\,;\,(c'')\,;\\(e)\,::\,(g)\,;\,(g')\,;\,(g'')\,:\,(h);\,(h');\,(h'')\,;\end{smallmatrix}\;x,y,z\right]\\ &= \sum_{m,n,p=0}^{\infty}\wedge(m,n,p)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\end{aligned} \tag{2.8}$$

where

$$\wedge(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{m+p}\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{m+p}\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p} \tag{2.9}$$

and (a) abbreviates the array of A parameters $a_1,a_2,\dots,a_A$ etc. The region of convergence of the general triple hypergeometric series (2.8) is given in the literature [8].

For the sake of convenience, we denote the following two special cases of (2.8) as:

$$\begin{aligned}&\rho^{(1)}_{(1)}\left(\begin{smallmatrix}a_1,b_1,c_1\,;\,a_2,a_3,a_4\\ d_1,\,d_2,\,d_3\end{smallmatrix}\middle|;x_1,x_2,x_3\right)\\ &= F^{(3)}\left[\begin{smallmatrix}a_4\,::\,a_3;\,-\,;\,-\,:\,a_1,a_2\,;\,b_1\,;\,c_1\,;\\ d_3\,::\,d_2\,;\,-\,;\,-\,:\,d_1\,;\,-\,;\,-\,;\end{smallmatrix}\;x_1,x_2,x_3\right],\end{aligned} \tag{2.10}$$

$$\rho^{(1)}_{(2)}\left(\begin{matrix} a_1,a_2,a_3;b_1;c_1,c_2 \\ b'_1;c'_1,c'_2\end{matrix}\middle|;x_1,x_2,x_3\right)$$
$$= F^{(3)}\left[\begin{matrix} c_2 :: c_1 ; b_1 ; - : a_1 ; a_2 ; a_3 ; \\ c'_2 :: c'_1 ; b'_1 ; - : -; -; - ;\end{matrix} x_1,x_2,x_3\right]. \tag{2.11}$$

The motivation for these special cases of $F^3[x,y,z]$, namely $\rho^{(1)}_{(1)}$ and $\rho^{(1)}_{(2)}$, follows from the Appell technique. Note that we can construct series $\rho^{(1)}_{(1)}$ and $\rho^{(1)}_{(2)}$ by application of the rule used by Appell to the product of the hypergeometric functions ${}_4F_3\ {}_3F_2\ {}_2F_1$ and ${}_3F_2\ {}_4F_3\ {}_3F_2$ respectively. We start with

$${}_4F_3\left(\begin{matrix} a_1,a_2,a_3,a_4 \\ d_1,d_2,d_3\end{matrix}\middle|;x_1\right){}_3F_2\left(\begin{matrix} b_1,b_2,b_3 \\ e_1,e_2\end{matrix}\middle|;x_2\right){}_2F_1\left(\begin{matrix} c_1,c_2 \\ f_1\end{matrix}\middle|;x_3\right)$$
$$= \sum_{m_1,m_2,m_3=0}^{\infty} \frac{\prod_{i=1}^4 (a_i)_{m_1} \prod_{i=1}^3 (b_i)_{m_2} \prod_{i=1}^2 (c_i)_{m_3}}{\prod_{i=1}^3 (d_i)_{m_i} \prod_{i=1}^2 (e_i)_{m_2} (f_1)_{m_3}} \prod_{i=1}^{3} \frac{x_i^{m_i}}{m_i!}. \tag{2.12}$$

Using the notation $M_j = \sum_{i=1}^j m_i$ and the replacements

$$(a_3)_{m_1}(b_2)_{m_2} \longrightarrow (a_3)_{M_2},\quad (a_4)_{m_1}(b_3)_{m_2}(c_2)_{m_3} \longrightarrow (a_4)_{M_3},$$
$$(d_2)_{m_1}(e_1)_{m_2} \longrightarrow (d_2)_{M_2},\quad (d_3)_{m_1}(e_2)_{m_2}(f_1)_{m_3} \longrightarrow (d_3)_{M_3}$$

the following coefficient is generated:

$$\frac{\prod_{i=1}^4 (a_i)_{m_1} \prod_{i=1}^3 (b_i)_{m_2} \prod_{i=1}^2 (c_i)_{m_3}}{\prod_{i=1}^3 (d_i)_{m_i} \prod_{i=1}^2 (e_i)_{m_2} (f_1)_{m_3}} \longrightarrow \frac{(a_1)_{m_1}(b_1)_{m_2}(c_1)_{m_3}(a_2)_{M_1}(a_3)_{M_2}(a_4)_{M_3}}{(d_1)_{M_1}(d_2)_{M_2}(d_3)_{M_3}}.$$

So the series (2.12) is represented as

$$\rho^{(1)}_{(1)}\left(\begin{matrix} a_1,b_1,c_1;a_2,a_3,a_4 \\ d_1,d_2,d_3\end{matrix}\middle|;x_1,x_2,x_3\right)$$
$$= \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{m_1}(b_1)_{m_2}(c_1)_{m_3}(a_2)_{M_1}(a_3)_{M_2}(a_4)_{M_3}}{(d_1)_{M_1}(d_2)_{M_2}(d_3)_{M_3}} \prod_{i=1}^{3} \frac{x_i^{m_i}}{m_i!}. \tag{2.13}$$

Similarly, $\rho^{(1)}_{(2)}$ is generated by applying Appell technique to the product of the hypergeometric functions ${}_3F_2\ {}_4F_3\ {}_3F_2$. Iteratively, we can construct $\rho^{(n)}_{(1)}$, $\rho^{(n)}_{(2)}$, $\rho^{(n)}_{(r+2)}$, $r = 1,2,\ldots,n-1$ and $\rho^{(n)}$, by applying Appell technique on product of the hypergeometric functions ${}_{n+3}F_{n+2}\ {}_{n+2}F_{n+1}\ \ldots\ {}_2F_1$, ${}_{n+2}F_{n+1}\ {}_{n+3}F_{n+2}\ {}_{n+2}F_{n+1}\ \ldots\ {}_3F_2$, ${}_{n+3}F_{n+2}\cdots{}_{r+3}F_{r+2}\ {}_{r+3}F_{r+2}\ {}_{r+2}F_{r+1}\cdots{}_3F_2$ and ${}_2F_1\ {}_{n+3}F_{n+2}\ {}_{n+2}F_{n+1}\ \ldots\ {}_3F_2$ respectively. This leads to the following hypergeometric functions in $(n+2)$–variables:

$$\rho^{(n)}_{(1)}\left(\begin{matrix} a_1,\ldots,a_{n+2};c_1,\ldots,c_{n+2} \\ c'_1,\ldots,c'_{n+2}\end{matrix}\middle|;x_1,\ldots,x_{n+2}\right)$$
$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i} \prod_{i=1}^{n+2}(c_i)_{M_i}}{\prod_{i=1}^{n+2}(c'_i)_{M_i}} \prod_{i=1}^{n+2} \frac{x_i^{m_i}}{m_i!}; \tag{2.14}$$

$$\rho^{(n)}_{(2)}\left(\begin{matrix} a_1,\ldots,a_{n+2};b_1;c_1,\ldots,c_{n+1} \\ b'_1;c'_1,\ldots,c'_{n+1}\end{matrix}\middle|;x_1,\ldots,x_{n+2}\right)$$
$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+2}-M_1} \prod_{i=1}^{n+1}(c_i)_{M_{i+1}}}{(b'_1)_{M_{n+2}-M_1}\prod_{i=1}^{n+1}(c'_i)_{M_{i+1}}} \prod_{i=1}^{n+2} \frac{x_i^{m_i}}{m_i!}; \tag{2.15}$$

$$\rho^{(n)}_{(r+2)}\left(\begin{matrix} a_1,\ldots,a_{n+2};b_1,b_2;c_1,\ldots,c_{n-r};c_{n+1-r},\ldots,c_{n+1} \\ b'_1,b'_2 ; c'_1,\ldots,c'_{n-r} ; c'_{n+1-r},\ldots,c'_{n+1}\end{matrix}\middle|;x_1,\ldots,x_{n+2}\right)$$

$$= \sum_{m_1,\dots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+1-r}}(b_2)_{M_{n+2}-M_{n+1-r}}\prod_{i=1}^{n-r}(c_i)_{M_i}}{(b'_1)_{M_{n+1-r}}(b'_2)_{M_{n+2}-M_{n+1-r}}\prod_{i=1}^{n-r}(c'_i)_{M_i}} \times \frac{\prod_{i=n+1-r}^{n+1}(c_i)_{M_{i+1}}}{\prod_{i=n+1-r}^{n+1}(c'_i)_{M_{i+1}}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \quad (2.16)$$

$$\rho^{(n)}\left(\begin{smallmatrix} a_1,\dots,a_{n+2};\, b_1;\, c_1,\dots,c_{n+1} \\ d'_1;\, d'_2;\, c'_1,\dots,c'_n \end{smallmatrix}\middle|; x_1,\dots,x_{n+2}\right) = \sum_{m_1,\dots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+2}}\prod_{i=1}^{n+1}(c_i)_{M_{i+1}-M_1}}{(d'_1)_{M_{n+2}-M_1}(d'_2)_{M_{n+2}}\prod_{i=1}^{n}(c'_i)_{M_{i+1}-M_1}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}. \quad (2.17)$$

As we will see, the nth-order derivative of F_1 with respect to parameters are expressible in terms of $\rho_{(1)}^{(n)}$, $\rho_{(2)}^{(n)}$, $\rho_{(r+2)}^{(n)}$ and $\rho^{(n)}$.

Similarly, for expressing derivatives of remaining three Appell functions with respect to the parameters, we introduce the following $(n+2)$–variable hypergeomteric functions:

$$\sigma_{(1)}^{(n)}\left(\begin{smallmatrix} a_1,\dots,a_{n+2};\, c_1,\dots,c_{n+2} \\ c'_{n+3};\, c'_{n+2};\, c'_1,\dots,c'_{n+1} \end{smallmatrix}\middle|; x_1,\dots,x_{n+2}\right) = \sum_{m_1,\dots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}\prod_{i=1}^{n+2}(c_i)_{M_i}}{\prod_{i=1}^{n+1}(c'_i)_{M_i}(c'_{n+2})_{M_{n+1}}(c'_{n+3})_{m_{n+2}}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \quad (2.18)$$

$$\sigma_{(2)}^{(n)}\left(\begin{smallmatrix} a_1,\dots,a_{n+2};\, b_1;\, c_1,\dots,c_{n+1} \\ b'_1, b'_2;\, c'_1,\dots,c'_{n+1} \end{smallmatrix}\middle|; x_1,\dots,x_{n+2}\right) = \sum_{m_1,\dots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+2}-M_1}\prod_{i=1}^{n+1}(c_i)_{M_{i+1}}}{(b'_1)_{M_{n+2}-M_1}(b'_2)_{M_{n+2}-M_1}\prod_{i=1}^{n+1}(c'_i)_{M_i}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \quad (2.19)$$

$$\sigma_{(r+2)}^{(n)}\left(\begin{smallmatrix} a_1,\dots,a_{n+2};\, b_1, b_2;\, c_1,\dots,c_{n-r};\, c_{n+1-r},\dots,c_{n+1} \\ b'_1, b'_2;\, b'_3, b'_4;\, c'_1,\dots,c'_{n-r};\, c_{n+1-r}\dots,c'_n \end{smallmatrix}\middle|; x_1,\dots,x_{n+2}\right) = \sum_{m_1,\dots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+1-r}}(b_2)_{M_{n+2}-M_{n+1-r}}}{(b'_1)_{M_{n+1-r}}(b'_2)_{M_{n+2}-M_{n+1-r}}(b'_3)_{M_{n+1-r}}(b'_4)_{M_{n+2}-M_{n+1-r}}} \times \frac{\prod_{i=1}^{n-r}(c_i)_{M_i}\prod_{i=n+1-r}^{n+1}(c_i)_{M_{i+1}}}{\prod_{i=1}^{n-r}(c'_i)_{M_i}\prod_{i=n+1-r}^{n}(c'_i)_{M_{i+1}}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \quad (2.20)$$

$$\sigma^{(n)}\left(\begin{smallmatrix} a_1,\dots,a_{n+2};\, c_1,\dots,c_{n+1};\, c_{n+2} \\ b'_1, b'_2;\, b'_3;\, c'_1,\dots,c'_n \end{smallmatrix}\middle|; x_1,\dots,x_{n+2}\right) = \sum_{m_1,\dots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}\prod_{i=1}^{n+1}(c_i)_{M_{i+1}-M_1}(c_{n+2})_{M_{n+2}}}{(b'_1)_{M_{n+2}-M_1}(b'_2)_{M_{n+2}-M_1}(b'_3)_{M_1}\prod_{i=1}^{n}(c'_i)_{M_{i+1}-M_1}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \quad (2.21)$$

$$\Delta_{(1)}^{(n)}\left(\begin{smallmatrix} a_1,\dots,a_{n+2};\, b_1;\, b_2;\, c_1,\dots,c_{n+1} \\ c'_1,\dots,c'_{n+2} \end{smallmatrix}\middle|; x_1,\dots,x_{n+2}\right) = \sum_{m_1,\dots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{m_{n+2}}(b_2)_{M_{n+1}}\prod_{i=1}^{n+1}(c_i)_{M_i}}{\prod_{i=1}^{n+2}(c'_i)_{M_i}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \quad (2.22)$$

$$\Delta_{(2)}^{(n)}\left(\begin{smallmatrix} a_1,\dots,a_{n+2};\, b_1, b_2;\, c_1,\dots,c_{n+1} \\ b'_1;\, c'_1,\dots,c'_{n+1} \end{smallmatrix}\middle|; x_1,\dots,x_{n+2}\right)$$

$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+2}-M_1}(b_2)_{M_{n+2}-M_1}\prod_{i=1}^{n+1}(c_i)_{M_i}}{(b'_1)_{M_{n+2}-M_1}\prod_{i=1}^{n+1}(c'_i)_{M_i+1}} \prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \tag{2.23}$$

$$\Delta^{(n)}_{(r+2)}\left(\begin{matrix} a_1,\ldots,a_{n+2}\,;b_1,b_2\ ;\ b_3,b_4\ ;c_1,\ldots,c_{n-r}\,;c_{n-r+1},\ldots,c_n \\ b'_1,b'_2\,;c'_1,\ldots,c_{n-r}\,;c_{n+1-r},\ldots,c'_{n+1}\end{matrix}\middle|\,;x_1,\ldots,x_{n+2}\right)$$
$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+1-r}}(b_2)_{M_{n+2}-M_{n+1-r}}(b_3)_{M_{n+1-r}}}{(b'_1)_{M_{n+1-r}}(b'_2)_{M_{n+2}-M_{n+1-r}}}$$
$$\times \frac{(b_4)_{M_{n+2}-M_{n+1-r}}\prod_{i=1}^{n-r}(c_i)_{M_i}\prod_{i=n+1-r}^{n}(c_i)_{M_{i+1}}}{\prod_{i=1}^{n-r}(c'_i)_{M_i}\prod_{i=n+1-r}^{n+1}(c'_i)_{M_{i+1}}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \tag{2.24}$$

$$\Delta^{(n)}\left(\begin{matrix} a_1,\ldots,a_{n+2}\,;b_1\,;c_1\,;c_2\ldots,c_{n+2} \\ b'_1\,;,c'_1\ldots,c'_n\,;c'_{n+1}\end{matrix}\middle|\,;x_1,\ldots,x_{n+2}\right)$$
$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+2}-M_1}(c_1)_{M_1}\prod_{i=1}^{n+1}(c_{i+1})_{M_{i+1}-M_1}}{(b'_1)_{M_{n+2}-M_1}\prod_{i=1}^{n}(c'_i)_{M_i+1-M_1}(c'_{n+1})_{M_{n+2}}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \tag{2.25}$$

$$\vee^{(n)}_{(1)}\left(\begin{matrix} a_1,\ldots,a_{n+1}\,;b_1,c_{n+1}\,;c_1,\ldots,c_n \\ b'_1\,;b'_3\,;c'_1,\ldots,c'_{n+1}\end{matrix}\middle|\,;x_1,\ldots,x_{n+2}\right)$$
$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+1}(a_i)_{m_i}(b_1)_{M_{n+2}}\prod_{i=1}^{n}(c_i)_{M_i}(c_{n+1})_{M_{n+2}}}{(b'_1)_{M_{n+1}}(b'_3)_{m_{n+2}}\prod_{i=1}^{n+1}(c'_i)_{M_i}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \tag{2.26}$$

$$\vee^{(n)}_{(2)}\left(\begin{matrix} a_1,\ldots,a_{n+1}\,;b_1\,;c_1,\ldots,c_{n+1} \\ b'_1,b'_2\,;b'_3\,;c'_1,\ldots,c'_n\end{matrix}\middle|\,;x_1,\ldots,x_{n+2}\right)$$
$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+1}(a_i)_{m_{i+1}}(b_1)_{M_{n+2}}\prod_{i=1}^{n+1}(c_i)_{M_{i+1}}}{(b'_1)_{M_{n+2}-M_1}(b'_2)_{M_{n+2}-M_1}(b'_3)_{m_1}\prod_{i=1}^{n}(c'_i)_{M_{i+1}}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \tag{2.27}$$

$$\vee^{(n)}_{(r+2)}\left(\begin{matrix} a_1,\ldots,a_{n+2}\,;b_1\,;c_1,\ldots,c_{n-r}\,;c_{n+1-r},\ldots,c_{n+1} \\ b'_1,b'_2\,;\,b'_3,b'_4\,;\,c'_1,\ldots,c'_n\end{matrix}\middle|\,;x_1,\ldots,x_{n+2}\right)$$
$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+2}(a_i)_{m_i}(b_1)_{M_{n+2}}\prod_{i=1}^{n-r}(c_i)_{M_i}}{(b'_1)_{M_{n+1-r}}(b'_2)_{M_{n+2}-M_{n+1-r}}(b'_3)_{M_{n+1-r}}(b'_4)_{M_{n+2}-M_{n+1-r}}}$$
$$\times \frac{\prod_{i=n+1-r}^{n+1}(c_i)_{M_{i+1}}}{\prod_{i=1}^{n}(c'_i)_{M_i}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}; \tag{2.28}$$

$$\vee^{(n)}\left(\begin{matrix} a_1,\ldots,a_{n+1}\,;b_1,c_1\,;c_2,\ldots,c_{n+1} \\ b'_1\,;c'_1\,;c'_2,\ldots,c'_{n+2}\end{matrix}\middle|\,;x_1,\ldots,x_{n+2}\right)$$
$$= \sum_{m_1,\ldots,m_{n+2}=0}^{\infty} \frac{\prod_{i=1}^{n+1}(a_i)_{m_{i+1}}(b_1)_{M_{n+2}}(c_1)_{M_{n+2}}\prod_{i=2}^{n+1}(c_i)_{M_i-M_1}}{(b'_1)_{M_{n+2}-M_1}(c'_1)_{M_1}\prod_{i=2}^{n+2}(c'_i)_{M_i-M_1}}\prod_{i=1}^{n+2}\frac{x_i^{m_i}}{m_i!}. \tag{2.29}$$

where $r = 1, 2, \ldots, n-1$.

We remark that $\sigma^{(n)}_{(1)}$, $\Delta^{(n)}_{(1)}$ and $\vee^{(n)}_{(1)}$ are obtained by applying Appell technique to the product of the hypergeometric functions ${}_{n+3}F_{n+2}\ {}_{n+2}F_{n+1}\ \ldots\ {}_2F_1$; $\sigma^{(n)}_{(2)}$, $\Delta^{(n)}_{(2)}$ and $\vee^{(n)}_{(2)}$ are obtained from ${}_{n+2}F_{n+1}\ {}_{n+3}F_{n+2}\ {}_{n+2}F_{n+1}\ \ldots\ {}_3F_2$; $\sigma^{(n)}_{(r+2)}$, $\Delta^{(n)}_{(r+2)}$

and $\vee^{(n)}_{(r+2)}$ are obtained from ${}_{n+3}F_{n+2}\cdots {}_{r+3}F_{r+2}\ {}_{r+3}F_{r+2}\ {}_{r+2}F_{r+1}\cdots {}_{3}F_{2}$ and $\sigma^{(n)}$, $\Delta^{(n)}$ and $\vee^{(n)}$ are obtained from the product of hypergeometric functions ${}_2F_1\ {}_{n+3}F_{n+2}\ {}_{n+2}F_{n+1}\ \ldots\ {}_3F_2$, suitably.

3. First order derivatives of Appell functions with respect to parameters

For brevity, we denote the four Appell functions by F_1, F_2, F_3 and F_4. The notation $F^{(n)}_{1,a}$ is used to express nth-order derivative of F_1 with respect to the parameter a, etc.

We begin with finding first order derivative of F_1 with respect to parameter a. Clearly

$$F^{(1)}_{1,a} = \sum_{m_1,m_2=0}^{\infty} [\psi(a+M_2)-\psi(a)]A(m_1,m_2)\frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!},$$

where

$$A(m_1,m_2) = \frac{(a)_{m_1+m_2}(b)_{m_1}(b')_{m_2}}{(c)_{m_1+m_2}}. \tag{3.1}$$

This gives

$$\begin{aligned}
F^{(1)}_{1,a} &= \sum_{m_1,m_2=0}^{\infty} A(m_1,m_2)[\psi(a+M_2)-\psi(a+M_1)]\frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!}\\
&\quad+\sum_{m_1,m_2=0}^{\infty} A(m_1,m_2)[\psi(a+M_1)-\psi(a)]\frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!}\\
&= \sum_{m_1,m_2=0}^{\infty} A(m_1,m_2+1)[\psi(a+M_2+1)-\psi(a+M_1)]\frac{x^{m_1}}{m_1!}\frac{y^{m_2+1}}{(m_2+1)!}\\
&\quad+\sum_{m_1,m_2=0}^{\infty} A(m_1+1,m_2)[\psi(a+1+M_1)-\psi(a)]\frac{x^{m_1+1}}{(m_1+1)!}\frac{y^{m_2}}{m_2!}\\
&= \sum_{m_1,m_2=0}^{\infty} A(m_1,m_2+1)\frac{x^{m_1}}{m_1!}\frac{y^{m_2+1}}{(m_2+1)!}\sum_{s_2=0}^{m_2}\frac{1}{(s_2+M_1+a)}\\
&\quad+\sum_{m_1,m_2=0}^{\infty} A(m_1+1,m_2)\frac{x^{m_1+1}}{(m_1+1)!}\frac{y^{m_2}}{m_2!}\sum_{s_1=0}^{m_1}\frac{1}{(s_1+a)}.
\end{aligned}$$

Applying the series rearrangement [6, 8]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k), \tag{3.2}$$

we get

$$\begin{aligned}
F^{(1)}_{1,a} &= \frac{y}{a}\sum_{m_1,m_2,s_2=0}^{\infty} A(m_1,m_2+s_2+1)\frac{(a)_{s_2+M_1}}{(a+1)_{s_2+M_1}}\frac{x^{m_1}}{m_1!}\frac{y^{m_2+s_2}}{(m_2+s_2+1)!}\\
&\quad+\frac{x}{a}\sum_{m_1,m_2,s_1=0}^{\infty} A(m_1+s_1+1,m_2)\frac{(a)_{s_1}}{(a+1)_{s_1}}\frac{x^{m_1+s_1}}{(m_1+s_1+1)!}\frac{y^{m_2}}{m_2!}.
\end{aligned}$$

We replace $s_1, s_2 \to m_3$ in both expressions. Now $m_2 \leftrightarrow m_3$ in first expression and $m_2 \leftrightarrow m_3$ then $m_2 \leftrightarrow m_1$ in second expression gives after some simplification:

$$\begin{aligned} F^{(1)}_{1,a} = x\frac{b}{c} \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(1)_{m_1}(1)_{m_2}(b')_{m_3}(a)_{M_1}(b+1)_{M_2}(a+1)_{M_3}}{(a+1)_{M_1}(2)_{M_2}(c+1)_{M_3}} \frac{x^{m_1}}{m_1!}\frac{x^{m_2}}{m_2!}\frac{y^{m_3}}{m_3!} \\ + y\frac{b'}{c} \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(b)_{m_1}(1)_{m_2}(1)_{m_3}(a)_{M_2}(b'+1)_{M_3-M_1}(a+1)_{M_3}}{(a+1)_{M_2}(2)_{M_3-M_1}(c+1)_{M_3}} \frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!}\frac{y^{m_3}}{m_3!}. \end{aligned} \tag{3.3}$$

Using the definition of Srivastava's triple hypergeometric function (2.8), the above derivative formula can be expressed as

$$\begin{aligned} F^{(1)}_{1,a} &= x\frac{b}{c} F^{(3)} \left[\begin{matrix} a+1 :: b+1; - ; - : 1,a ; 1 ; b' ; \\ c+1 :: 2 ; - ; - : a+1; -; - ; \end{matrix} x, x, y\right] \\ &+ y\frac{b'}{c} F^{(3)} \left[\begin{matrix} a+1 :: a ; b'+1 ; - : b ; 1 ; 1 ; \\ c+1 :: a+1 ; 2 ; - : -; -; - ; \end{matrix} x, y, y\right] \end{aligned} \tag{3.4}$$

The first order derivatives of F_1 with respect to numerator parameters b and b' are given by

$$F^{(1)}_{1,b} = x\frac{a}{c} F^{(3)} \left[\begin{matrix} a+1 :: b+1 ; - ; - : 1,b ; 1 ; b' ; \\ c+1 :: 2 ; - ; - : b+1; -; - ; \end{matrix} x, x, y\right] \tag{3.5}$$

and

$$F^{(1)}_{1,b'} = y\frac{a}{c} F^{(3)} \left[\begin{matrix} a+1 :: -; b'+1 ; - : b ; 1,b' ; 1 ; \\ c+1 :: - ; 2 ; - : - ; b'+1 ; - ; \end{matrix} x, y, y\right] \tag{3.6}$$

respectively.

Next we find first order derivative of Appell Function F_1 with respect to denominator parameter c. We have

$$F^{(1)}_{1,c} = (-1) \sum_{m_1,m_2=0}^{\infty} [\psi(c+M_2) - \psi(c)] A(m_1,m_2) \frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!},$$

where $A(m_1, m_2)$ is given by (3.1). Now

$$\begin{aligned} F^{(1)}_{1,c} = (-1) \sum_{m_1,m_2=0}^{\infty} A(m_1,m_2+1)[\psi(c+M_2+1) - \psi(c+M_1)] \frac{x^{m_1}}{m_1!}\frac{y^{m_2+1}}{(m_2+1)!} \\ + (-1) \sum_{m_1,m_2=0}^{\infty} A(m_1+1,m_2)[\psi(c+1+M_1) - \psi(c)] \frac{x^{m_1+1}}{(m_1+1)!}\frac{y^{m_2}}{m_2!}. \end{aligned}$$

Using eqs. (2.3) and (3.2), we get

$$\begin{aligned} F^{(1)}_{1,c} = \frac{(-1)y}{c} \sum_{m_1,m_2,s_2=0}^{\infty} A(m_1,m_2+s_2+1) \frac{(c)_{s_2+M_1}}{(c+1)_{s_2+M_1}} \frac{x^{m_1}}{m_1!}\frac{y^{m_2+s_2}}{(m_2+s_2+1)!} \\ + \frac{(-1)x}{c} \sum_{m_1,m_2,s_1=0}^{\infty} A(m_1+s_1+1,m_2) \frac{(c)_{s_1}}{(c+1)s_1} \frac{x^{m_1+s_1}}{(m_1+s_1+1)!}\frac{y^{m_2}}{m_2!}. \end{aligned}$$

If $s_1, s_2 \to m_3$ in both expressions. Now $m_2 \leftrightarrow m_3$ in first expression and $m_2 \leftrightarrow m_3$ then $m_2 \leftrightarrow m_1$ in second expression gives after some simplification:

$$F^{(1)}_{1,c} = \frac{(-1)xab}{c^2} \sum_{m_1,m_2,m_3=0}^{\infty} \frac{\prod_{i=1}^{2}(1)_{m_i}(b')_{m_3}(c)_{M_1}(b+1)_{M_2}(a+1)_{M_3}}{(c+1)_{M_1}(2)_{M_2}(c+1)_{M_3}} \prod_{i=1}^{2}\frac{x^{m_i}}{m_i!}\frac{y^{m_3}}{m_3!}$$

$$+\frac{(-1)yab'}{c^2}\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(b)_{m_1}\prod_{i=2}^{3}(1)_{m_i}(c)_{M_2}(b'+1)_{M_3-M_1}(a+1)_{M_3}}{(c+1)_{M_2}(2)_{M_3-M_1}(c+1)_{M_3}}\frac{x^{m_1}}{m_1!}\prod_{i=2}^{3}\frac{y^{m_i}}{m_i!}. \tag{3.7}$$

Now, using the definition of Srivastava's triple hypergeometric function (2.8), the above derivative formula can be expressed as

$$\begin{aligned} F^{(1)}_{1,c} = & -\frac{xab}{c^2}F^{(3)}\left[\begin{matrix} a{+}1 :: b{+}1\,;\ -\ ;\ -\ :\ 1,c\,;\ 1\,;\ b'\,; \\ c{+}1 :: 2\,;\ -\ ;\ -\ :\ c{+}1\,;\ -\ ;\ -\ ; \end{matrix}\ x,x,y\right] \\ & -\frac{ya\,b'}{c^2}F^{(3)}\left[\begin{matrix} a{+}1 :: c\,;\ b'{+}1\,;\ -\ :\ b\,;\ 1\,;\ 1\,; \\ c{+}1 :: c{+}1\,;\ 2\,;\ -\ :\ -\,;\ -\,;\ -\,; \end{matrix}\ x,y,y\right]. \end{aligned} \tag{3.8}$$

First order derivatives of remaining three Appell functions F_2, F_3, F_4 with respect to parameters are given in the following theorems. We omit the proofs.

Theorem 3.1. *The following derivative formulas hold for* F_2*:*

$$\begin{aligned} F^{(1)}_{2,a} = & \ x\frac{b}{c}F^{(3)}\left[\begin{matrix} a{+}1 :: b{+}1;\ -\ ;\ -\ :\ a,1\,;\ 1\,;\ b'\,; \\ - :: 2,c{+}1\,;\ -\ ;\ -\ :\ a{+}1\,;\ -\,;\ c'\,; \end{matrix}\ x,x,y\right] \\ & +y\frac{b'}{c'}F^{(3)}\left[\begin{matrix} a{+}1 :: a\,;\ b'{+}1\,;\ -\ :\ b\,;\ 1\,;\ 1\,; \\ - :: a{+}1\,;\ 2,c'{+}1\,;\ -\ :\ c;\ -\,;\ -\,; \end{matrix}\ x,y,y\right], \end{aligned} \tag{3.9}$$

$$F^{(1)}_{2,b} = x\frac{a}{c}F^{(3)}\left[\begin{matrix} a{+}1 :: b{+}1;\ -\ ;\ -\ :\ b,1\,;\ 1\,;\ b'\,; \\ - :: 2,c{+}1\,;\ -\ ;\ -\ :\ b{+}1\,;\ -\,;\ c'\,; \end{matrix}\ x,x,y\right], \tag{3.10}$$

$$F^{(1)}_{2,b'} = y\frac{a}{c'}F^{(3)}\left[\begin{matrix} a{+}1 :: -\,;\ b'{+}1\,;\ -\ :\ b\,;\ 1,b'\,;\ 1\,; \\ - :: -,\ ;\ 2,c'{+}1\,;\ -\ :\ c\,;\ b'{+}1\,;\ -\,; \end{matrix}\ x,y,y\right], \tag{3.11}$$

$$F^{(1)}_{2,c} = -x\frac{a\,b}{c^2}F^{(3)}\left[\begin{matrix} a{+}1 :: b{+}1\,;\ -\ ;\ -\ :\ 1,c\,;\ 1\,;\ b'\,; \\ - :: 2,c{+}1\,;\ -\ ;\ -\ :\ c{+}1\,;\ -\,;\ c'\,; \end{matrix}\ x,x,y\right], \tag{3.12}$$

$$F^{(1)}_{2,c'} = -y\frac{ab'}{c'^2}F^{(3)}\left[\begin{matrix} a{+}1 :: -\,;\ b'{+}1\,;\ -\ :\ b\,;\ 1,c'\,;\ 1\,; \\ - :: -\,;\ 2,c'{+}1\,;\ -\ :\ c\,;\ c'{+}1\,;\ -\,; \end{matrix}\ x,y,y\right]. \tag{3.13}$$

Theorem 3.2. *The following derivative formulas hold for* F_3*:*

$$F^{(1)}_{3,a} = x\frac{b}{c}F^{(3)}\left[\begin{matrix} - :: a{+}1,b{+}1\,;\ -\ ;\ -\ :\ 1,a\,;\ 1\,;\ a',b'\,; \\ c{+}1 :: 2\,;\ -\ ;\ -\ :\ a{+}1\,;\ -\,;\ -\,; \end{matrix}\ x,x,y\right], \tag{3.14}$$

$$F^{(1)}_{3,a'} = y\frac{b'}{c}F^{(3)}\left[\begin{matrix} - :: -\,;\ a'{+}1,b'{+}1\,;\ -\ :\ a,b\,;\ 1,a'\,;\ 1\,; \\ c{+}1 :: -\,;\ 2\,;\ -\ :\ -\,;\ a'{+}1\,;\ -\,; \end{matrix}\ x,y,y\right], \tag{3.15}$$

$$F^{(1)}_{3,b} = x\frac{a}{c}F^{(3)}\left[\begin{matrix} - :: a{+}1,b{+}1\,;\ -\ ;\ -\ :\ b,1\,;\ 1\,;\ a',b'\,; \\ c{+}1 :: 2\,;\ -\ ;\ -\ :\ b{+}1\,;\ -\,;\ -\,; \end{matrix}\ x,x,y\right], \tag{3.16}$$

$$F^{(1)}_{3,b'} = y\frac{a'}{c}F^{(3)}\left[\begin{matrix} - :: -\,;\ a'{+}1,b'{+}1\,;\ -\ :\ a,b\,;\ 1,b'\,;\ 1\,; \\ c{+}1 :: -\,;\ 2\,;\ -\ :\ -\,;\ b'{+}1\,;\ -\,; \end{matrix}\ x,y,y\right], \tag{3.17}$$

$$\begin{aligned} F^{(1)}_{3,c} = & -\frac{x\,ab}{c^2}F^{(3)}\left[\begin{matrix} - :: a{+}1,b{+}1\,;\ -\ ;\ -\ :\ c,1\,;\ 1\,;\ a',b'\,; \\ c{+}1 :: 2\,;\ -\ ;\ -\ :\ c{+}1\,;\ -\,;\ -\,; \end{matrix}\ x,x,y\right] \\ & -\frac{y\,a'b'}{c^2}F^{(3)}\left[\begin{matrix} - :: c\,;\ a'{+}1,b'{+}1\,;\ -\ :\ a,b\,;\ 1\,;\ 1\,; \\ c{+}1 :: c{+}1\,;\ 2\,;\ -\ :\ -\,;\ -\,;\ -\,; \end{matrix}\ x,y,y\right]. \end{aligned} \tag{3.18}$$

Theorem 3.3. *The following derivative formulas hold for* F_4*:*

$$\begin{aligned} F^{(1)}_{4,a} = & \ x\frac{b}{c}F^{(3)}\left[\begin{matrix} a{+}1,b{+}1 :: -\ ;\ -\ ;\ -\ :\ a,1\,;\ 1\,;\ -\,; \\ - :: 2,c{+}1\,;\ -\ ;\ -\ :\ a{+}1\,;\ -\,;\ c'\,; \end{matrix}\ x,x,y\right] \\ & +\frac{y\,b}{c'}F^{(3)}\left[\begin{matrix} a{+}1,b{+}1 :: a\ ;\ -\ ;\ -\ :\ -\,;\ 1\,;\ 1\,; \\ - :: a{+}1\,;\ 2,c'{+}1;\ -\ :\ c\,;\ -\,;\ -\,; \end{matrix}\ x,y,y\right], \end{aligned} \tag{3.19}$$

$$F_{4,b}^{(1)} = x\frac{a}{c}\,F^{(3)}\left[\begin{smallmatrix} a+1,b+1 :: & - ; & - ; & - : & b,1 ; & 1 ; & - ; \\ - :: & 2,c+1 ; & - ; & - : & b+1 ; & - ; & c' ; \end{smallmatrix} x,x,y\right] + \frac{y\,a}{c'}\,F^{(3)}\left[\begin{smallmatrix} a+1,b+1 :: & b ; & - ; & - : & - ; & 1 ; & 1 ; \\ - :: & b+1 ; & 2,c'+1 ; & - : & c ; & - ; & - ; \end{smallmatrix} x,y,y\right], \tag{3.20}$$

$$F_{4,c}^{(1)} = -x\frac{ab}{c^2}\,F^{(3)}\left[\begin{smallmatrix} a+1,b+1 :: & - ; & - ; & - : & c,1 ; & 1 ; & - ; \\ - :: & 2,c+1 ; & - ; & - : & c+1 ; & - ; & c' ; \end{smallmatrix} x,x,y\right], \tag{3.21}$$

$$F_{4,c'}^{(1)} = -y\frac{ab}{c'^2}\,F^{(3)}\left[\begin{smallmatrix} a+1,b+1 :: & - ; & - ; & - : & - ; & c'\,1 ; & 1 ; \\ - :: & - ; & 2,c'+1 ; & - : & c ; & c'+1 ; & - ; \end{smallmatrix} x,y,y\right]. \tag{3.22}$$

4. nth-order derivatives of Appell functions with respect to parameters

In this section, we give nth-order derivatives with respect to parameters of Appell functions. Before giving the formulation for $F_{1,a}^{(n)}$, it will be worthwhile to compute $F_{1,a}^{(2)}$. We have

$$F_{1,a}^{(2)} = \frac{d}{da}F_{1,a}^{(1)} = \frac{x(b)_1}{(c)_1}A_1 + \frac{y(b')_1}{(c)_1}A_2, \tag{4.1}$$

where

$$\begin{aligned} A_1 = \frac{yb'}{(c+1)} \sum_{m_1,\ldots,m_4=0}^{\infty} & \frac{\prod_{i=1}^{4}(1)_{m_i}(b'+1)_{M_4-M_2}(a)_{M_1}(b+1)_{M_2}(a+1)_{M_3}(a+2)_{M_4}}{(a+1)_{M_1}(2)_{M_2}(2)_{M_4-M_2}(a+2)_{M_3}(c+2)_{M_4}} \\ & \times \prod_{i=1}^{2}\frac{x^{m_i}}{m_i!}\prod_{i=3}^{4}\frac{y^{m_i}}{m_i!} \\ + \frac{x(b+1)}{(c+1)} \sum_{m_1,\ldots,m_4=0}^{\infty} & \frac{\prod_{i=1}^{3}(1)_{m_i}(b')_{m_4}(a)_{M_1}(a+1)_{M_2}(b+2)_{M_3}(a+2)_{M_4}}{(a+1)_{M_1}(a+2)_{M_2}(3)_{M_3}(c+2)_{M_4}} \\ & \times \prod_{i=1}^{3}\frac{x^{m_i}}{m_i!}\frac{y^{m_4}}{m_4!} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} A_2 = \frac{y(b'+1)}{(c+1)} \sum_{m_1,\ldots,m_4=0}^{\infty} & \frac{(b)_{m_1}\prod_{i=2}^{4}(1)_{m_i}(a)_{M_2}(a+1)_{M_3}(b'+2)_{M_4-M_1}(a+2)_{M_4}}{(a+1)_{M_2}(a+2)_{M_3}(3)_{M_4-M_1}(c+2)_{M_4}} \\ & \times \frac{x^{m_1}}{m_1!}\prod_{i=2}^{4}\frac{y^{m_i}}{m_i!} \\ + \frac{xb}{(c+1)} \sum_{m_1,\ldots,m_4=0}^{\infty} & \frac{\prod_{i=1}^{4}(1)_{m_i}(a)_{M_1}(a+1)_{M_3}(b+1)_{M_2}(b'+1)_{M_4-M_2}(a+2)_{M_4}}{(a+1)_{M_1}(2)_{M_2}(2)_{M_4-M_2}(a+2)_{M_3}(c+2)_{M_4}} \\ & \times \prod_{i=1}^{2}\frac{x^{m_i}}{m_i!}\prod_{i=3}^{4}\frac{y^{m_i}}{m_i!}. \end{aligned} \tag{4.3}$$

Putting the value of A_1 and A_2 in eq. (4.1), we get after some simplification

$$F_{1,a}^{(2)} = x^2\frac{(b)_2}{(c)_2} \sum_{m_1,\ldots,m_4=0}^{\infty} \frac{\prod_{i=1}^{3}(1)_{m_i}(b')_{m_4}(a)_{M_1}(a+1)_{M_2}(b+2)_{M_3}(a+2)_{M_4}}{(a+1)_{M_1}(a+2)_{M_2}(3)_{M_3}(c+2)_{M_4}}$$

$$\times \prod_{i=1}^{3} \frac{x^{m_i}}{m_i!} \frac{y^{m_4}}{m_4!}$$

$$+ y^2 \frac{(b')_2}{(c)_2} \sum_{m_1,\ldots,m_4=0}^{\infty} \frac{(b)_{m_1} \prod_{i=2}^{4} (1)_{m_i} (a)_{M_2} (a+1)_{M_3} (b'+2)_{M_4-M_1} (a+2)_{M_4}}{(a+1)_{M_2} (a+2)_{M_3} (3)_{M_4-M_1} (c+2)_{M_4}}$$

$$\times \frac{x^{m_1}}{m_1!} \prod_{i=2}^{4} \frac{y^{m_i}}{m_i!}$$

$$+ 2xy \frac{(b)_1 (b')_1}{(c)_2} \sum_{m_1,\ldots,m_4=0}^{\infty} \frac{\prod_{i=1}^{4} (1)_{m_i} (a)_{M_1} (a+1)_{M_3} (b+1)_{M_2} (b'+1)_{M_4-M_2}}{(a+1)_{M_1} (2)_{M_2} (2)_{M_4-M_2} (a+2)_{M_3}}$$

$$\times \frac{(a+2)_{M_4}}{(c+2)_{M_4}} \prod_{i=1}^{2} \frac{x^{m_i}}{m_i!} \prod_{i=3}^{4} \frac{y^{m_i}}{m_i!}. \tag{4.4}$$

This result can be iteratively generalized to give nth-order derivative of F_1 with respect to parameter a, as follows:

$$F_{1,a}^{(n)}$$

$$= x^n \frac{(b)_n}{(c)_n} \rho_{(1)}^{(n)} \left({}^{1,1,\ldots,1,b'\,;\,a,a+1,\ldots,a+n-1,b+n,a+n}_{a+1,\ldots\ldots,a+n,n+1,c+n} \Big| ; x, \ldots, x, y \right)$$

$$+ y^n \frac{(b')_n}{(c)_n} \rho_{(2)}^{(n)} \left({}^{b,1,\ldots,1\,;b'+n\,;a,a+1\ldots,a+n}_{n+1\,;\,a+1,\ldots,a+n,c+n} \Big| ; x, y, \ldots, y \right) + \sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r \frac{(b)_{n-r} (b')_r}{(c)_n}$$

$$\times \rho_{(r+2)}^{(n)} \left({}^{1,\ldots,1\,;b+1,b'+1\,;a,a+1,\ldots,a+n-r-1\,;\,a+n-r,\ldots,a+n}_{n-r+1,1+r\,;\,a+1,\ldots,a+n-r\,;\,a+n-r+1,\ldots,a+n,c+n} \Big| ; \underbrace{x, \ldots, x}_{(n+1-r)\text{-times}}, \underbrace{y, \ldots, y}_{(r+1)\text{-times}} \right). \tag{4.5}$$

A simple proof of (4.5) may be given using induction method. The nth-order derivatives of F_1 with respect to numerator parameters b and b' are given by

$$F_{1,b}^{(n)} = x^n \frac{(a)_n}{(c)_n} \rho_{(1)}^{(n)} \left({}^{1,1,\ldots,1,b'\,;\,b,b+1,\ldots,b+n-1,b+n,a+n}_{b+1,\ldots\ldots,b+n,n+1,c+n} \Big| ; x, \ldots, x, y \right) \tag{4.6}$$

and

$$F_{1,b'}^{(n)} = y^n \frac{(a)_n}{(c)_n} \rho^{(n)} \left({}^{b,1,\ldots,\,1\,;\,a+n\,;\; b',b'+1,\ldots,\, b'+n}_{n+1\,;\,c+n\,;\; b'+1,b'+2,\ldots,b'+n} \Big| ; x, y, \ldots, y \right). \tag{4.7}$$

respectively.

Next we find nth-order derivative of Appell Function F_1 with respect to denominator parameter c. We start with calculating $F_{1,c}^{(2)}$. By using (3.7), $F_{1,c}^{(2)}$ can be evaluated as

$$F_{1,c}^{(2)} = \frac{2!(-1)^2\, xab}{c^3} \sum_{m_1,..,m_4=0}^{\infty} \frac{\prod_{i=1}^{3} (1)_{m_i} (b')_{m_4} (c)_{M_1} (c)_{M_2} (b+1)_{M_3} (a+1)_{M_4}}{(c+1)_{M_1} (c+1)_{M_2} (2)_{M_3} (c+1)_{M_4}}$$

$$\times \prod_{i=1}^{3} \frac{x^{m_i}}{m_i!} \frac{y^{m_4}}{m_4!}$$

$$+\frac{2!(-1)^2y\,ab'}{c^3}\sum_{m_1,\ldots,m_4=0}^{\infty}\frac{(b)_{m_1}\prod_{i=2}^{4}(1)_{m_i}(c)_{M_2}(c)_{M_3}(b'+1)_{M_4-M_1}(a+1)_{M_4}}{(c+1)_{M_2}(c+1)_{M_3}(2)_{M_4-M_1}(c+1)_{M_4}}$$
$$\times\frac{x^{m_1}}{m_1!}\prod_{i=2}^{4}\frac{y^{m_i}}{m_i!}$$
$$+\frac{2!(-1)^2\,xya(a+1)bb'}{c^2(c+1)^2}\sum_{m_1,\ldots,m_4=0}^{\infty}\frac{\prod_{i=1}^{4}(1)_{m_i}(b'+1)_{M_4-M_2}(c)_{M_1}(b+1)_{M_2}}{(c+1)_{M_1}(2)_{M_2}(2)_{M_4-M_2}(c+1)_{M_3}(c+2)_{M_3}}$$
$$\times\frac{(a+2)_{M_4}}{(c+2)_{M_4}}\prod_{i=1}^{2}\frac{x^{m_i}}{m_i!}\prod_{i=3}^{4}\frac{y^{m_i}}{m_i!}. \tag{4.8}$$

Iteratively, we get nth-order derivative of Appell function F_1 with respect to c, as follows:

$$F_{1,c}^{(n)}=(-1)^n n!\left[\frac{xab}{c^{n+1}}\,\rho_{(1)}^{(n)}\left(\begin{matrix}1,1,\ldots,1,b'\,;\,c,c,\ldots,c,b+1,a+1\\ c+1,c+1,\ldots,c+1,2,c+1\end{matrix}\middle|;x,\ldots,x,y\right)\right.$$
$$+\frac{yab'}{c^{n+1}}\,\rho_{(2)}^{(n)}\left(\begin{matrix}b,1,\ldots,1\,;b'+1\,;c,c\ldots,c,a+1\\ 2\,;\,c+1,\ldots,c+1,c+1\end{matrix}\middle|;x,y\ldots,y\right)$$
$$+\sum_{r=1}^{n-1}\frac{xya(a+1)bb'}{c^{n+1-r}(c+1)^{r+1}}$$
$$\left.\times\rho_{(r+2)}^{(n)}\left(\begin{matrix}1,\ldots,1\,;\,b+1,b'+1\,;\,c,\ldots,c\,;\,c+1,\ldots,c+1,a+2\\ 2,2\,;\,c+1,\ldots,c+1\,;\,c+2,\ldots,c+2\end{matrix}\middle|;\underbrace{x,\ldots,x}_{(n+1-r)\text{-times}},\underbrace{y,\ldots,y}_{(r+1)\text{-times}}\right)\right]. \tag{4.9}$$

The nth-order derivatives of the remaining three Appell functions with respect to parameters are given in the following theorems. We omit the proofs.

Theorem 4.1. *The following nth-order derivative formulas hold for F_2:*

$$F_{2,a}^{(n)}=x^n\frac{(b)_n}{(c)_n}\,\sigma_{(1)}^{(n)}\left(\begin{matrix}1,1,\ldots,1,b'\,;\,a,a+1,\ldots,a+n-1,b+n,a+n\\ c'\,;\,n+1\,;\,a+1,\ldots\ldots,a+n,c+n\end{matrix}\middle|;x,\ldots,x,y\right)$$
$$+y^n\frac{(b')_n}{(c')_n}\,\sigma_{(2)}^{(n)}\left(\begin{matrix}b,1,\ldots,1\,;\,b'+n\,;\,a,a+1\ldots,a+n-1,a+n\\ n+1,\,c'+n\;\;;\;\;c,a+1,\ldots,a+n\end{matrix}\middle|;x,y,\ldots,y\right)$$
$$+\sum_{r=1}^{n-1}\binom{n}{r}x^{n-r}y^r\frac{(b)_{n-r}(b')_r}{(c)_{n-r}(c')_r}$$
$$\times\sigma_{(r+2)}^{(n)}\left(\begin{matrix}1\ldots,1\,;\,b+n-r\;b'+r\,;\,a,a+1,\ldots,a+n-r-1,\ldots,a+n\\ n-r+1,1+r\,;\,c+n-r,c'+r\,;\,a+1,\ldots,a+n-r\,;\,a+n-r+1,\ldots,a+n\end{matrix}\middle|;\underbrace{x,\ldots,x}_{(n+1-r)\text{-}times},\underbrace{y,\ldots,y}_{(r+1)\text{-}times}\right); \tag{4.10}$$

$$F_{2,b}^{(n)}=x^n\frac{(a)_n}{(c)_n}\,\sigma_{(1)}^{(n)}\left(\begin{matrix}1,1,\ldots,1,b'\,;\,b,b+1,\ldots,b+n,a+n\\ c'\,;\,n+1\,;\,b+1,\ldots\ldots,b+n,c+n\end{matrix}\middle|;x,\ldots,x,y\right); \tag{4.11}$$

$$F_{2,b'}^{(n)}=y^n\frac{(a)_n}{(c')_n}\,\sigma^{(n)}\left(\begin{matrix}b,1,\ldots,1\,;\,b',\ldots,b'+n\,;\,a+n\\ n+1,c'+n\,;\,c\,;\,b'+1,\ldots,b'+n\end{matrix}\middle|;x,y,\ldots,y\right); \tag{4.12}$$

$$F_{2,c}^{(n)} = (-1)^n n! \frac{xab}{c^{n+1}}\, \sigma_{(1)}^{(n)} \left({}^{1,1,\ldots,1,b'\,;\,c,\ldots,c,b+1,a+1}_{c'\,;\,2\,;\,c+1,\ldots\ldots,c+1} \middle| ; x,\ldots,x,y \right); \tag{4.13}$$

$$F_{2,c'}^{(n)} = (-1)^n n! \frac{yab'}{c'^{\,n+1}}\, \sigma^{(n)} \left({}^{b,1,\ldots,1\,;\,c',\ldots,c',b'+1\,;\,a+1}_{2,c'+1\,;\,c\,;\,c'+1,\ldots,c'+1} \middle| ; x,y,\ldots,y \right). \tag{4.14}$$

Theorem 4.2. *The following nth-order derivative formulas hold for F_3:*

$$F_{3,a}^{(n)} = x^n \frac{(b)_n}{(c)_n}\, \Delta_{(1)}^{(n)} \left({}^{1,\ldots,1,b'\,;\,a'\,;\,b+n\,;\,a,a+1,\ldots,a+n}_{a+1,\ldots,a+n,n+1,c+n} \middle| ; x,\ldots,x,y \right); \tag{4.15}$$

$$F_{3,a'}^{(n)} = y^n \frac{(b')_n}{(c)_n}\, \Delta^{(n)} \left({}^{b,1\ldots,1\,;\,b'+n\,;\,a\,;\,a',a'+1,\ldots,a'+n}_{n+1\,;\,a'+1,\ldots,a'+n\,;\,c+n} \middle| ; x,y,\ldots,y \right); \tag{4.16}$$

$$F_{3,b}^{(n)} = x^n \frac{(a)_n}{(c)_n}\, \Delta_{(1)}^{(n)} \left({}^{1,\ldots,1,b'\,;\,a'\,;\,a+n\,;\,b,b+1,\ldots,b+n}_{b+1,\ldots,b+n,n+1,c+n} \middle| ; x,\ldots,x,y \right); \tag{4.17}$$

$$F_{3,b'}^{(n)} = y^n \frac{(a')_n}{(c)_n}\, \Delta^{(n)} \left({}^{b,1\ldots,1\,;\,a'+n\,;\,a\,;\,b',b'+1,\ldots,b'+n}_{n+1\,;\,b'+1,\ldots,b'+n\,;\,c+n} \middle| ; x,y,\ldots,y \right); \tag{4.18}$$

$$\begin{aligned} F_{3,c}^{(n)} &= (-1)^n n! \frac{xab}{c^{n+1}} \Delta_{(1)}^{(n)} \left({}^{1,\ldots,1,b'\,;\,a'\,;\,b+1\,;\,c,c,\ldots,c,a+1}_{c+1,\ldots,c+1,2,c+1} \middle| ; x,\ldots,x,y \right) \\ &\quad + (-1)^n n! \frac{ya'b'}{c^{n+1}} \Delta_{(2)}^{(n)} \left({}^{b,1,\ldots,1\,;\,b'+1,a'+1\,;\,a,c,c,\ldots,c}_{2\,;\,c+1,c+1,\ldots,c+1} \middle| ; x,y,\ldots,y \right) \\ &\quad + \sum_{r=1}^{n-1} (-1)^n n! \frac{xyaba'b'}{c^{n+1-r}(c+1)^{r+1}} \\ &\quad \times \Delta_{(r+2)}^{(n)} \left({}^{1,\ldots,1\,;\,a+1,a'+1\,;\,b+1,b'+1\,;\,c,c\ldots,c\,;\,c+1,\ldots,c+1}_{2,2\,;\,c+1,c+1,\ldots,c+1\,;\,c+2,\ldots,c+2} \middle| ; \underbrace{x,\ldots,x}_{(n+1-r)\text{-}times}, \underbrace{y,\ldots,y}_{(r+1)\text{-}times} \right). \end{aligned} \tag{4.19}$$

Theorem 4.3. *The following nth-order derivative formulas hold for F_4:*

$$\begin{aligned} F_{4,a}^{(n)} &= x^n \frac{(b)_n}{(c)_n} \vee_{(1)}^{(n)} \left({}^{1,\ldots,1\,;\,b+n,a+n\,;\,a,a+1,\ldots,a+n-1}_{n+1\,;\,c'\,;\,a+1,a+2,\ldots,a+n,\,c+n} \middle| ; x,\ldots,x,y \right) \\ &+ y^n \frac{(b)_n}{(c')_n} \vee_{(2)}^{(n)} \left({}^{1,\ldots,1\,;\,b+n\,;\,a,a+1\ldots,a+n}_{n+1,c'+n\,;\,c\,;\,a+1,\ldots,a+n} \middle| ; x,y,\ldots,y \right) + \sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r \frac{(b)_n}{(c)_{n-r}(c')_r} \\ &\times \vee_{(r+2)}^{(n)} \left({}^{1,\ldots,1\,;\,b+n\,;\,a,a+1,\ldots,a+n-r-1\,;\,a+n-r,\ldots,a+n}_{n+1-r,r+1\,;\,c+n-r,c'+r\,;\,a+1,\ldots,a+n-r\,;\,a+n-r+1,\ldots,a+n} \middle| ; \underbrace{x,\ldots,x}_{(n+1-r)\text{-}times}, \underbrace{y,\ldots,y}_{(r+1)\text{-}times} \right); \end{aligned} \tag{4.20}$$

$$\begin{aligned} F_{4,b}^{(n)} &= x^n \frac{(a)_n}{(c)_n} \vee_{(1)}^{(n)} \left({}^{1,\ldots,1\,;\,a+n,b+n\,;\,b,b+1,\ldots,b+n-1}_{n+1\,;\,c'\,;\,b+1,b+2,\ldots,b+n,c+n} \middle| ; x,\ldots,x,y \right) \\ &+ y^n \frac{(a)_n}{(c')_n} \vee_{(2)}^{(n)} \left({}^{1,\ldots,1\,;\,a+n\,;\,b,b+1\ldots,b+n}_{n+1,c'+n\,;\,c\,;\,b+1,\ldots,b+n} \middle| ; x,y,\ldots,y \right) + \sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r \frac{(a)_n}{(c)_{n-r}(c')_r} \end{aligned}$$

$$\times \vee_{(r+2)}^{(n)} \left(\begin{matrix} 1,\ldots,1\,;a+n\,;b,b+1,\ldots,b+n-r-1\,;b+n-r,\ldots,b+n \\ n+1-r,r+1\,;c+n-r,c'+r\,;b+1,\ldots,b+n-r\,;b+n-r+1,\ldots,b+n \end{matrix} \middle|; \underbrace{x,\ldots,x}_{(n+1-r)\text{-}times}, \underbrace{y,\ldots,y}_{(r+1)\text{-}times} \right); \tag{4.21}$$

$$F_{4,c}^{(n)} = (-1)^n n! \frac{xab}{c^{n+1}} \vee_{(1)}^{(n)} \left(\begin{matrix} 1,\ldots,1\,;b+1\;a+1;\,c,c,\ldots,c \\ 2,c+1,c',c+1,c+1,\ldots,c+1 \end{matrix} \middle|; x,\ldots,x,y \right); \tag{4.22}$$

$$F_{4,c'}^{(n)} = (-1)^n n! \frac{yab}{c'^{n+1}} \vee^{(n)} \left(\begin{matrix} 1,\ldots,1,b+1,a+1,c',\ldots,c' \\ 2\,;c\,;c'+1,c'+1,\ldots,c'+1 \end{matrix} \middle|; x,\ldots,x,y \right). \tag{4.23}$$

5. Some applications

Let $a = c$. Then F_1 is independent from a and c parameters and $F_{1,a}^{(1)} = F_{1,c}^{(1)} = 0$. Also if we put $a = c$ in (3.3) and (3.7) we get

$$F_{1,a}^{(1)} + F_{1,c}^{(1)} = 0. \tag{5.1}$$

Using these derivative of Appell functions with respect to the parameters, we can write Taylor expansion of Appell functions around some given values of numerator (or denominator) parameters. For example, Taylor expansion of F_1 around some given values $\overline{a}$ is given by

$$F_1 = \sum_{n=0}^{\infty} \frac{(a-\overline{a})^n}{n!} F_{1,a}^{(n)}\big|_{a=\overline{a}}.$$

We now apply these derivatives to certain contiguous relations satisfied by Appell functions. We know that F_1 satisfies the relation [5]

$$cF_1(a,b,b';c;x,y) - (c-a)F_1(a,b,b';c+1;x,y) - aF_1(a+1,b,b';c+1;x,y) = 0. \tag{5.2}$$

Taking derivative of (5.2) with respect to a and c respectively yield

$$\begin{aligned} &F_1(a,b,b';c+1;x,y) - F_1(a+1,b,b';c+1;x,y) \\ &= (c-a)F_{1,a}^{(1)}(a,b,b';c+1;x,y) - cF_{1,a}^{(1)}(a,b,b';c;x,y) \\ &+ aF_{1,a}^{(1)}(a+1,b,b';c+1;x,y); \end{aligned} \tag{5.3}$$

$$\begin{aligned} &F_1(a,b,b';c;x,y) - F_1(a,b,b';c+1;x,y) \\ &= (c-a)F_{1,c}^{(1)}(a,b,b';c+1;x,y) - cF_{1,c}^{(1)}(a,b,b';c;x,y) \\ &+ aF_{1,c}^{(1)}(a+1,b,b';c+1;x,y). \end{aligned} \tag{5.4}$$

Similarly from the following relations satisfied by F_1:

$$\begin{aligned} cF_1(a,b,b';c;x,y) + c(x-1)F_1(a,b+1,b';c;x,y) \\ = (c-a)xF_1(a,b+1,b';c+1;x,y), \end{aligned} \tag{5.5}$$

$$\begin{aligned} cF_1(a,b,b';c;x,y) + c(y-1)F_1(a,b,b'+1;c;x,y) \\ = (c-a)yF_1(a,b,b'+1;c+1;x,y), \end{aligned} \tag{5.6}$$

$$\begin{aligned} &(a-b-b')F_1(a,b,b';c;x,y) - aF_1(a+1,b,b';c;x,y) \\ &+ bF_1(a,b+1,b';c;x,y) + b'F_1(a,b,b'+1;c;x,y) = 0, \end{aligned} \tag{5.7}$$

we obtain by differentiating (5.5), (5.6) and (5.7) with respect to a and c the following results:

$$xF_1(a,b+1,b';c+1;x,y) = (c-a)xF^{(1)}_{1,a}(a,b+1,b';c+1;x,y) - c(x-1)F^{(1)}_{1,a}(a,b+1,b';c;x,y) - cF^{(1)}_{1,a}(a,b,b';c;x,y), \tag{5.8}$$

$$F_1(a,b,b';c;x,y) + (x-1)F_1(a,b+1,b';c;x,y) - xF_1(a,b+1,b';c+1;x,y) = (c-a)xF^{(1)}_{1,c}(a,b+1,b';c+1;x,y) - c(x-1)F^{(1)}_{1,c}(a,b+1,b';c;x,y) - cF^{(1)}_{1,c}(a,b,b';c;x,y), \tag{5.9}$$

$$yF_1(a,b,b'+1;c+1;x,y) = (c-a)yF^{(1)}_{1,a}(a,b,b'+1;c+1;x,y) - c(y-1)F^{(1)}_{1,a}(a,b,b'+1;c;x,y) - cF^{(1)}_{1,a}(a,b,b';c;x,y), \tag{5.10}$$

$$F_1(a,b,b';c;x,y) + (y-1)F_1(a,b,b'+1;c;x,y) - yF_1(a,b,b'+1;c+1;x,y) = (c-a)yF^{(1)}_{1,c}(a,b,b'+1;c+1;x,y) - c(y-1)F^{(1)}_{1,c}(a,b,b'+1;c;x,y) - cF^{(1)}_{1,c}(a,b,b';c;x,y), \tag{5.11}$$

$$F_1(a,b,b';c;x,y) - F_1(a+1,b,b';c;x,y) = aF^{(1)}_{1,a}(a+1,b,b';c;x,y) - (a-b-b')F^{(1)}_{1,a}(a,b,b';c;x,y) - bF^{(1)}_{1,a}(a,b+1,b';c;x,y) - b'F^{(1)}_{1,a}(a,b,b'+1;c;x,y), \tag{5.12}$$

$$(a-b-b')F^{(1)}_{1,c}(a,b,b';c;x,y) - aF^{(1)}_{1,c}(a+1,b,b';c;x,y) + bF^{(1)}_{1,c}(a,b+1,b';c;x,y) + b'F^{(1)}_{1,c}(a,b,b'+1;c;x,y) = 0. \tag{5.13}$$

Similarly, from the contiguous relations for F_3 [5]

$$cF_3(a,a',b,b';c;x,y) - cF_3(a+1,a',b,b';c;x,y) + bxF_3(a+1,a',b+1,b';c+1;x,y) = 0, \tag{5.14}$$

$$cF_3(a,a',b,b';c;x,y) - cF_3(a,a'+1,b,b';c;x,y) + b'yF_3(a,a'+1,b,b'+1;c+1;x,y) = 0, \tag{5.15}$$

$$aF_3(a+1,a',b,b';c;x,y) - bF_3(a,a',b+1,b';c;x,y) + (b-a)F_3(a,a',b,b';c;x,y) = 0, \tag{5.16}$$

$$a'F_3(a,a'+1,b,b';c;x,y) - b'F_3(a,a',b,b'+1;c;x,y) + (b'-a')F_3(a,a',b,b';c;x,y) = 0, \tag{5.17}$$

we get on differentiating (5.14) to (5.17) with respect to c the following relations,

$$F_3(a,a',b,b';c;x,y) = F_3(a+1,a',b,b';c;x,y) + cF^{(1)}_{3,c}(a+1,a',b,b';c;x,y) - cF^{(1)}_{3,c} - bxF^{(1)}_{3,c}(a+1,a',b+1,b';c+1;x,y), \tag{5.18}$$

$$F_3(a,a',b,b';c;x,y) = F_3(a,a'+1,b,b';c;x,y) + cF^{(1)}_{3,c}(a,a'+1,b,b';c;x,y) - cF^{(1)}_{3,c} - b'yF^{(1)}_{3,c}(a,a'+1,b,b'+1;c+1;x,y), \tag{5.19}$$

$$aF^{(1)}_{3,c}(a+1,a',b,b';c;x,y)-bF^{(1)}_{3,c}(a,a',b+1,b';c;x,y)$$
$$+(b-a)F^{(1)}_{3,c}(a,a',b,b';c;x,y)=0, \tag{5.20}$$
$$a'F^{(1)}_{3,c}(a,a'+1,b,b';c;x,y)-b'F^{(1)}_{3,c}(a,a',b,b'+1;c;x,y)$$
$$+(b'-a')F^{(1)}_{3,c}(a,a',b,b';c;x,y)=0. \tag{5.21}$$

Also from the contiguous relations for F_2 and F_4 given below:

$$aF_2(a+1,b,b';c,c';x,y)-bF_2(a,b+1,b';c,c';x,y)-b'F_2(a,b,b'+1;c,c';x,y)$$
$$=(a-b-b')F_2(a,b,b';c,c';x,y), \tag{5.22}$$
$$c'bxF_2(a+1,b+1,b';c+1,c';x,y)+b'cyF_2(a+1,b,b'+1;c,c'+1;x,y)$$
$$=cc'F_2(a+1,b,b';c,c';x,y)-cc'F_2(a,b,b';c,c';x,y), \tag{5.23}$$
$$aF_4(a+1,b;c,c';x,y)-bF_4(a,b+1;c,c';x,y)=(a-b)F_4(a,b;c,c';x,y), \tag{5.24}$$
$$\frac{b}{c}xF_4(a+1,b+1;c+1,c';x,y)+\frac{b}{c'}yF_4(a+1,b+1;c,c'+1;x,y)$$
$$=F_4(a+1,b;c,c';x,y)-F_4(a,b;c,c';x,y), \tag{5.25}$$

we get on differentiating (5.22) and (5.23) with respect to a the following relations,

$$F_2(a+1,b,b';c,c';x,y)-F_2(a,b,b';c,c';x,y)=(a-b-b')F^{(1)}_{2,a}(a,b,b';c,c';x,y)$$
$$+bF^{(1)}_{2,a}(a,b+1,b';c,c';x,y)+b'F^{(1)}_{2,a}(a,b,b'+1;c,c';x,y)$$
$$-aF^{(1)}_{2,a}(a+1,b,b';c,c';x,y), \tag{5.26}$$
$$c'bxF^{(1)}_{2,a}(a+1,b+1,b';c+1,c';x,y)+b'cyF^{(1)}_{2,a}(a+1,b,b'+1;c,c'+1;x,y)$$
$$=cc'F^{(1)}_{2,a}(a+1,b,b';c,c';x,y)-cc'F^{(1)}_{2,a}(a,b,b';c,c';x,y). \tag{5.27}$$

Also differentiating (5.24) and (5.25) with respect to a and b gives following relations:

$$F_4(a+1,b;c,c';x,y)-F_4(a,b;c,c';x,y)$$
$$=(a-b)F^{(1)}_{4,a}(a,b;c,c';x,y)-aF^{(1)}_{4,a}(a+1,b;c,c';x,y)+bF^{(1)}_{4,a}(a,b+1;c,c';x,y), \tag{5.28}$$
$$F_4(a,b;c,c';x,y)-F_4(a,b+1;c,c';x,y)$$
$$=(a-b)F^{(1)}_{4,b}(a,b;c,c';x,y)-aF^{(1)}_{4,b}(a+1,b;c,c';x,y)+bF^{(1)}_{4,b}(a,b+1;c,c';x,y), \tag{5.29}$$
$$\frac{b}{c}xF^{(1)}_{4,a}(a+1,b+1;c+1,c';x,y)+\frac{b}{c'}yF^{(1)}_{4,a}(a+1,b+1;c,c'+1;x,y)$$
$$=F^{(1)}_{4,a}(a+1,b;c,c';x,y)-F^{(1)}_{4,a}(a,b;c,c';x,y), \tag{5.30}$$
$$\frac{x}{c}F_4(a+1,b+1;c+1,c';x,y)+\frac{y}{c'}F_4(a+1,b+1;c,c'+1;x,y)$$
$$+\frac{b}{c}xF^{(1)}_{4,b}(a+1,b+1;c+1,c';x,y)+\frac{b}{c'}yF^{(1)}_{4,b}(a+1,b+1;c,c'+1;x,y)$$
$$=F^{(1)}_{4,b}(a+1,b;c,c';x,y)-F^{(1)}_{4,b}(a,b;c,c';x,y). \tag{5.31}$$

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