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AN EXAMPLE FOR THE GENERALIZATION OF THE INTEGRATION OF SPECIAL FUNCTIONS BY USING THE LAPLACE TRANSFORM

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Abstract. By using the convolution theorem of the Laplace transform (Faltung theorem), we can generalize the calculation of integrals involving special functions. In this paper, the following integral involving the modified Bessel function is calculated, $\int_0^t \tau^{\alpha+n} I_\alpha(a\tau)(t-\tau)^{\beta+m} I_\beta(a(t-\tau))\,d\tau$. As a consistency test of the result obtained, setting the parameters n, m to particular values, some integrals reported in the literature are recovered.

1. Introduction

This paper uses the convolution theorem of the Laplace transform in order to generalize some integrals of special functions given in the literature. Despite the fact we are going to consider an integral of a special type, involving only the modified Bessel function of the first kind, the method described here can be extended to many others types of integrands.

In this way, in the literature we can find the following integral [1, Eqn. 2.15.19 (2)]

$$\begin{aligned} &\int_0^t \tau^\alpha I_\alpha(a\tau)(t-\tau)^{\beta+(1\pm1)/2} I_\beta(a(t-\tau))\,d\tau \\ = \quad &\frac{t^{\alpha+\beta+1\pm1/2}}{\sqrt{2\pi a}} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+1\pm\frac{1}{2}\right)}{\Gamma\left(\alpha+\beta+\frac{3\pm1}{2}\right)} I_{\alpha+\beta+\frac{1}{2}}(at), \\ &\operatorname{Re}\alpha > -\frac{1}{2},\ \operatorname{Re}\beta > -\left(\frac{3\pm1}{4}\right),\ t>0, \end{aligned} \tag{1.1}$$

where $I_\nu(z)$ denotes the *modified Bessel function of the first kind* [2, Chap. 50] and $\Gamma(z)$ the *gamma function* [2, Chap. 43]. By using the *beta function* [3, Eqn. 5.12.1]

$$\mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{1.2}$$

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it is worth noting that we can rewrite (1.1) as

$$\begin{aligned}&\int_0^t \tau^{\alpha} I_{\alpha}(a\tau)(t-\tau)^{\beta+(1\pm 1)/2} I_{\beta}(a(t-\tau))\,d\tau\\ =\ &\frac{t^{\alpha+\beta+1\pm 1/2}}{\sqrt{2\pi a}}\,\mathrm{B}\left(\alpha+\frac{1}{2},\beta+1\pm\frac{1}{2}\right) I_{\alpha+\beta+\frac{1}{2}}(at),\\ &\operatorname{Re}\alpha>-\frac{1}{2},\ \operatorname{Re}\beta>-\left(\frac{3\pm 1}{4}\right), t\in\mathbb{R},\end{aligned}$$

where we can extend the parameter t to non-positive values. The scope of this paper is just to generalize (1.1) to the following integral

$$\int_0^t \tau^{\alpha+n} I_{\alpha}(a\tau)(t-\tau)^{\beta+m} I_{\beta}(a(t-\tau))\,d\tau, \quad n,m=0,1,2,\ldots \tag{1.3}$$

by using the convolution theorem of the Laplace transform [4, Eqn. 17.12.5]

$$\mathcal{L}^{-1}\left\{\mathcal{L}[f(t)]\ \mathcal{L}[g(t)]\right\}=\int_0^t f(\tau)\,g(t-\tau)\,d\tau, \tag{1.4}$$

where the Laplace transform and the inverse Laplace transform are defined as [4, Sect. 17.11]

$$F(s)=\mathcal{L}[f(t)]=\int_0^{\infty} e^{-st} f(t)\,dt, \tag{1.5}$$

and

$$f(t)=\mathcal{L}^{-1}[F(s)]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds \tag{1.6}$$

being γ a constant that exceeds the real part of all the singularities of $F(s)$.

This paper is organized as follows. Section 2 performs the calculation of the integral (1.3) by means of the convolution theorem (1.4). The result is expressed as a finite sum of terms. As a consistency test, Section 3 particularizes the integral calculated in the previous Section for the cases $n=m=0$ and $n=0$, $m=1$, so we recover the integral reported in the literature (1.1).

2. Calculation of the integral

Theorem 2.1. *For n,m non-negative integers,* $\operatorname{Re}\alpha>-\frac{n+1}{2}$, $\operatorname{Re}\beta>-\frac{m+1}{2}$ *and* $t\in\mathbb{R}$,

$$\begin{aligned}&\int_0^t \tau^{\alpha+n} I_{\alpha}(a\tau)(t-\tau)^{\beta+m} I_{\beta}(a(t-\tau))\,d\tau \qquad (2.1)\\ =\ & b^{\mu-1}\,\Gamma(2(\alpha+\tilde{n})+u+1)\,\Gamma(2(\beta+\tilde{m})+v+1)\\ &\times\frac{(2(\tilde{n}+u))!\,(2(\tilde{m}+u))!}{(u\tilde{n}+1)(v\tilde{m}+1)}\,t^{\mu+2r}\\ &\times\sum_{k=0}^{r} b^{2k}\,\frac{c_k(\tilde{n},u,\tilde{m},v,\alpha,\beta)}{\Gamma(2(\mu+r+k))}\,{}_1F_2\left(\left.\begin{array}{c}\mu+2r\\ \mu+r+k,\frac{1}{2}+\mu+r+k\end{array}\right|b^2\right),\end{aligned}$$

where

$$\tilde{n} = \left\lfloor \frac{n}{2} \right\rfloor, \tag{2.2}$$

$$u = n - 2\tilde{n} = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}, \tag{2.3}$$

$$\tilde{m} = \left\lfloor \frac{m}{2} \right\rfloor, \tag{2.4}$$

$$v = m - 2\tilde{m} = \begin{cases} 1, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}, \tag{2.5}$$

$$\mu = \alpha + \beta + 1, \tag{2.6}$$

$$r = \tilde{n} + \tilde{m} + \frac{u+v}{2}, \tag{2.7}$$

$$b = \frac{at}{2}, \tag{2.8}$$

and

$$c_k(\tilde{n}, u, \tilde{m}, v, \alpha, \beta) = \sum_{l=\max(k-\tilde{m},0)}^{\min(\tilde{n},k)} b_l(\tilde{n}, u, \alpha)\ b_{k-l}(\tilde{m}, v, \beta), \tag{2.9}$$

being

$$b_l(\tilde{n}, u, \alpha) = \frac{u(\tilde{n} - l) + 1}{l!\,(2(\tilde{n} + u - l))!\,\Gamma(1 + \alpha + l)}. \tag{2.10}$$

Proof. According to (1.4), we have

$$\begin{aligned} &\int_0^t \tau^{\alpha+n} I_\alpha(a\tau)(t-\tau)^{\beta+m} I_\beta(a(t-\tau))\,d\tau \\ = \ & \mathcal{L}^{-1}\left\{\mathcal{L}\left[t^{\alpha+n} I_\alpha(at)\right]\ \mathcal{L}\left[t^{\beta+m} I_\beta(at)\right]\right\}, \end{aligned} \tag{2.11}$$

thus, according to (1.5), let us calculate first

$$\begin{aligned} &\mathcal{L}\left[t^{\alpha+n} I_\alpha(at)\right] \\ = \ & \int_0^\infty e^{-st} t^{\alpha+n} I_\alpha(at)\,dt \\ = \ & \left(\frac{a}{2}\right)^\alpha \frac{\Gamma(2\alpha+2n+1)}{s^{2\alpha+n+1}\Gamma(1+\alpha)}\, {}_2F_1\left(\begin{array}{c} \alpha + \frac{n+1}{2}, \alpha + \frac{n}{2} + 1 \\ \alpha + 1 \end{array} \middle| \frac{a^2}{s^2}\right), \\ & \operatorname{Re}\alpha > -\frac{n+1}{2}, \end{aligned} \tag{2.12}$$

where we have applied the following result [1, Eqn. 2.15.3 (2)]

$$\begin{aligned} &\int_0^\infty x^{\lambda-1} e^{-px} I_\nu(cx)\,dx \\ = \ & p^{-\lambda-\nu}\left(\frac{c}{2}\right)^\nu \frac{\Gamma(\nu+\lambda)}{\Gamma(\nu+1)}\, {}_2F_1\left(\begin{array}{c} \frac{\lambda+\nu}{2}, \frac{\lambda+\nu+1}{2} \\ \nu+1 \end{array} \middle| \frac{c^2}{p^2}\right), \\ & \operatorname{Re}(\lambda+\nu) > 0,\ \operatorname{Re}p > |\operatorname{Re}c|, \end{aligned} \tag{2.13}$$

being ${}_2F_1(a, b; c; z)$ the *Gauss hypergeometric function* [2, Chap. 60]. According to (2.2) and (2.3), let us set for convenience

$$n = 2\tilde{n} + u,$$

thus

$$\begin{aligned}
&\mathcal{L}\left[t^{\alpha+n} I_\alpha(at)\right] \qquad (2.14)\\
=\ & \left(\frac{a}{2}\right)^\alpha \frac{\Gamma(2(\alpha+\tilde{n})+u+1)}{s^{2(\alpha+\tilde{n})+u+1}\Gamma(1+\alpha)}\, {}_2F_1\left(\begin{array}{c}\alpha+\tilde{n}+u+\frac{1}{2}, \alpha+\tilde{n}+1\\ \alpha+1\end{array}\middle| \frac{a^2}{s^2}\right).
\end{aligned}$$

In order to reduce the above hypergeometric function to an elementary function, let us apply [2, Eqn. 60:4:10]

$$\begin{aligned}
&{}_2F_1\left(\begin{array}{c}a, c+n\\ c\end{array}\middle| x\right) = \frac{1}{(1-x)^{a+n}} \sum_{k=0}^{n} \binom{n}{k} \frac{(c-a)_k}{(c)_k} \ (-x)^k,\\
&n = 0, 1, 2, \ldots
\end{aligned}$$

where

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}, \qquad (2.15)$$

denotes the *Pochhammer symbol*, which has the following property [2, Eqn. 18:5:1]

$$(-x)_k = (-1)^k (x-k+1)_k .$$

Therefore, we have

$$\begin{aligned}
&{}_2F_1\left(\begin{array}{c}\alpha+\tilde{n}++u+\frac{1}{2}, \alpha+\tilde{n}+1\\ \alpha+1\end{array}\middle| \frac{a^2}{s^2}\right) \qquad (2.16)\\
=\ & \frac{s^{2\alpha+4\tilde{n}+2u+1}}{(s^2-a^2)^{\alpha+2\tilde{n}+u+1/2}} \sum_{k=0}^{\tilde{n}} \binom{\tilde{n}}{k} \frac{\left(\tilde{n}+u+\frac{1}{2}-k\right)_k}{(1+\alpha)_k} \left(\frac{a}{s}\right)^{2k}.
\end{aligned}$$

Substituting back (2.16) in (2.12) and taking into account the definition of the Pochhammer symbol (2.15) and the following formula [2, Eqn. 49:4:3]

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}, \qquad (2.17)$$

we can rewrite (2.12) as

$$\begin{aligned}
&\mathcal{L}\left[t^{\alpha+n} I_\alpha(at)\right] \qquad (2.18)\\
=\ & \left(\frac{a}{2}\right)^\alpha s^u \frac{\Gamma(2(\alpha+\tilde{n})+u+1)}{(s^2-a^2)^{\alpha+2\tilde{n}+u+1/2}} \frac{(2(\tilde{n}+u))!\tilde{n}!}{(\tilde{n}+u)!}\\
&\times \sum_{k=0}^{\tilde{n}} \frac{(\tilde{n}+u-k)!\left(\frac{a}{2}\right)^{2k} s^{2(\tilde{n}-k)}}{k!\,(\tilde{n}-k)!\,(2(\tilde{n}+u-k))!\,\Gamma(1+\alpha+k)},\\
&\operatorname{Re}\alpha > -\frac{n+1}{2}.
\end{aligned}$$

Notice that

$$\frac{x!}{(x+u)!} = \left\{\begin{array}{cc} 1, & u=0\\ \frac{1}{x+1}, & u=1 \end{array}\right\} = \frac{1}{ux+1},$$

so we can simplify (2.18), arriving at

$$\begin{aligned}
&\mathcal{L}\left[t^{\alpha+n} I_\alpha(at)\right] \qquad (2.19)\\
=\;& \left(\frac{a}{2}\right)^{\alpha} s^u \frac{\Gamma(2(\alpha+\tilde{n})+u+1)}{(s^2-a^2)^{\alpha+2\tilde{n}+u+1/2}} \frac{(2(\tilde{n}+u))!}{u\tilde{n}+1} R^{\alpha}_{\tilde{n},u}\left(s^2\right),\\
&\operatorname{Re}\alpha > -\frac{n+1}{2},
\end{aligned}$$

where we have defined the following polynomial

$$R^{\alpha}_{\tilde{n},u}\left(s^2\right) = \sum_{k=0}^{\tilde{n}} \frac{\left[u(\tilde{n}-k)+1\right]\left(\frac{a}{2}\right)^{2k} s^{2(\tilde{n}-k)}}{k!\,(2(\tilde{n}+u-k))!\,\Gamma(1+\alpha+k)}.$$

Therefore, substituting in (2.11) the result given in (2.19), we get

$$\begin{aligned}
&\int_0^t \tau^{\alpha+n} I_\alpha(a\tau)(t-\tau)^{\beta+m} I_\beta(a(t-\tau))\,d\tau\\
=\;& \left(\frac{a}{2}\right)^{\alpha+\beta} \Gamma(2(\alpha+\tilde{n})+u+1)\;\Gamma(2(\beta+\tilde{m})+v+1)\\
&\times \frac{(2(\tilde{n}+u))!\,(2(\tilde{m}+u))!}{(u\tilde{n}+1)(v\tilde{m}+1)}\\
&\times \mathcal{L}^{-1}\left[\frac{s^{u+v} R^{\alpha}_{\tilde{n},u}\left(s^2\right) R^{\beta}_{\tilde{m},v}\left(s^2\right)}{(s^2-a^2)^{\alpha+\beta+2(\tilde{n}+\tilde{m})+u+v+1}}\right], \qquad (2.20)\\
&\operatorname{Re}\alpha > -\frac{n+1}{2},\ \operatorname{Re}\beta > -\frac{m+1}{2},
\end{aligned}$$

where, similar to (2.2) and (2.3), we have defined (2.4) and (2.5). According to the result given in the Appendix (A.3), we have

$$R^{\alpha}_{\tilde{n},u}\left(s^2\right) R^{\beta}_{\tilde{m},v}\left(s^2\right) = \sum_{k=0}^{\tilde{n}+\tilde{m}} \left(\frac{a}{2}\right)^{2k} s^{2(\tilde{n}+\tilde{m}-k)} c_k(\tilde{n},u,\tilde{m},v,\alpha,\beta), \qquad (2.21)$$

where we have defined the coefficient c_k as the finite sum given in (2.9)-(2.10). Substituting now (2.21) in (2.20) we arrive at

$$\begin{aligned}
&\int_0^t \tau^{\alpha+n} I_\alpha(a\tau)(t-\tau)^{\beta+m} I_\beta(a(t-\tau))\,d\tau\\
=\;& \left(\frac{a}{2}\right)^{\alpha+\beta} \Gamma(2(\alpha+\tilde{n})+u+1)\,\Gamma(2(\beta+\tilde{m})+v+1)\\
&\times \frac{(2(\tilde{n}+u))!\,(2(\tilde{m}+u))!}{(u\tilde{n}+1)(v\tilde{m}+1)}\\
&\times \sum_{k=0}^{r} \left(\frac{a}{2}\right)^{2k} c_k(\tilde{n},u,\tilde{m},v,\alpha,\beta)\;\mathcal{L}^{-1}\left[\frac{s^{2(r-k)}}{(s^2-a^2)^{\mu+2r}}\right], \qquad (2.22)\\
&\operatorname{Re}\alpha > -\frac{n+1}{2},\ \operatorname{Re}\beta > -\frac{m+1}{2},
\end{aligned}$$

where we have set μ and r as (2.6) and (2.7) respectively. In order to calculate the inverse Laplace transform given in (2.22), we apply [5, Eqn. 2.1.5(8)]

$$\begin{aligned} & \mathcal{L}^{-1}\left[s^{\lambda}\left(s^2-a^2\right)^{\nu}\right] \\ = \; & \frac{t^{-\lambda-2\nu-1}}{\Gamma\left(-\lambda-2\nu\right)}\, {}_1F_2\left(\begin{array}{c} -\nu \\ -\frac{\lambda}{2}-\nu, \frac{1-\lambda}{2}-\nu \end{array} \middle| \frac{a^2t^2}{4} \right), \\ & \mathrm{Re}\left(\lambda+2\nu\right)<0,\ \mathrm{Re}\, s>\mathrm{Re}\, a, \end{aligned}$$

where ${}_pF_q$ is the *generalized hypergeometric function* [3, Eqn. 16.2.1]

$${}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{z^k}{k!}. \tag{2.23}$$

Therefore, we finally obtain (2.1), as we wanted to prove. □

It is worth noting that the numerical integration of (2.1) is slower than the computation of the above formula. For instance, taking random parameters such as $\alpha=1.1+i$, $\beta=2.3-2i$, $n=2$, $m=5$, $a=-1.4+2.1i$ and $t=-2.9$ the numerical integration is ≈ 9 times slower.

3. Particular cases

As a consistency test, we are going to particularize (2.1) in order to recover the result reported in the literature and stated in (1.1).

3.1. **Example 1.** Taking the "$-$" sign in (1.1), we have

$$\begin{aligned} & \int_0^t \tau^{\alpha} I_{\alpha}\left(a\tau\right)\left(t-\tau\right)^{\beta} I_{\beta}\left(a\left(t-\tau\right)\right)d\tau \\ = \; & \frac{t^{\alpha+\beta+1/2}}{\sqrt{2\pi a}}\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma\left(\alpha+\beta+1\right)} I_{\alpha+\beta+\frac{1}{2}}\left(at\right), \\ & \mathrm{Re}\,\alpha>-\frac{1}{2},\ \mathrm{Re}\,\beta>-\frac{1}{2}, \end{aligned} \tag{3.1}$$

which corresponds to the case $n=m=0$ (i.e. $\tilde{n}=\tilde{m}=u=v=r=0$) in (2.1), so that, taking into account (2.6), (2.8) and (2.23), we have

$$\begin{aligned} & \int_0^t \tau^{\alpha} I_{\alpha}\left(a\tau\right)\left(t-\tau\right)^{\beta} I_{\beta}\left(a\left(t-\tau\right)\right)d\tau \\ = \; & \left(\frac{at}{2}\right)^{\alpha+\beta}\Gamma\left(2\alpha+1\right)\Gamma\left(2\beta+1\right)t^{\alpha+\beta+1} \\ & \times \frac{c_0\left(0,0,0,0,\alpha,\beta\right)}{\Gamma\left(2\alpha+2\beta+2\right)}\, {}_0F_1\left(\begin{array}{c} - \\ \frac{3}{2}+\alpha+\beta \end{array} \middle| \frac{a^2t^2}{4} \right), \\ & \mathrm{Re}\,\alpha>-\frac{1}{2},\ \mathrm{Re}\,\beta>-\frac{1}{2}. \end{aligned}$$

Applying now the formula [6, Sect. 9.14]

$${}_0F_1\left(\begin{array}{c} - \\ \gamma \end{array} \middle| z \right) = z^{(1-\gamma)/2}\,\Gamma\left(\gamma\right)\ I_{\gamma-1}\left(2\sqrt{z}\right), \tag{3.2}$$

and knowing that, according to (2.9), we have

$$c_0\left(0,0,0,0,\alpha,\beta\right)=\frac{1}{\Gamma\left(1+\alpha\right)\Gamma\left(1+\beta\right)},$$

then

$$\begin{aligned}&\int_0^t \tau^\alpha I_\alpha\left(a\tau\right)\left(t-\tau\right)^\beta I_\beta\left(a\left(t-\tau\right)\right)d\tau \qquad (3.3)\\ =\;&\sqrt{\frac{2}{a}}\frac{\Gamma\left(2\alpha+1\right)\Gamma\left(2\beta+1\right)}{\Gamma\left(2\alpha+2\beta+2\right)}\frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{\Gamma\left(1+\alpha\right)\Gamma\left(1+\beta\right)}t^{\alpha+\beta+1/2}I_{\frac{1}{2}+\alpha+\beta}\left(at\right),\\ &\operatorname{Re}\alpha>-\frac{1}{2},\ \operatorname{Re}\beta>-\frac{1}{2}.\end{aligned}$$

Taking into account now the duplication formula of the gamma function [6, Eqn. 1.2.3]

$$\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right)=\sqrt{\pi}2^{1-2z}\ \Gamma\left(2z\right), \qquad (3.4)$$

we have

$$\frac{\Gamma\left(2\alpha+1\right)}{\Gamma\left(\alpha+1\right)}=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}2^{-2\alpha}}, \qquad (3.5)$$

and

$$\frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{\Gamma\left(2\alpha+2\beta+2\right)}=\frac{\sqrt{\pi}2^{-1-2\alpha-2\beta}}{\Gamma\left(\alpha+\beta+1\right)},$$

so that (3.3) is reduced to (3.1), as we wanted to check.

3.2. **Example 2.** Taking the "+" sign in (1.1), we have

$$\begin{aligned}&\int_0^t \tau^\alpha I_\alpha\left(a\tau\right)\left(t-\tau\right)^\beta I_\beta\left(a\left(t-\tau\right)\right)d\tau \qquad (3.6)\\ =\;&\frac{t^{\alpha+\beta+3/2}}{\sqrt{2\pi a}}\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{3}{2}\right)}{\Gamma\left(\alpha+\beta+2\right)}I_{\alpha+\beta+\frac{1}{2}}\left(at\right),\\ &\operatorname{Re}\alpha>-\frac{1}{2},\ \operatorname{Re}\beta>-\frac{3}{2},\end{aligned}$$

which corresponds to the case $n=0$, $m=1$ (i.e. $\tilde{n}=\tilde{m}=u=0$, $v=1$, $r=\frac{1}{2}$) in (2.1), so that, taking into account (2.6), (2.8) and (2.23), we have

$$\begin{aligned}&\int_0^t \tau^\alpha I_\alpha\left(a\tau\right)\left(t-\tau\right)^{\beta+1} I_\beta\left(a\left(t-\tau\right)\right)d\tau \qquad (3.7)\\ =\;&2!\left(\frac{at}{2}\right)^{\alpha+\beta}\Gamma\left(2\alpha+1\right)\Gamma\left(2\beta+2\right)t^{\alpha+\beta+2}\\ &\times\frac{c_0\left(0,0,0,1,\alpha,\beta\right)}{\Gamma\left(2\alpha+2\beta+3\right)}\,{}_0F_1\left(\left.\begin{array}{c}-\\ \frac{3}{2}+\alpha+\beta\end{array}\right|\frac{a^2t^2}{4}\right),\\ &\operatorname{Re}\alpha>-\frac{1}{2},\ \operatorname{Re}\beta>-\frac{3}{2}.\end{aligned}$$

Applying now (3.2) and knowing that, according to (2.9), we have

$$c_0\left(0,0,0,1,\alpha,\beta\right)=\frac{1}{2!\,\Gamma\left(1+\alpha\right)\Gamma\left(1+\beta\right)},$$

then

$$\begin{aligned}
&\int_0^t \tau^{\alpha} I_{\alpha}(a\tau)(t-\tau)^{\beta} I_{\beta}(a(t-\tau))\,d\tau \\
=\;& \sqrt{\frac{2}{a}}\frac{\Gamma(2\alpha+1)\Gamma(2\beta+2)}{\Gamma(2\alpha+2\beta+3)}\frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{\Gamma(1+\alpha)\Gamma(1+\beta)}\,t^{\alpha+\beta+3/2} I_{\frac{1}{2}+\alpha+\beta}(at), \\
& \operatorname{Re}\alpha > -\frac{1}{2},\ \operatorname{Re}\beta > -\frac{3}{2}.
\end{aligned}$$

Taking into account now the duplication formula of the gamma function (3.4), we have (3.5) and also

$$\frac{\Gamma(2\beta+2)}{\Gamma(\beta+1)} = \frac{\Gamma\left(\beta+\frac{3}{2}\right)}{\sqrt{\pi}2^{-2\beta-1}},$$

and

$$\frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{\Gamma(2\alpha+2\beta+3)} = \frac{\sqrt{\pi}2^{-2-2\alpha-2\beta}}{\Gamma(\alpha+\beta+2)},$$

so that (3.7) is reduced to (3.6) as we wanted to check.

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Appendix A. Product of two finite sums

Let us consider the product of two finite sums

$$P = \left(\sum_{i=0}^{n} a_i\right)\left(\sum_{j=0}^{m} b_j\right) = a_0 b_0 + \cdots + a_n b_m. \tag{A.1}$$

We can view the summands of the above product as the elements of following rectangular matrix

$$\begin{array}{c|cccc}
 & a_0 & a_1 & \cdots & a_n \\
\hline
b_0 & a_0 b_0 & a_1 b_0 & & a_n b_0 \\
b_1 & a_0 b_1 & a_1 b_1 & & a_n b_1 \\
\vdots & & & \ddots & \\
b_m & a_0 b_m & a_1 b_m & & a_n b_m
\end{array} \tag{A.2}$$

We can group these summands by "antidiagonals" as follows

$$P=\underbrace{a_0b_0}_{i+j=0}+\underbrace{a_0b_1+a_1b_0}_{i+j=1}+\cdots+\underbrace{a_nb_m}_{i+j=n+m},$$

thus we can rewrite (A.1) as a double finite sum, taking as first index $k=i+j$

$$P=\sum_{k=0}^{n+m}\sum_{l=l_{\min}}^{l_{\max}}a_lb_{k-l}.$$

Notice that in the matrix (A.2) we have

$$\begin{aligned} l_{\min} &= \begin{cases} 0, & k\leq m \\ k-m, & k>m \end{cases} \\ &= \max\left(k-m,0\right), \end{aligned}$$

and

$$\begin{aligned} l_{\max} &= \begin{cases} k, & k\leq n \\ n, & k>n \end{cases} \\ &= \min\left(k,n\right). \end{aligned}$$

Therefore, we can write finally

$$\left(\sum_{i=0}^{n}a_i\right)\left(\sum_{j=0}^{m}b_j\right)=\sum_{k=0}^{n+m}\sum_{l=\min(k-m,0)}^{\max(k,n)}a_lb_{k-l}. \tag{A.3}$$

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