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ON GENERALIZATION OF RAMANUJAN'S PARTIAL THETA FUNCTION IDENTITIES

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Abstract. In the present paper, using a known three term relation of ${}_3\varphi_2$'s, a partial theta function identity of Ramanujan has been generalized. Some application our new identity has also been shown.

1. Introduction

Ramanujan [4] has given many partial theta function identities of the type

$$\sum_{n=0}^{\infty}\frac{q^n}{(-\alpha q)_n(-\beta q)_n}+\frac{\alpha^{-1}\sum\limits_{n=0}^{\infty}q^{n(n+1)/2}(-\beta/\alpha)^n}{\prod\limits_{j=1}^{\infty}(1+\alpha q^j)(1+\beta q^j)} = (1+\alpha^{-1})\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\beta/\alpha)^n}{(-\beta q)_n}, \quad (1.1)$$

which have been deduced by Andrew's from one of his key formula [2, Eq. (3.1), p. 141]. Later, Agarwal [1] showed that Andrews key formula which houses many identities of the type (1.1) can be placed in a well known basic hypergeometric setting, and is open for further extension. Agarwal, in the same paper, also showed the effectiveness of Sears [5] three terms and four terms relations in the study of partial theta function identities. Agarwal [1], in fact, suggested a systematic study of all such three terms and four terms relations and remarked that because of application of these identities in partition it is expected that such a study would be very fruitful.

The presents work is motivated by Agarwal's remark and supplements his work [1]. In Section 2, we give a generalization of (1.1) in the form of identity (2.2). In Section 3, we give some applications of our new identity (2.2).

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As usual, we shall use the standard notation, a basic hypergeometric series is defined as

$$ {}_{r+1}\varphi_r(a_1, a_2, a_3....a_{r+1}; b_1, b_2, b_3......b_r; q, z) = \sum_{n=0}^{\infty} \frac{(a_1;q^k)_n (a_2;q^k)_n....(a_{r+1};q^k)_n}{(q^k;q^k)_n (b_1;q^k)_n (b_2;q^k)_n....(b_r;q^k)_n} z^n, $$

which converges for $|q| < 1$, $|z| < 1$. In the above definition $(a;q^k)_n$ is $q-$shifted factorial, defined for $|q^k| < 1$ as

$$ (a;q^k)_n = (1-a)(1-aq)..........(1-aq^{k(n-1)}); \quad (a;q^k)_0 = 1. $$

Also

$$ (a;q^k)_\infty = \prod_{j=0}^{\infty}(1-aq^j). $$

To abbreviate our notations, we shall write

$$ (a_1, a_2, ..., a_r; q^k) = (a_1;q^k)_n (a_2;q^k)_n....(a_r;q^k)_n, $$

and when $q = 1$, we shall write $(a;q) = (a)_n$.

2. A Generalization of Ramanujan's Partial Theta Function Identity

Let us consider the following three term relation [3, Ex. 3.6, p. 92]

$$ \begin{aligned} &{}_3\varphi_2(a, b, c; d, f; q; q) \\ &\quad + \frac{(q/f, a, b, c, dq/f)_\infty}{(f/q, aq/f, bq/f, cq/f, d)_\infty} \; {}_3\varphi_2(aq/f, bq/f, cq/f; q^2/f, dq/f; q; q) \\ &= \frac{(q/f, abq/f, acq/f, d/a)_\infty}{(d, aq/f, bq/f, cq/f)_\infty} \; {}_3\varphi_2(a, aq/f, abcq/df; abq/f, acq/f; q; d/a). \end{aligned} \tag{2.1} $$

If we take $a = q$ in (2.1), we get

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n q^n}{(d)_n (f)_n} &+ \frac{(q/f, q, b, c, dq/f)_\infty}{(f/q, q^2/f, bq/f, cq/f, d)_\infty} \; {}_2\varphi_1(bq/f, cq/f; dq/f; q; q) \\ &= \frac{(1-q/f)(1-d/q)}{(1-bq/f)(1-cq/f)} \sum_{n=0}^{\infty} \frac{(q^2/f)_n (bcq^2/df)_n (d/q)^n}{(bq^2/f)_n (cq^2/f)_n}. \end{aligned} $$

Now, transforming the ${}_2\varphi_1(..)$ on the left using the following transformation due to Sears [5]

$$ {}_2\varphi_1(a, b; c; q; x) = \frac{(b, ax)_\infty}{(x, c)_\infty} \; {}_2\varphi_1(b/c, x; ax; q; b), $$

we obtain

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n q^n}{(d)_n (f)_n} &+ \frac{(1-q/f)}{(1-bq/f)} \frac{(b, c)_\infty}{(f/q, d)_\infty} \sum_{n=0}^{\infty} \frac{(d/c)_n (cq/f)^n}{(bq^2/f)_n} \\ &= \frac{(1-q/f)(1-d/q)}{(1-bq/f)(1-cq/f)} \sum_{n=0}^{\infty} \frac{(q^2/f)_n (bcq^2/df)_n (d/q)^n}{(bq^2/f)_n (cq^2/f)_n}. \end{aligned} \tag{2.2} $$

Also, transforming the series on right in (2.2) using a known transformation [5, Eq. (10.1), p.174]

$$\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(b_1)_n(b_2)_n}(b_1b_2/a_1a_2q)^n=\frac{(b_2/q,b_1b_2/a_1a_2)_\infty}{(b_2,b_1b_2/a_1a_2q)_\infty}\sum_{n=0}^{\infty}\frac{(b_1/a_1)_n(b_1/a_2)_n}{(b_1)_n(b_1b_2/a_1a_2)_n}(b_2/q)^n,$$

we get

$$\sum_{n=0}^{\infty}\frac{(b)_n(c)_nq^n}{(d)_n(f)_n}+\frac{(1-q/f)(b,c)_\infty}{(1-bq/f)(f/q,d)_\infty}\sum_{n=0}^{\infty}\frac{(d/c)_n(cq/f)^n}{(bq^2/f)_n}$$
$$=\frac{(1-q/f)}{(1-bq/f)}\sum_{n=0}^{\infty}\frac{(b)_n(d/c)_n(cq/f)^n}{(bq^2/f)_n(d)_n}. \quad (2.3)$$

In (2.3) taking $b\to 0$ and $c\to 0$, and then $d=-\beta q$ and $f=-\alpha q$, we get

$$\sum_{n=0}^{\infty}\frac{q^n}{(-\alpha q)_n(-\beta q)_n}+\frac{\alpha^{-1}\sum_{n=0}^{\infty}q^{n(n+1)/2}(-\beta/\alpha)^n}{\prod_{j=1}^{\infty}(1+\alpha q^j)(1+\beta q^j)}=(1+\alpha^{-1})\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\beta/\alpha)^n}{(-\beta q)_n},$$

which is precisely Ramanujan's partial theta function identity appearing in his 'Lost' Notebook (see[2, Eq. (3.6), p. 144]).

3. SOME APPLICATIONS OF (2.2)-(2.3)

Replacing d and f respectively by Aq and q/A in (2.2), we get

$$\sum_{n=0}^{\infty}\frac{(b)_n(c)_nq^n}{(-Aq)_n(-q/A)_n}+\frac{A(1+Ab)^{-1}(b,c)_\infty}{(-q/A,-Aq)_\infty}\sum_{n=0}^{\infty}\frac{(-Aq/c)_n(-Ac)^n}{(-Abq)_n}$$
$$=\frac{(1+A)^2}{(1+Ab)(1+Ac)}\sum_{n=0}^{\infty}\frac{(-Aq)_n(bc)_n(-A)^n}{(-Abq)_n(-Acq)_n}. \quad (3.1)$$

For $b\to 0$ and $c\to 0$ in (3.1), we get

$$\sum_{n=0}^{\infty}\frac{q^n}{(-Aq)_n(-q/A)_n}+\frac{A}{(-q/A,-Aq)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)/2}A^{2n}$$
$$=(1+A)^2\sum_{n=0}^{\infty}(-Aq)_n(-A)^n. \quad (3.2)$$

In (3.1), if b and c are respectively replaced by B and $-B$, we get

$$\sum_{n=0}^{\infty}\frac{(B^2;q^2)_nq^n}{(-Aq)_n(-q/A)_n}+\frac{A(1+AB)^{-1}(B^2;q^2)_\infty}{(-q/A,-Aq)_\infty}\sum_{n=0}^{\infty}\frac{(Aq/B)_n(AB)^n}{(-ABq)_n}$$
$$=\frac{(1+A)^2}{(1-A^2B^2)}\sum_{n=0}^{\infty}\frac{(-Aq)_n(-B^2)_n(-A)^n}{(-ABq)_n(ABq)_n}. \quad (3.3)$$

In (3.3), if we take $B\to 0$, we get (3.2).

Lastly, taking $b = ABq$, $c = -ABq$, $d = -Aq$, $f = -Bq$ in (2.1), we get after simplification

$$\sum_{n=0}^{\infty} \frac{(ABq;q^2)_n q^n}{(-Aq)_n(-Bq)_n} + \frac{(1+1/B)(ABq,-ABq)_\infty}{(1+Aq)(-B,-Aq)_\infty} \sum_{n=0}^{\infty} \frac{(1/B)_n (Aq)^n}{(-Aq^2)_n}$$
$$= \frac{(1+1/B)(1+A)}{(1+Aq)(1-Aq)} \sum_{n=0}^{\infty} \frac{(-q/B)_n(-ABq^2)_n(-A)^n}{(Aq)_n(-Aq^2)_n}. \quad (3.4)$$

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