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THE TWO-POINT CONNECTION PROBLEM FOR A SUB-CLASS OF THE HEUN EQUATION

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Abstract. The present article discusses the two-point connection problem for Heun's differential equation. We employ a contour integral method to derive connection matrices for a sub-class of the Heun equation containing three free parameters. Explicit expressions for the connection coefficients are found.

1. Introduction

In [1], [2], and [3], R. Schäfke and D. Schmidt studied two point connection problems between pairs of solutions around neighboring singularities using a contour integral approach based on the Cauchy Integral Formula. In [3], they studied in particular the connection problem between pairs of solutions around regular singularities. They obtained expressions for the connection coefficients as a limit of a sequence involving the coefficients in the power series expansion of the Frobenius solution around zero with vanishing exponent. One limitation of this approach is that it assumes that these coefficients are known. For the hypergeometric equation [4], this is not a problem as the coefficients satisfy a two-term recurrence relation which is easy to solve. However this is not true for the Heun equation. The required coefficients are solutions of a three-term recurrence relation for which there is no known explicit solution in the general case. In this paper, we modify the methods used in [3] to fully solve the connection problem for a subclass of the Heun Equation for which this recurrence relation can be solved explicitly. We give explicit expressions for the connection coefficients.

The Heun equation is an increasingly important equation which appears more and more frequently in the literature. Much of the work surrounding the Heun function involves finding integral representations. Several integral representations for the hypergeometric function are known. These provide a successful strategy for solving the two-point connection problem for the hypergeometric equation (see [4] for details). In this paper we solve the two-point connection problem for a subclass of the Heun equation without using any integral representations, thus illustrating the power of the approach employed by Schäfke and Schmidt.

We consider the two-point connection problem for the Huen equation [5] given

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$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0, \tag{1.1}$$

with $a, q \in \mathbb{C}$ and where $\epsilon = \alpha + \beta + 1 - \gamma - \delta$ and $a \neq 0, 1$. It is well known that equation (1.1) has regular singularities at 0, 1, a, and ∞. Furthermore, equation (1.1) has a Frobenius solution which is regular for $|z| < \min\{1, |a|\}$ and is denoted $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$, the local Heun function. Note that Hl is normalized so that $Hl(a, q; \alpha, \beta, \gamma, \delta; 0) = 1$. The coefficients in the expansion

$$Hl(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{k=0}^{\infty} A_k z^k, \quad |z| < \min\{1, |a|\}$$

satisfy the well known [5] recurrence relation

$$\begin{aligned} 0 &= aA_1\gamma - qA_0, \\ 0 &= aP_kA_{k+1} - (Q_k + q)A_k + R_kA_{k-1}, \qquad k \geq 1, \end{aligned} \tag{1.2}$$

where $P_k = (k+1)(k+\gamma)$, $Q_k = k(k-1+\gamma)(1+a)+k(a\delta+\epsilon)$, $R_k = (k-1+\alpha)(k-1+\beta)$, and $A_0 = 1$. Maier [6] described the fundamental pairs of local Frobenius solutions to equation (1.1) and gave various relations satisfied by the local Heun function. We donote these pairs of solutions by $\{y_{01}, y_{02}\}$, $\{y_{11}, y_{12}\}$, $\{y_{a1}, y_{a2}\}$, and $\{y_{\infty 1}, y_{\infty 2}\}$ where

$$\begin{aligned}
y_{01}(z) &= Hl(a, q; \alpha, \beta, \gamma, \delta; z), \\
y_{02}(z) &= z^{1-\gamma} Hl(a, q + (1-\gamma)(\alpha + \beta + 1 - \gamma + (a-1)\delta); \\
&\quad 1 + \alpha - \gamma, 1 + \beta - \gamma, 2 - \gamma, \delta; z), \\
y_{11}(z) &= Hl(1 - a, \alpha\beta - q; \alpha, \beta, \delta, \gamma; 1 - z), \\
y_{12}(z) &= (1-z)^{1-\delta} Hl(1 - a, \alpha\beta - q + (1-\delta)(\epsilon + (1-a)\gamma); 1 + \alpha - \delta; \\
&\quad 1 + \beta - \delta, 2 - \delta, \gamma; 1 - z), \\
y_{a1}(z) &= Hl\left(\frac{a}{a-1}, \frac{a\alpha\beta - q}{a-1}; \alpha, \beta, \alpha + \beta + 1 - \gamma - \delta, \delta; \frac{z-a}{1-a}\right), \\
y_{a2}(z) &= (z-a)^{1-\epsilon} Hl\left(\frac{a}{a-1}, \frac{-q + a(\gamma + \delta - \alpha)(\gamma + \delta - \beta) + \gamma(\epsilon - 1)}{a-1};\right. \\
&\quad \left.\gamma + \delta - \beta, \gamma + \delta - \alpha, 2 - \epsilon, \delta; \frac{z-a}{1-a}\right), \\
y_{\infty 1}(z) &= z^{-\alpha} Hl\left(\frac{1}{a}, \frac{f(\alpha, \beta, \gamma, \delta)}{a}; \alpha, \alpha + 1 - \gamma, 1 + \alpha - \beta, \delta; \frac{1}{z}\right), \\
y_{\infty 2}(z) &= z^{-\beta} Hl\left(\frac{1}{a}, \frac{f(\beta, \alpha, \gamma, \delta)}{a}; \beta, \beta + 1 - \gamma, 1 + \beta - \alpha, \delta; \frac{1}{z}\right).
\end{aligned}$$

with $f(\alpha, \beta, \gamma, \delta) = q + \alpha[a(\alpha + 1 - \gamma - \delta) + \delta - \beta]$. In order for these pairs to be linearly independent and well-defined, we require that $\gamma,\ \delta,\ \epsilon,\ \alpha - \beta \notin \mathbb{Z}$.

2. Preliminaries

In this paper we consider the subclass of equation (1.1) where $\delta = (\alpha+\beta+1-\gamma)/2$, $q = 0$, and $a = -1$. That is, we consider the Fuchsian equation

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\delta}{z+1}\right)\frac{dy}{dz} + \frac{\alpha\beta}{(z-1)(z+1)}y = 0. \tag{2.1}$$

Remark. *Note that for this subclass of the Heun Equation, the methods employed in* [3] *cannot be immediately applied since they assumed that the differential equation has no other singularity in the closed disk* $\{z \in \mathbb{C} : |z| \leq 1\}$ *besides zero and one, whereas (2.1) has singularities at minus one, zero, and one.*

It is not difficult to see that (1.2) becomes

$$A_{k+1} = \frac{(k-1+\alpha)(k-1+\beta)}{(k+1)(k+\gamma)} A_{k-1}, \quad k \geq 1.$$

Whence we obtain

$$A_{2n} = \frac{(\frac{\alpha}{2})_n(\frac{\beta}{2})_n}{n!(\frac{\gamma+1}{2})_n}, \quad n \geq 0, \tag{2.2}$$

and $A_{2n+1} = 0,\ \forall n \geq 0$. Hence,

$$Hl(-1,0;\alpha,\beta,\gamma,\delta;z) = \sum_{n=0}^{\infty} \frac{(\frac{\alpha}{2})_n(\frac{\beta}{2})_n}{n!(\frac{\gamma+1}{2})_n} z^{2n}. \tag{2.3}$$

Equation (2.1) has fundamental pairs of solutions given by

$$\begin{aligned}
y_{01}(z) &= Hl(-1,0;\alpha,\beta,\gamma,\delta;z),\\
y_{02}(z) &= z^{1-\gamma}Hl(-1,0;1+\alpha-\gamma,1+\beta-\gamma,2-\gamma,\delta;z),\\
y_{+1}(z) &= Hl(2,\alpha\beta;\alpha,\beta,\delta,\gamma;1-z),\\
y_{+2}(z) &= (1-z)^{1-\delta}Hl(2,\alpha\beta+(1-\delta)(\delta+2\gamma);\\
&\quad 1+\alpha-\delta;1+\beta-\delta,2-\delta,\gamma;1-z),\\
y_{-1}(z) &= Hl(1/2,\alpha\beta/2;\alpha,\beta,\alpha+\beta+1-\gamma-\delta,\delta;(z+1)/2),\\
y_{-2}(z) &= (z+1)^{1-\delta}Hl(1/2,[(\gamma+\delta-\alpha)(2\gamma+\delta-\beta)-\gamma\beta]/2;\\
&\quad \gamma+\delta-\beta,\gamma+\delta-\alpha,2-\delta,\delta;(z+1)/2),\\
y_{\infty 1}(z) &= z^{-\alpha}Hl\left(-1,0;\alpha,\alpha+1-\gamma,1+\alpha-\beta,\delta;z^{-1}\right),\\
y_{\infty 2}(z) &= z^{-\beta}Hl\left(-1,0;\beta,\beta+1-\gamma,1+\beta-\alpha,\delta;z^{-1}\right).
\end{aligned}$$

Remark. *Note that any solution of (1.1) may be analytically continued along any path in* $\mathbb{C} - \{0,1,a\}$ *and the analytic continuation is a solution (see for example Theorem 3.7.2 in* [7] *or 10.1 in* [8]*). If two paths are homotopic then the continuation is unique by the Monodromy Theorem (see* [9] *or* [10]*). Thus, if the domain* $D \subset \mathbb{C} - \{0,1,a\}$ *is simply connected and has non-empty intersection with the open disc* $\{z \in \mathbb{C} : |z| < 1\}$*, then in particular it is not difficult to see that* y_{01} *has a unique analytic extension to* D*.*

We will also find the following results regarding the Euler-Beta function and the ratio of Gamma functions helpful in proving our main result.

Lemma 2.1. *If* $\Re\alpha,\ \Re\beta > 0$*, then*

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\ dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Proof. This is a standard result about the Euler-Beta function. For a proof, see, for example, Section 1.5 in [4]. □

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{C}$. Then for $|\arg z| < \pi/2$ the following asymptotic approximation is valid*

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim z^{\alpha-\beta}\left[1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+\mathcal{O}(|z|^{-2})\right], \quad z\to\infty. \tag{2.4}$$

Proof. See [11] or Section 5.1, Chapter 4 of [12]. □

In the following lemma, we give asymptotic expressions for some integrals which will appear in the proof of our main result.

Lemma 2.3. *Let $1<\rho<2$ and $\alpha \in \mathbb{C}$. If $k > \Re\alpha$ and $\Re\alpha > 0$, then*

$$\int_1^\rho (z-1)^\alpha z^{-k-1}\mathrm{d}z = \frac{\Gamma(k-\alpha)\Gamma(\alpha+1)}{\Gamma(k+1)} + \mathcal{O}(\rho^{-k}), \quad k\to\infty.$$

Let F be a holomorphic complex-valued function defined in a disc of radius one centered at one. Then,

$$\int_1^\rho (z-1)^\alpha z^{-k-1}F(z)\mathrm{d}z = \mathcal{O}(k^{-\Re\alpha-1}), \quad k\to\infty.$$

where the powers in the above integral take their principal values.

Proof. Observe that

$$\int_1^\rho (z-1)^\alpha z^{-k-1}\mathrm{d}z = \int_1^\infty (z-1)^\alpha z^{-k-1}\mathrm{d}z - \int_\rho^\infty (z-1)^\alpha z^{-k-1}\mathrm{d}z.$$

Since $k>\Re\alpha$ and $\Re\alpha>0$, using the transformation $z=1/t$ we obtain

$$\left|\int_\rho^\infty (z-1)^\alpha z^{-k-1}\mathrm{d}z\right| = \left|\int_0^{\frac{1}{\rho}} t^{k-1-\alpha}(1-t)^\alpha \mathrm{d}t\right| \le \int_0^{\frac{1}{\rho}} x^{k-1-\Re\alpha}\mathrm{d}x = \frac{\rho^{-k+\Re\alpha}}{k-\Re\alpha}.$$

Hence,

$$\int_\rho^\infty (z-1)^\alpha z^{-k-1}\mathrm{d}z = \mathcal{O}(\rho^{-k}), \quad k\to\infty.$$

Also, using Lemma 2.1

$$\int_1^\infty (z-1)^\alpha z^{-k-1}\mathrm{d}z \overset{z=1/t}{=} \int_0^1 (1-t)^\alpha t^{k-\alpha-1}\mathrm{d}t = \frac{\Gamma(k-\alpha)\Gamma(\alpha+1)}{\Gamma(k+1)}.$$

We prove now the second part of the lemma. Notice that since F is holomorphic we may find an $M\in\mathbb{R}^+$ such that

$$\left|\int_1^\rho (z-1)^\alpha z^{-k-1}F(z)\mathrm{d}z\right| \le M\int_1^\rho x^{-k-1}(x-1)^{\Re\alpha}\mathrm{d}x \le M\int_1^\infty x^{-k-1}(x-1)^{\Re\alpha}\mathrm{d}x,$$
$$= M\int_0^1 t^{k-1-\Re\alpha}(1-t)^{\Re\alpha}\mathrm{d}t = M\frac{\Gamma(k-\Re\alpha)\Gamma(\Re\alpha+1)}{\Gamma(k+1)}.$$

where we employed the transformation $x=1/t$ and Lemma 2.1. Using Lemma 2.4, we get

$$\int_1^\rho (z-1)^\alpha z^{-k-1}F(z)\mathrm{d}z = \mathcal{O}(k^{-\Re\alpha-1}), \quad k\to\infty.$$

This concludes the proof. □

3. Solution of The Two-Point Connection Problem

We consider simultaneously the two-point connection problem between 0 and 1 and between 0 and -1. That is we seek coefficients $c_{11}^+,\ c_{12}^+,\ c_{11}^-,\ c_{12}^-$ such that

$$y_{01} = c_{11}^+ y_{+1} + c_{12}^+ y_{+2} \tag{3.1}$$

$$y_{01} = c_{11}^- y_{-1} + c_{12}^- y_{-2} \tag{3.2}$$

Note however, that c_{11}^+ and c_{11}^- may be easily found. We recall the well-known result [4]

$$\lim_{z\to 1+} F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

Using this relation and taking the limit of (3.1) as $z \to 1$ and assuming $\Re(1-\delta) > 0$ we obtain

$$c_{11}^+ = \sum_{n=0}^{\infty} \frac{(\frac{\alpha}{2})_n(\frac{\beta}{2})_n}{n!(\frac{\gamma+1}{2})_n} = \lim_{z\to 1+} F\left(\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma+1}{2};z\right) = \frac{\Gamma(\frac{\gamma+1}{2}-\frac{\alpha}{2}-\frac{\beta}{2})\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\gamma+1}{2}-\frac{\alpha}{2})\Gamma(\frac{\gamma+1}{2}-\frac{\beta}{2})},$$

and similarly taking the limit as $z \to -1$ of (3.2) we obtain

$$c_{11}^- = \frac{\Gamma(\frac{\gamma+1}{2}-\frac{\alpha}{2}-\frac{\beta}{2})\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\gamma+1}{2}-\frac{\alpha}{2})\Gamma(\frac{\gamma+1}{2}-\frac{\beta}{2})} = c_{11}^+.$$

We compute c_{12}^+ and c_{12}^- with the following theorem.

Theorem 3.1. *Let $y_{01},\ y_{+1},\ y_{+2},\ y_{-1},\ y_{-2}$ be the fundamental pairs of solutions associated to equation (2.1) and $c_{11}^+,\ c_{12}^+,\ c_{11}^-,\ c_{12}^-$ be as in (3.1) and (3.2). Furthermore, let $\Re(1-\delta) > 0$. Then, we have*

$$c_{12}^+ = 2^{1-\delta}\frac{\Gamma(\delta-1)\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})} = c_{12}^-,$$

Proof. For simplification, first observe that y_{01} is of the form $y_{01}(z) = \sum_{k=0}^{\infty} d_k z^k$. Let $C_{\sigma,\phi} = C_{\sigma,\phi}^0 + L_{\sigma,\phi}^1 - C_{\sigma,\phi}^+ - L_{\sigma,\phi}^2 + L_{\sigma,\phi}^4 - C_{\sigma,\phi}^- - L_{\sigma,\phi}^3$ be the contour shown in Fig. 1 where $C_{\sigma,\phi}^0 = H_{\sigma,\phi}^0 + H_{\sigma,\phi}^1$ and $1 < \rho < 2$. By the Cauchy integral formula we have for any $k \in \mathbb{N}_0$, and $\sigma,\ \phi > 0$ sufficiently small

$$d_k = \frac{1}{2\pi i}\int_{C_{\sigma,\phi}} z^{-k-1}\hat{y}_{01}(z)dz \tag{3.3}$$

where $\hat{y}_{01}$ is the unique analytic extension of y_{01} to the simply connected set $\mathbb{C} - \{(-\infty,-1] \cup [1,+\infty)\}$ guaranteed to exist by Remark 2. In particular, we have $\hat{y}_{01}(z) = c_{11}^+ y_{+1}(z) + c_{12}^+ y_{+2}(z)$ for $z \in L_{\sigma,\phi}^1 \cup C_{\sigma,\phi}^+ \cup L_{\sigma,\phi}^2$ and $\hat{y}_{01}(z) = c_{11}^- y_{-1}(z) + c_{12}^- y_{-2}(z)$ for $z \in L_{\sigma,\phi}^3 \cup C_{\sigma,\phi}^- \cup L_{\sigma,\phi}^4$ where we take the principle value of the powers occurring in y_{+2} and y_{-2}. Notice that the l.h.s. of (3.3) does not depend on σ or ϕ. Hence, we consider the limit of the expression on the right hand side as $\sigma,\ \phi \to 0$. This limit if it exists should be equal to d_k. So

$$2\pi i d_k = \lim_{\sigma\to 0}\lim_{\phi\to 0}\int_{C_{\sigma,\phi}} z^{-k-1}\hat{y}_{01}(z)dz = I_k^0 + I_k^1 + I_k^2$$

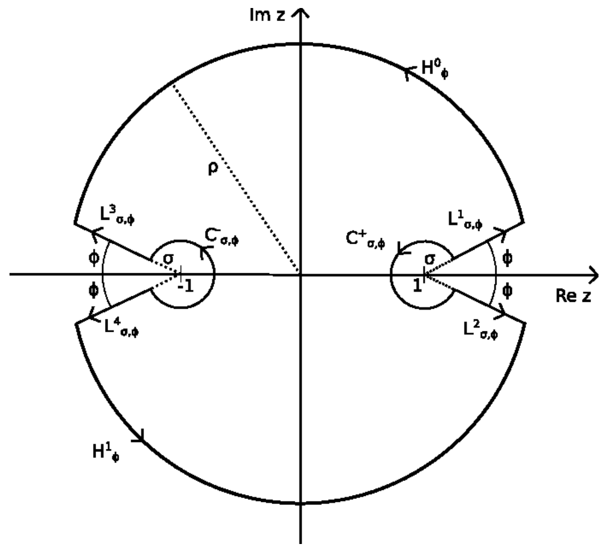

FIGURE 1. Integration contour showing components of $C_{\sigma,\phi}$

where $I_k^0 = I_{k,1}^0 - I_{k,2}^0 - I_{k,3}^0$ and

$$\begin{aligned} I_k^1 &= \lim_{\sigma\to 0}\lim_{\phi\to 0}\left(\int_{L^1_{\sigma,\phi}} z^{-k-1}(c^+_{11}y_{+1}+c^+_{12}y_{+2})(z)dz - \right.\\ &\qquad \left.\int_{L^2_{\sigma,\phi}} z^{-k-1}(c^+_{11}y_{+1}+c^+_{12}y_{+2})(z)dz\right),\\ I_k^2 &= \lim_{\sigma\to 0}\lim_{\phi\to 0}\left(\int_{L^4_{\sigma,\phi}} z^{-k-1}(c^-_{11}y_{-1}+c^-_{12}y_{-2})(z)dz - \right.\\ &\qquad \left.\int_{L^3_{\sigma,\phi}} z^{-k-1}(c^-_{11}y_{-1}+c^-_{12}y_{-2})(z)dz\right),\\ I_{k,1}^0 &= \lim_{\sigma\to 0}\lim_{\phi\to 0}\int_{C^0_{\sigma,\phi}} z^{-k-1}\hat{y}_{01}(z)dz,\\ I_{k,2}^0 &= \lim_{\sigma\to 0}\lim_{\phi\to 0}\int_{C^+_{\sigma,\phi}} z^{-k-1}(c^+_{11}y_{+1}+c^+_{12}y_{+2})(z)dz,\\ I_{k,3}^0 &= \lim_{\sigma\to 0}\lim_{\phi\to 0}\int_{C^-_{\sigma,\phi}} z^{-k-1}(c^-_{11}y_{-1}+c^-_{12}y_{-2})(z)dz). \end{aligned}$$

First we deal with I_k^0. In particular, we will show that $I_k^0 = \mathcal{O}(\rho^{-k})$. Note that we have the parameterizations $C^+_{\sigma,\phi}$ with $z(\theta) = 1+\sigma e^{i\theta}$ and $\phi \le \theta \le 2\pi - \phi$, and $C^-_{\sigma,\phi}$ with $z(\theta) = -1+\sigma e^{i\theta}$ and $\phi - \pi \le \theta \le \pi - \phi$. Since y_{+1} is holomorphic in $\{z : |z-1| < 1\}$ and y_{-1} is holomorphic in $\{z : |z+1| < 1\}$ we obtain

$$I_{k,2}^0 = c^+_{12}\lim_{\sigma\to 0}\lim_{\phi\to 0}\int_{C^+_{\sigma,\phi}} z^{-k-1}y_{+2}(z)dz, \quad I_{k,3}^0 = c^-_{12}\lim_{\sigma\to 0}\lim_{\phi\to 0}\int_{C^-_{\sigma,\phi}} z^{-k-1}y_{-2}(z)dz.$$

Expressing y_{+2} and y_{-2} as $y_{+2}(z) = (1-z)^{1-\delta} f_1(z)$ and $y_{-2}(z) = (z+1)^{1-\delta} f_2(z)$ where $f_1,\ f_2$ are holomorphic functions in the discs of radius 1 centered at 1, and -1, respectively, yields $I^0_{k,2} = c^+_{12} \lim_{\sigma\to 0} \lim_{\phi\to 0} \int_{C^+_{\sigma,\phi}} z^{-k-1}(1-z)^{1-\delta} f_1(z)dz$. Since f_1 and f_2 are holomorphic there exists a positive number M such that $|f_1(z)|, |f_2(z)| < M$ for all $z \in \{z \in \mathbb{C} : |z-1| \leq \rho - 1 < 1\}$. Hence, with the help of the parameterization introduced above we have that $\left|\int_{C^+_{\sigma,\phi}} z^{-k-1}(1-z)^{1-\delta} f_1(z)dz\right| \leq \int_\phi^{2\pi-\phi} M|1+\sigma e^{i\theta}|^{-k-1}\sigma^{2-\Re\delta} d\theta \leq M2^{k+1}\int_\phi^{2\pi-\phi} \sigma^{2-\Re\delta} d\theta = M2^{k+2}(\pi-\phi)\sigma^{2-\Re\delta}$. The second inequality follows from the fact that for σ sufficiently small $|1+\sigma e^{i\theta}| > 1/2$. Since $\Re(1-\delta) > 0$, this implies that $I^0_{k,2} = 0$. It may similarly be shown that $I^0_{k,3} = 0$. Note that y_{01} may be extended analytically to simply connected domains D_1 and D_2 containing H^0_0 and H^1_0, respectively. By the continuity of these extensions, their absolute values have a common upper bound $M \in \mathbb{R}^+$ on H^0_0 and H^1_0. Since these extensions also extend $\hat{y}_{01}$ and since $H^0_\phi \subset H^0_0$ and $H^1_\phi \subset H^1_0$, this bound also holds for $z \in H^0_\phi \cup H^1_\phi = C^0_{\sigma,\phi}$. Thus by the M-L Formula (see for example (9) page 83 in [9]) we have $\left|\int_{C^0_{\sigma,\phi}} z^{-k-1}\hat{y}_{01}(z)dz\right| \leq 2\pi M\rho^{-k}$. Hence, $I^0_k = I^0_{k,1} = \mathcal{O}(\rho^{-k})$. Introducing the parameterizations $L^1_{\sigma,\phi} : z(r) = 1 + re^{i\phi}$, $L^2_{\sigma,\phi} : z(r) = 1+re^{-i\phi}$, $L^3_{\sigma,\phi} : z(r) = -1-re^{-i\phi}$, and $L^4_{\sigma,\phi} : z(r) = -1-re^{i\phi}$ where $\sigma \leq r \leq \sqrt{\cos^2\phi + \rho^2 - 1} - \cos\phi$. Using the above parametrizations, and taking limits we see that

$$I^1_k = c^+_{12}\int_0^{\rho-1} (1+r)^{-k-1}(e^{-2\pi i(1-\delta)} - 1)y_{+2}(1+r)dr = c^+_{12}\left[1 - e^{-2\pi i(1-\delta)}\right] J^1_k$$

where we used the substitution $x = 1 + r$ and similarly but with the substitution $x = -(1+r)$ we obtain

$$I^2_k = c^-_{12}\int_0^{\rho-1} (-1-r)^{-k-1}(1-e^{-2\pi i(1-\delta)})y_{-2}(-1-r)dr = c^-_{12}\left[1 - e^{-2\pi i(1-\delta)}\right] J^2_k$$

with

$$J^1_k = \int_\rho^1 x^{-k-1}y_{+2}(x)dx, \quad J^2_k = \int_{-\rho}^{-1} x^{-k-1}y_{-2}(x)dx.$$

Now, we rewrite y_{+2} and y_{-2} as follows

$$y_{+2}(z) = \sum_{j=0}^m G_j(1-z)^{\tau_j} + (1-z)^{\xi_m}F_1(z), \quad y_{-2}(z) = \sum_{j=0}^m H_j(1+z)^{\tau_j} + (1+z)^{\xi_m}F_2(z) \tag{3.4}$$

where $F_1,\ F_2$ are analytic functions in the discs of radii 1 centered at 1, and-1, respectively, $G_0 = 1 = H_0$, $\tau_j = 1-\delta+j$ for all $j = 0, \cdots, m$ and $\xi_m = 2+m-\delta$. Using the representation for y_{+2} and y_{-2} found in (3.4), we obtain

$$\begin{aligned} J^1_k &= \sum_{j=0}^m G_j \int_\rho^1 x^{-k-1}(1-x)^{\tau_j}dx + \int_\rho^1 x^{-k-1}(1-x)^{\xi_m}F_1(x)dx, \\ J^2_k &= \sum_{j=0}^m H_j \int_{-\rho}^{-1} x^{-k-1}(1+x)^{\tau_j}dx + \int_{-\rho}^{-1} x^{-k-1}(1+x)^{\xi_m}F_2(x)dx. \end{aligned}$$

Hence, if $\Re(1-\delta)>0$, we may apply Lemma 2.3 to obtain

$$J_k^1=-\sum_{j=0}^m G_j e^{\pi i\tau_j}\frac{\Gamma(k-\tau_j)\Gamma(1+\tau_j)}{\Gamma(k+1)}+\mathcal{O}(\rho^{-k})+\mathcal{O}(k^{-\Re(3+m-\delta)})$$

and similarly

$$\begin{aligned}J_k^2 &= \sum_{j=0}^m H_j(-1)^{-k-1}\int_1^\rho v^{-k-1}(1-v)^{\tau_j}\,dv+\\ &\quad +(-1)^{-k-1}\int_1^\rho v^{-k-1}(1-v)^{\xi_m}F_2(-v)dv,\\ &= \sum_{j=0}^m H_j(-1)^{-k-1}e^{\pi i\tau_j}\frac{\Gamma(k-\tau_j)\Gamma(1+\tau_j)}{\Gamma(k+1)}+\mathcal{O}(\rho^{-k})+\mathcal{O}(k^{-\Re(3+m-\delta)}),\end{aligned}$$

where we used the substitution $z=-v$. Hence, when we multiply by the ratio $\Gamma(k+1)/\Gamma(k-1+\delta)$ and use (2.4) we end up with

$$\begin{aligned}\frac{\Gamma(k+1)}{\Gamma(k-1+\delta)}d_k &= \frac{\Gamma(k+1)(I_k^1+I_k^2)}{2\pi i\Gamma(k-1+\delta)}+\mathcal{O}(\rho^{-k}),\\ &= \frac{\Gamma(k+1)[1-e^{-2\pi i(1-\delta)}]}{2\pi i\Gamma(k-1+\delta)}\left[c_{12}^+J_k^1+c_{12}^-J_k^2\right]+\mathcal{O}(\rho^{-k})+\mathcal{O}(k^{-m-1})\\ &= \frac{\sin[\pi(\delta-1)]}{\pi}\Bigg[c_{12}^+\sum_{j=0}^m\frac{G_je^{j\pi i}\Gamma(1+\tau_j)}{\prod_{\sigma=1}^j(k-1+\delta-\sigma)}+\\ &\quad (-1)^{-k}c_{12}^-\sum_{j=0}^m\frac{H_je^{j\pi i}\Gamma(1+\tau_j)}{\prod_{\sigma=1}^j(k-1+\delta-\sigma)}\Bigg]+\mathcal{O}(\rho^{-k})+\mathcal{O}(k^{-m-1})\end{aligned}$$

If k is even, we have when we take the limit $n\to\infty$ and using the gamma reflection formula (see (6) at page 3 in [4])

$$\lim_{n\to\infty}\frac{\Gamma(2n+1)}{\Gamma(2n-1+\delta)}d_{2n}=\frac{(c_{12}^++c_{12}^-)}{\Gamma(\delta-1)}.$$

Moreover, taking k odd we obtain

$$\lim_{n\to\infty}\frac{\Gamma(2n+2)}{\Gamma(2n+\delta)}d_{2n+1}=\frac{(c_{12}^+-c_{12}^-)}{\Gamma(\delta-1)}$$

Employing (2.2) yields

$$c_{12}^++c_{12}^-=\lim_{n\to\infty}\frac{\Gamma(2n+1)\Gamma(\delta-1)}{\Gamma(2n-1+\delta)}\frac{(\frac{\alpha}{2})_n(\frac{\beta}{2})_n}{n!(\frac{\gamma+1}{2})_n},\quad c_{12}^+-c_{12}^-=0$$

and it follows that $c_{12}^+=c_{12}^-$. Furthermore, since $(\alpha)_n=\Gamma(\alpha+n)/\Gamma(\alpha)$ we get

$$\begin{aligned}c_{12}^++c_{12}^- &= \frac{\Gamma(\delta-1)\Gamma(\gamma/2+1/2)}{\Gamma(\alpha/2)\Gamma(\beta/2)}\lim_{n\to\infty}\frac{\Gamma(2n+1)\Gamma(\alpha/2+n)\Gamma(\beta/2+n)}{\Gamma(2n-1+\delta)\Gamma(n+1)\Gamma(\gamma/2+1/2+n)},\\ &= 2^{2-\delta}\frac{\Gamma(\delta-1)\Gamma(\gamma/2+1/2)}{\Gamma(\alpha/2)\Gamma(\beta/2)},\end{aligned}$$

where we used (2.4). This completes the proof. □

4. The Connection Matrices

Having computed the coefficients c_{11}^+, c_{12}^+, c_{11}^-, c_{12}^- in (3.1) and (3.2) we will now show that this is enough to compute all connection matrices for equation (2.1). Maier [6] proved the validity of the relations $y_{11}(z) = z^{1-\gamma} Hl(1-a, -q + \alpha\beta + (\gamma-1)(1-a)\delta; 1+\alpha-\gamma, 1+\beta-\gamma, \delta, 2-\gamma; 1-z)$,

$$y_{a1}(z) = \left(\frac{z}{a}\right)^{1-\gamma} Hl\left(\frac{a}{a-1}, Q; 1+\alpha-\gamma, 1+\beta-\gamma, \alpha+\beta+1-\gamma-\delta, \delta; \frac{z-a}{1-a}\right),$$

where

$$Q = \frac{a(1+\alpha-\gamma)(1+\beta-\gamma) - q - (1-\gamma)(\alpha+\beta+1-\gamma+(a-1)\delta)}{a-1},$$

and

$$y_{11}(z) = z^{-\alpha} Hl\left(1 - \frac{1}{a}, \frac{-q + \alpha[(a-1)\delta+\beta]}{a}; \alpha, \alpha+1-\gamma, \delta, 1+\alpha-\beta; 1-z^{-1}\right).$$

Furthermore, in the special case $a = -1$, we can assert that

$$y_{a1}(z) = (-z)^{-\alpha} Hl\left(\frac{1}{1-a}, \frac{g(\alpha,\beta,\gamma,\delta)}{1-a}; \alpha, \alpha+1-\gamma, \epsilon, \delta; \frac{z^{-1}-a}{1-a}\right).$$

with $g(\alpha,\beta,\gamma,\delta) = -q + \alpha[(\alpha+1-\gamma-\delta)(1-a)+\beta]$. Indeed, the transformation $z = 1/t$, $w(z) = f(t)$, and $f(t) = t^{\alpha}\phi(t)$ is a symmetry of equation (1.1). Applying the transformation, (1.1) is transformed into a Heun equation with singularities at $\{0, 1, 1/a, \infty\}$ and the r.h.s. above is a solution to (1.1) in a neighborhood of $1/a$. In particular if $a = -1$ then we have another solution in a neighborhood of minus one. Hence, the r.h.s. above must be a linear combination of y_{a1} and y_{a2} in the case $a = -1$. Comparing the behaviours of the functions, it becomes clear that the assertion above is true. Let

$$q_1(\alpha,\beta,\gamma) = \frac{\Gamma(\frac{\gamma+1}{2} - \frac{\alpha}{2} - \frac{\beta}{2})\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\gamma+1}{2} - \frac{\alpha}{2})\Gamma(\frac{\gamma+1}{2} - \frac{\beta}{2})}, \quad q_2(\alpha,\beta,\gamma) = 2^{1-\delta}\frac{\Gamma(\delta-1)\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}.$$

Then

$$Hl(-1, 0; \alpha, \beta, \gamma, \delta; z) = q_1(\alpha,\beta,\gamma) Hl(2, \alpha\beta; \alpha, \beta, \delta, \gamma; 1-z) +$$
$$+ q_2(\alpha,\beta,\gamma)(1-z)^{1-\delta} Hl(2, \alpha\beta + (1-\delta)(\delta+2\gamma); 1+\alpha-\delta; 1+\beta-\delta, 2-\delta, \gamma; 1-z),$$

and

$$Hl(-1, 0; \alpha, \beta, \gamma, \delta; z) = q_1(\alpha,\beta,\gamma) Hl\left(\frac{1}{2}, \frac{\alpha\beta}{2}; \alpha, \beta, \delta, \delta; \frac{z+1}{2}\right) + q_2(\alpha,\beta,\gamma)\cdot$$
$$\cdot\left(\frac{z+1}{2}\right)^{1-\delta} Hl\left(\frac{1}{2}, \frac{\alpha\beta}{2} + (1-\delta)\left(\gamma + \frac{\delta}{2}\right); 1+\alpha-\delta, 1+\beta-\delta, 2-\delta, \delta; \frac{z+1}{2}\right).$$

Using the above relations it follows, after some computations, that the connection matrices we seek are given by the following expressions

$$C_{0+} = \begin{pmatrix} q_1(\alpha,\beta,\gamma) & q_2(\alpha,\beta,\gamma) \\ q_1(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma}) & q_2(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma}) \end{pmatrix},$$

$$C_{0-} = \begin{pmatrix} q_1(\alpha,\beta,\gamma) & q_2(\alpha,\beta,\gamma) \\ (-1)^{1-\gamma} q_1(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma}) & (-1)^{1-\gamma} q_2(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma}) \end{pmatrix},$$

$$C_{\infty+} = \begin{pmatrix} q_1(\alpha, \alpha+1-\gamma, 1+\alpha-\beta) & (-1)^{\delta-1} q_2(\alpha, \alpha+1-\gamma, 1+\alpha-\beta) \\ q_1(\beta, \beta+1-\gamma, 1+\beta-\alpha) & (-1)^{\delta-1} q_2(\beta, \beta+1-\gamma, 1+\beta-\alpha) \end{pmatrix},$$

$$C_{\infty -} = \begin{pmatrix} (-1)^{-\alpha} q_1(\alpha, \widetilde{\alpha}, 1+\alpha-\beta) & (-1)^{\delta-\alpha-1} q_2(\alpha, \widetilde{\alpha}, 1+\alpha-\beta) \\ (-1)^{-\beta} q_1(\beta, \widetilde{\beta}, 1+\beta-\alpha) & (-1)^{\delta-\beta-1} q_2(\beta, \widetilde{\beta}, 1+\beta-\alpha) \end{pmatrix}.$$

where $\widetilde{\alpha} = 1+\alpha-\gamma$, $\widetilde{\beta} = 1+\beta-\gamma$, and $\widetilde{\gamma} = 2-\gamma$.

Remark. *These matrices remarkably provide a way for us to express y_{+1} and y_{-1} linearly in terms of y_{01} and y_{02}. This is interesting because the coefficients appearing in y_{+1} and y_{-1} satisfy a much more general three-term recurrence relation than that of the coefficients of y_{01} and y_{02}. Hence it would be difficult to express y_{+1} and y_{-1} in closed form if one only had (1.2) to rely on. Thus our matrices enable us to express y_{+1} and y_{-1} in closed form since y_{01} and y_{02} have closed form expressions.*

5. Conclusion

In this paper we have solved the two-point connection problem for equation (2.1), a three-parameter sub-class of the Heun equation. We were able to explicitly compute all connection matrices, by modifying the methods used by [3], thus providing a concrete application of these techniques. To the best of our knowledge the two-point connection problem for this equation has not yet been treated explicitly in the literature. Thus we expect that our results will shed some new light on problems where an equation can be transformed into equation (2.1). For example, in [13] the most general class of potential was found such that the solution of the one-dimensional time-independent Schrödinger equation may be expressed in terms of Heun functions. Thus for a sub-class of the potential derived in Theorem 4.1 of [13], our results should enable some further analysis, such as the computation of bound states and energy eigenvalues, and the study of scattering and tunneling phenomena. We do not expect, however, that the applicability of our results to be limited to this as we have treated a class of equation with three free parameters.

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