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COMPACT WEIGHTED DIFFERENCE OPERATORS

SUSHMA CHIB[1], B.S. KOMAL[2]

Abstract. The main purpose of this paper is to characterize compact weighted difference operators on $L^2(\mathcal{R})$.

1. Introduction and Preliminaries

Let (X, s, λ) be a σ-finite measure space. For $1 \leq p < \infty$, the space $L^p(X, s, \lambda)$ defined as $L^p(\lambda) = \{f | f : X \to \mathcal{C}$ is measurable and $\int |f|^p d\lambda < \infty\}$ is a Banach space under the norm $||f||_p = (\int |f|^p d\lambda)^{1/p}$. For p = 2, $L^2(\lambda)$ is a Hilbert space under the inner product $\langle f, g\rangle = \int f\bar{g}d\lambda$. If X = N, λ is a counting measure on P(N), the power set of N, then $L^p(\lambda) = \ell^p$, the sequence space of pth summable sequences of complex numbers . In this paper, we study weighted difference operators on $L^2(\mathcal{R})$, where $\lambda = \mu$, the Lebesgue measure. For $h \in \mathcal{R}$, the difference operators $D_\mu : L^2(\mathcal{R}) \to L^2(\mathcal{R})$ is defined as $(D_h f)(x) = f(x) - f(x - h)$ $\forall f \in L^2(\mathcal{R})$ and $x \in \mathcal{R}$. Let $\theta : \mathcal{R} \to \mathcal{R}$ be measurable function. Then the weighted difference operator $W_{\theta,D_h} : L^2(\mathcal{R}) \to L^2(\mathcal{R})$ is defined as

$$(W_{\theta,D_h} f)(x) = \theta(x)[f(x) - f(x - h)]$$

The function θ is known as weight function.
The difference operators are widely used in the numerical methods . For example in Newton's forward and Backward formulae for interpolation and etc. For more details we refer the [1] [2] [3] [5] [6]. In this paper we characterize bounded and compact weighted difference operators on $L^2(\mathcal{R})$ and $\ell^2(N)$. The adjoint of the weighted difference operator is also obtained.

2. Bounded Weighted Difference Operators on $\ell^2(N)$

In this section we investigate a necessary and suficient condition for a weighted difference operator to be bounded on $\ell^2(N)$.

Theorem 2.1. *Let $\theta : N \to \mathcal{C}$ be a mapping. Then $W_{\theta,D}$ is a bounded operator if and only if θ is a bounded function.*

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Proof. We first suppose that θ is a bounded function. Then there exists $M > 0$ such that $|\theta(x)| \leq M \ \forall \ x \in N$. For any $f \in \ell^2(N)$,
Consider

$$\begin{aligned} ||W_{\theta,D}f|| &= (\sum_{n=1}^{\infty} |\theta(n)|^2 |f_n - f_{n-1}|^2)^{1/2}, \ f_0 = 0 \\ &\leq M[(\sum_{n=1}^{\infty} |f_n|^2)^{1/2} + (\sum_{n=1}^{\infty} |f_{n-1}|^2)^{1/2}] \\ &\leq 2M||f|| \end{aligned}$$

Hence $W_{\theta,D}$ is a bounded operator.
Conversely, suppose that $W_{\theta,D}$ is a bounded operator. Then $\exists \ M > 0$ such that

$$||W_{\theta,D}e_n|| \leq M||e_n|| \ \forall \ n \in N$$

or

$$(\sum_{p=1}^{\infty} |\theta(p)[e_n(p) - e_n(p-1)]|^2)^{1/2} \leq M \ \forall \ n \in N$$

or

$$[|\theta(n)|^2 + |\theta(n+1)|^2]^{1/2} \leq M \ \forall \ n \in N$$

Hence θ is a bounded function.
Let $B : \ell^2(N) \to \ell^2(N)$ be defined by

$$(Bf)(n) = \bar{\theta}(n)f(n) - \overline{\theta(n+1)}f(n+1)$$

In the following theorem we prove that B is the adjoint of $W_{\theta,D}$. □

Theorem 2.2. *Let $W_{\theta,D} \in B(\ell^2(N))$. Then $W^*_{\theta,D} = B$.*

Proof. For $f, g \in \ell^2(N)$, we have

$$\begin{aligned} \langle f, M_{\theta,D}g \rangle &= \sum_{n=1}^{\infty} f(n)[\overline{\theta(n)} \ \overline{(g(n) - g(n-1))}] \\ &= \sum_{n=1}^{\infty} f(n)\overline{\theta(n)} \ \overline{g(n)} - \sum_{n=1}^{\infty} f(n)\overline{\theta(n)} \ \overline{g(n-1)} \\ &= \sum_{n=1}^{\infty} f(n)\overline{\theta(n)} \ \overline{g(n)} - \sum_{n=2}^{\infty} f(n)\overline{\theta(n)} \ \overline{g(n-1)}, \ g_0 = 0 \\ &= \sum_{n=1}^{\infty} [\overline{\theta(n)}f(n) - \overline{\theta(n+1)}f(n+1)]\overline{g(n)} \\ &= \langle Af, g \rangle \end{aligned}$$

This proves that $W^*_{\theta,D} = A$. □

Theorem 2.3. *Let $W_{\theta,D} \in B(\ell^2(N))$. Thus $W_{\theta,D}$ is a compact operator if and only if the set $\{n \in N : |\theta(n)| \geq \epsilon\}$ is a finte set for each $\epsilon > 0$.*

Proof. Suppose $W_{\theta,D}$ is a compact operator we know that $e_n \to 0$ weakly as $n \to \infty$. Therefore $W_{\theta,D}e_n \to 0$ strongly as $n \to \infty$. In other words $||W_{\theta,D}e_n|| \to 0$ as $n \to \infty$.
or

$$(|\theta(n)|^2 + |\theta(n+1)|^2)^{1/2} \to 0 \text{ as } n \to \infty$$

Hence for each $\epsilon > 0$ the set $\{n \in N : |\theta(n)| \geq \epsilon\}$ is a finite set.
Conversely, suppose that the condition is true. Let $\{f^{(p)} : p \in N\}$ be a bounded sequence in $\ell^2(N)$. Then there exists $k > 0$ such that $||f^{(p)}|| \leq k \forall\ p \in N$.
For $\epsilon > 0$, take $n_0 \in N$ such that

$$|\theta(n)| \geq \epsilon \ \forall\ n \leq n_0.$$

Consider

$$\begin{aligned}
||W_{\theta,D}f^{(p)}||^2 &= \sum_{n=1}^{\infty} |\theta(n)|^2 |[f^{(p)}(n) - f^{(p)}(n-1)]|^2 \\
&= \sum_{n=1}^{n_0} |\theta(n)|^2 |[f^{(p)}(n) - f^{(p)}(n-1)]|^2 \\
&\quad + \sum_{n=n_0+1}^{\infty} |\theta(n)|^2 |[f^{(p)}(n) - f^{(p)}(n-1)]|^2 \\
&\leq 2\sum_{n=1}^{n_0} |\theta(n)|^2 |f^{(p)}(n)|^2 + 2\sum_{n=1}^{n_0} |\theta(n)|^2 |f^{(p)}(n-1)|^2 \\
&\quad +2 \sum_{n=n_0+1}^{\infty} |\theta(n)|^2 |f^{(p)}(n)|^2 + 2 \sum_{n=n_0+1}^{\infty} |\theta(n)|^2 |f^{(p)}(n-1)|^2 \ (1)
\end{aligned}$$

Since every bounded sequence of real numbers contains a convergent subsequence, so for given $\epsilon > 0$ we can find a sequence $\{f^{p_\gamma})\}$ of $\{f^{(p)}\}$ such that

$$\sum_{n=1}^{n_0} |f^{(p_\gamma)}(n)|^2 < \epsilon \ \forall\ p_\gamma > p_0 > n_0$$

for some $p_0 \in N$.
Hence from (1),

$$||W_{\theta,D}f^{(p_\gamma)}||^2 \leq 4M^2\epsilon^2 + 4\epsilon^2 k^2 \ \forall\ p_\gamma > p_{\gamma_0}$$

This proves that $W_{\theta,D}$ is a compact operator. □

Example: Let $\theta : N \to \mathcal{C}$ be defined by $\theta(n) = e^{-n}$. Then the set $\{n \in N : |\theta(n)| \geq \epsilon\}$ is a finite set for each $\epsilon > 0$. Therefore $W_{\theta,D}$ is a compact operator in view of Theorem 2.3.

Example: Let $\theta : N \to \mathcal{C}$ be defined by $\theta(n) = e^{-in}$. Then $\{n \in N : |\theta(n)| \geq \epsilon\}$ is an infinite set so that $W_{\theta,D}$ is not a compact operator.

3. Weighted Difference Operators on $L^2(\mathcal{R})$

In this section we shall obtain a criterion for compact weighted difference operator on $L^2(\mathcal{R})$. For $h \in \mathcal{R}, \theta : \mathcal{R} \to \mathcal{C}$, the weighted difference operator $W_{\theta,D_h} : L^2\mathcal{R}) \to L^2(\mathcal{R})$ is defined by

$$(W_{\theta,D_h}f)(x) = \theta(x)[f(x) - f(x-h)]$$

for each $f \in L^2(\mathcal{R})$ and $x \in \mathcal{R}$.

Theorem 3.1. *Let $\theta : \mathcal{R} \to \mathcal{C}$ be a measurable function. Then for every $h \in \mathcal{R}, M_{\theta,D_h} : L^2(\mathcal{R}) \to L^2(\mathcal{R})$ is bounded if and only if θ is an essentially bounded function.*

Proof. If the condition of the theorem is true, then there exists $M > 0$ such that the set $E = \{x \in \mathcal{R} : |\theta(x)| > M\}$ is of measure zero.
Then for every $f \in L^2(\mathcal{R})$,

$$\begin{aligned} ||M_{\theta,D_h}f|| &= \int_{\mathcal{R}} |\theta(x)[f(x) - f(x-h)]|^2 d\mu(x) \\ &\leq M(\int_{\mathcal{R}} |f(x)|^2)^{1/2} + (\int_{\mathcal{R}} |f(x-h)|^2 d\mu(x))^{1/2}] \text{by using Minkowski's inequality} \\ &= M||f|| \end{aligned}$$

This proves that M_{θ,D_h} is a bounded operator.
Conversely, if θ is not essentially bounded, then for every positive integer n, the set $\{x \in \mathcal{R} : |\theta(x)| > n\}$ is of positive measure. Let $[a_n, b_n]$ be the interval contained in E_n such that $b_n - a_n < \infty$ and $b_n - a_n < h$. Let $F_n = [a_n, b_n]$ and $f_n = \frac{\chi_{F_n}}{\sqrt{\mu(F_n)}}$.
Then $||f_n|| = 1$ and

$$\begin{aligned} ||M_{\theta,D_h}f_n|| &= (\int |\theta(x)[f_n(x) - f_n(x-h)]|^2 d\mu(x))^{1/2} \\ &> \frac{n}{\sqrt{\mu(F_n)}}(\int |\chi_{F_n}(x) - \chi_{F_n}(x-h)|^2 d\mu(x))^{1/2} \\ &= \frac{n}{\sqrt{\mu(F_n)}}(\int |\chi_{F_n}(x) - \chi_{F_{n+h}}(x)|^2 d\mu(x))^{1/2} \\ &= 2n \to \infty \text{ as} n \to \infty. \end{aligned}$$

This contradicts the fact that W_{θ,D_h} is a bounded operator. Hence θ must be an essentially bounded function.
In the next theorem we compute the adjoint of weighted difference operator on $L^2(\mathcal{R})$. □

Theorem 3.2. *Let $W_{\theta,D_h} \in B(L^2(\mathcal{R}))$. Then $W^*_{\theta,D_h} = B$, where $(Bf)(x) = \bar{\theta(x)}f(x) - \overline{\theta(x+h)}f(x+h)$*

Proof. For $f, g \in L^2(\mathcal{R})$, we have

$$\begin{aligned} \langle f, W_{\theta,D_h}g \rangle &= \int_{\mathcal{R}} f(x)\theta(x)[g(x) - g(x-h)]d\mu(x) \\ &= \int_{\mathcal{R}} \bar{\theta}(x)f(x)\bar{g}(x)d\mu(x) - \int_{\mathcal{R}} \bar{\theta}(x)f(x)\bar{g}(x-h)d\mu(x) \\ &= \int_{\mathcal{R}} \bar{\theta}(x)f(x)\bar{g}(x)d\mu(x) \\ &\quad - \int_{\mathcal{R}} \bar{\theta}(x+h)f(x+h)\bar{g}(x)d\mu(x) \quad \text{because } \mu \text{ is translation invariant} \\ &= \int_{\mathcal{R}} [\bar{\theta}(x)f(x) - \bar{\theta}(x+h)f(x+h)]\bar{g}(x)d\mu(x) \\ &= \langle Bf, g \rangle. \end{aligned}$$

Hence $W^*_{\theta,D_h} = B$. □

Theorem 3.3. *Let $W_{\theta,D_h} \in B(L^2(\mathcal{R}))$. Then W_{θ,D_h} is a compact operator if and only if it is zero operator.*

Proof. If W_{θ,D_h} is the zero operator, then it is a compact operator.
Conversely, suppose W_{θ,D_h} is a compact operator. We prove that it is zero operator. For, if $W_{\theta,D_h} \neq 0$ a.e., then the set $E = \{x \in \mathcal{R} : |\theta(x)| \geq \epsilon\}$ is of positive mesure for some $\epsilon > 0$. Since Lebesgue measure is σ-finite, so there exists a sequence $\{E_n\}$ of disjoint measurable subsets of $\mathcal{R}$ such that $\cup_{n=1}^{\infty} E_n = \mathcal{R}$, and $0 < \mu(E_n) < \infty$, $E_n \cap E_{n+h} = \phi$. Let $F_n = E_n \cap E$. Take $f_n = \frac{\chi_{F_n}}{\sqrt{\mu(F_n)}}$. Then $||f_n|| = 1$ and

$$\begin{aligned}
||W_{\theta,D_h} f_n||^2 &= \int_{\mathcal{R}} |\theta(x)[f_n(x) - f_n(x-h)]|^2 d\mu(x) \\
&\geq \frac{\epsilon^2}{\mu(F_n)} \int_{\mathcal{R}} |\chi_{F_n}(x) - \chi_{F_n+h}(x)|^2 d\mu(x) \\
&= \frac{\epsilon^2}{\mu(F_n)} [\int_{\mathcal{R}} |\chi_{F_n}(x)|^2 d\mu(x) + \int_{\mathcal{R}} |\chi_{F_{n+h}}(x)|^2 d\mu(x) \\
&= \frac{\epsilon^2}{\mu(F_n)} 2\mu(F_n) \\
&= 2\epsilon^2
\end{aligned}$$

This proves that $\{W_{\theta,D_h} f_n\}$ cannot have a convergent subsequence. This is a contradiction since W_{θ,D_h} is assumed to be comapct. Then W_{θ,D_h} must be the zero operator. □

References

[1] A. M. Akhmedov, F. Basar, The fine spectra of the difference operator Δ over the sequence space bv_p, $(1 \leq p < \infty)$, Acta. Math. Sin. Eng. Ser., In press.

[2] B. Altay, F. Basar, On the fine spectra of the difference operator on c_0 and c, Inform. Sci., **168** (2004), 217-224.

[3] B. Altay, F. Basar, On the fine spectra of the generalized difference operator B(r, s) over the sequence spaces c_0 and c, Int. J. Math. Sci., **18** (2005), 3005-3013.

[4] B. S. Komal R. S. Pathania, Some results on multiplicaion operators induced by operator valued maps, Bull. Cal. Math. Soc., **83** (1981), 515-518.

[5] R. K. Singh, R. S. Dharamadhikari, Compact and Fredholm composition multiplication operators, Acta Sci. Math.(Szeged), **52** (1988), 437-441.

[6] Singh, R.K. amd Kumar, A: Multiplication Operators and composition operators with closed ranges, Bull. Austral. Math. Soc., 16(1977), 247-252.

[7] H. Takagi, Compact weighted composition operators on certain subspaces of C(X, E), Tokyo J. Math., **14** (1991), 121-127.

[8] H. Takagi, Compact weighted composition operators on L^p, Proc. Amer. Math. Soc., **116** (1992), 505-511.

Sushma Chib[1]
Department of Mathematics
Govt. M.A.M. P.G. College Jammu
E-mail address: sushmachib737@gmail.com

B. S. Komal[2]
MIET, Kot Bhalwal, Jammu
E-mail address: bskomal2@gmail.com