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A STUDY ON GENERALIZED MITTAG-LEFFLER FUNCTION VIA FRACTIONAL CALCULUS

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Abstract. In this paper, author introduced and developed here a new multivariable generalization of Mittag-Leffler type function. Certain properties of new multivariable generalization of Mittag-Leffler type function associated with fractional calculus are established. A composition of Riemann-Liouville fractional integral operators associated with multivariable Mittag-Leffler function in the kerenel has been obtained.

1. Introduction

In 1903, the Swedish mathematician Mittag-Leffler [2] introduced the function $E_\alpha(z)$ defined as

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad (\alpha,z\in\mathbb{C};Re(\alpha)>0) \tag{1.1}$$

A generalization of $E_\alpha(z)$ was studied by Wiman [10] in the following form

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\qquad (\alpha,\beta,z\in\mathbb{C};Re(\alpha)>0,Re(\beta)>0) \tag{1.2}$$

This is known as generalized Mittag-Leffler function or Wiman's function.

In 1971, Prabhakar [3] established the function

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad (\alpha,\beta,\gamma,z\in\mathbb{C};Re(\alpha)>0,Re(\beta)>0,Re(\gamma)>0) \tag{1.3}$$

this is a generalization of Wiman's function.

Another generalization of (1.3) was defined by Salim [4] as

$$E^{\gamma,\delta}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{(\delta)_n},(\alpha,\beta,\gamma,\delta,z\in\mathbb{C};Re(\alpha)>0,Re(\beta)>0,Re(\gamma)>0,Re(\delta)>0) \tag{1.4}$$

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A generalization of (1.4) is initiated by Salim and Faraj [5] as

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}\left(z\right)=\sum_{n=0}^{\infty}\frac{\left(\gamma\right)_{qn}}{\Gamma\left(\alpha n+\beta\right)}\frac{z^{n}}{\left(\delta\right)_{pn}} \tag{1.5}$$

where $\alpha,\beta,\gamma,\delta,z\in\mathbb{C};\min\left\{Re\left(\alpha\right),Re\left(\beta\right),Re\left(\gamma\right),Re\left(\delta\right)\right\}>0;p,q>0$ and $q\leq Re\left(\alpha\right)+p$

The multivariate analogue of generalized Mittag-Leffler function (1.3) is setup and studied by Saxena et al. [7] in the following form

$$E_{\rho_j,\lambda}^{\gamma_j}\left(z_1,...,z_m\right)=E_{(\rho_1,...,\rho_m),\lambda}^{(\gamma_1,...,\gamma_m)}\left(z_1,...,z_m\right)=\sum_{r_1,...,r_m=0}^{\infty}\frac{\left(\gamma_1\right)_{r_1}\cdots\left(\gamma_m\right)_{r_m}}{\Gamma\left(\lambda+\sum\limits_{j=1}^{m}\rho_j r_j\right)}\frac{z_1^{r_1}\cdots z_m^{r_m}}{r_1!\cdots r_m!} \tag{1.6}$$

where $\lambda,\rho_j,\gamma_j\in\mathbb{C}$ and $Re(\rho_j)>0;j=1,2,...,m.$

A further generalization of multivariate analogue of generalized Mittag-Leffler function (1.6) was also mentioned, Saxena et al. [7] in terms of the following multiple series:

$$E_{(\rho_1,...,\rho_m),\lambda}^{(\gamma_1,...,\gamma_m;l_1,...,l_m)}\left(z_1,...,z_m\right)=\sum_{r_1,...,r_m=0}^{\infty}\frac{\left(\gamma_1\right)_{l_1r_1}\cdots\left(\gamma_m\right)_{l_1r_m}}{\Gamma\left(\lambda+\sum\limits_{j=1}^{m}\rho_j r_j\right)}\frac{z_1^{r_1}\cdots z_m^{r_m}}{r_1!\cdots r_m!} \tag{1.7}$$

where $\lambda,\rho_j,\gamma_j,l_j\in\mathbb{C};Re(\rho_j)>0,Re(l_j)>0;j=1,2,...,m.$

In this paper, author introduce a new multivariable generalization of Mittag-Leffler type function $E_{(\rho_j;\lambda;q_j)}^{(\gamma_j;l_j;p_j)}\left(z_1,...,z_m\right)$, which is a generalization of (1.5)

$$\begin{aligned}E_{(\rho_j;\lambda;q_j)}^{(\gamma_j;l_j;p_j)}\left(z_1,...,z_m\right)&=E_{\rho_1,...,\rho_m;\;\;\lambda;\;\;q_1,...,q_m}^{\gamma_1,...,\gamma_m;l_1,...,l_m;p_1,...,p_m}\left(z_1,...,z_m\right)\\&=\sum_{r_1,...,r_m=0}^{\infty}\frac{\left(\gamma_1\right)_{p_1r_1}....\left(\gamma_m\right)_{p_mr_m}}{\Gamma\left(\lambda+\sum\limits_{j=1}^{m}\rho_j r_j\right)}\frac{z_1^{r_1}\cdots z_m^{r_m}}{\left(l_1\right)_{q_1r_1}....\left(l_m\right)_{q_mr_m}}\end{aligned} \tag{1.8}$$

where $\lambda,\rho_j,\gamma_j,l_j\in\mathbb{C};\min\limits_{1\leq j\leq m}\left\{Re\left(\lambda\right),Re\left(\rho_j\right),Re\left(\gamma_j\right),Re\left(l_j\right)\right\}>0$ and

$$p_j,q_j>0\,;\,p_j<q_j+Re\left(\rho_j\right);j=1,2,...,m \tag{1.9}$$

By putting suitable values of parameters in (1.8), we get (1.1) - (1.7).

To study, various properties of (1.8) , we need the following well-known definitions and results:

(1) Beta transform (Sneddon [9])

$$B\left\{f(z);a,b\right\}=\int_0^1 z^{a-1}\left(1-z\right)^{b-1}f\left(z\right)\,dz,\qquad\left(Re\left(a\right)>0,Re\left(b\right)>0\right) \tag{1.10}$$

(2) Laplace transform (Sneddon [9])

$$L\left\{f(z);s\right\}=\int_0^{\infty}e^{-sz}f\left(z\right)\,dz,\qquad\left(Re\left(s\right)>0\right) \tag{1.11}$$

(3) Fubini's theorem (Dirichlet formula) (Samko et al.[6])

$$\int_a^b dx \int_a^x f(x,t)\, dt = \int_a^b dt \int_t^b f(x,t)\, dx \tag{1.12}$$

(4) Riemann-Liouville fractional integral I_{a+}^{α} and derivative D_{a+}^{α} of order α defined as below (Samko et al.[6]):

$$\left(I_{a+}^{\alpha}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi(t)\, dt\,, \qquad (\alpha \in \mathbb{C}, Re(\alpha) > 0) \tag{1.13}$$

and

$$\left(D_{a+}^{\alpha}\varphi\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha}\varphi\right)(x)\ , \qquad (n = [Re(\alpha)] + 1; \alpha \in \mathbb{C}, Re(\alpha) > 0) \tag{1.14}$$

2. BASIC PROPERTIES

Theorem 2.1 If the condition (1.9) is satisfied, then for any $n \in \mathbb{N}$

$$\left(\frac{d}{dz}\right)^n \left[z^{\lambda-1} E_{(\rho_1,\ldots,\rho_m),\ \ \lambda,\ \ (q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)} \left(\omega_1 z^{\rho_1}, \ldots., \omega_m z^{\rho_m}\right)\right]$$

$$= z^{\lambda-n-1} E_{(\rho_1,\ldots,\rho_m),\ \ \lambda-n,\ \ (q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)} \left(\omega_1 z^{\rho_1}, \ldots., \omega_m z^{\rho_m}\right) \tag{2.1}$$

Proof: Using equation (1.8) and differentiating term by term under the summation sign, we find

$$\left(\frac{d}{dz}\right)^n \left[z^{\lambda-1} E_{(\rho_1,\ldots,\rho_m),\ \ \lambda,\ \ (q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)} \left(\omega_1 z^{\rho_1}, \ldots., \omega_m z^{\rho_m}\right)\right]$$

$$= \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1} \cdots (\gamma_m)_{p_m r_m}\, \omega_1^{r_1} \ldots \omega_m^{r_m}}{\Gamma\left(\lambda + \sum\limits_{j=1}^{m} \rho_j r_j\right) (l_1)_{q_1 r_1} \cdots (l_m)_{q_m r_m}} \left(\frac{d}{dz}\right)^n \left[z^{\lambda-1+\rho_1 r_1+\ldots+\rho_m r_m}\right]$$

$$= \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1} \cdots (\gamma_m)_{p_m r_m}\, \omega_1^{r_1} \ldots \omega_m^{r_m}}{\Gamma\left(\lambda - n + \sum\limits_{j=1}^{m} \rho_j r_j\right) (l_1)_{q_1 r_1} \cdots (l_m)_{q_m r_m}} \left[z^{\lambda-n-1+\rho_1 r_1+\ldots+\rho_m r_m}\right]$$

$$= z^{\lambda-n-1} E_{(\rho_1,\ldots,\rho_m),\ \ \lambda-n,\ \ (q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)} \left(\omega_1 z^{\rho_1}, \ldots., \omega_m z^{\rho_m}\right)$$

Remark 1: If we take $l_1 = \cdots = l_m = p_1 = \cdots p_m = q_1 = \cdots q_m = 1$ in equation (2.1), we get known result due to Saxena et al. [7].

Remark 2: If we take $m = 1, \gamma_1 = \gamma, l_1 = q, p_1 = q_1 = 1, \rho_1 = \alpha, \lambda = \beta, n = m$ and $\omega_1 = \omega$ in equation (2.1), we get known result due to Shukla and Prajapati [8].

Remark 3: If we take $m = 1, \gamma_1 = \gamma, l_1 = p_1 = q_1 = 1, \rho_1 = \rho, \lambda = \mu$ and $\omega_1 = \omega$ in equation (2.1), we get known result due to Kilbas et al. [1].

Theorem 2.2 If the condition (1.9) is satisfied, then
$\int_0^z t^{\lambda-1} E_{(\rho_1,\ldots,\rho_m),\quad \lambda \quad ,\ (q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)} (\omega_1 t^{\rho_1}, ..., \omega_m t^{\rho_m})\, dt$

$$= z^{\lambda} E_{(\rho_1,\ldots,\rho_m),\ \lambda+1,\ (q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)} (\omega_1 z^{\rho_1}, ..., \omega_m z^{\rho_m}) \tag{2.2}$$

Proof: Using equation (1.8) and changing the order of integration and summation, we get

$$\int_0^z \left[t^{\lambda-1} E_{(\rho_1,\ldots,\rho_m),\ \lambda,\ (q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)} (\omega_1 t^{\rho_1}, ..., \omega_m t^{\rho_m}) \right] dt$$

$$= \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1} \cdots (\gamma_m)_{p_m r_m} (\omega_1 z_1)^{r_1} ... (\omega_m z_m)^{r_m}}{\Gamma\left(\lambda + \sum\limits_{j=1}^{m} \rho_j r_j\right) (l_1)_{q_1 r_1} \cdots (l_m)_{q_m r_m}} \int_0^z \left[t^{\lambda+\rho_1 r_1+\ldots+\rho_m r_m-1} \right] dt$$

We get the require result after a little simplification.
Now, we consider the Riemann-Liouville fractional integral (1.13) and derivative (1.14) with the with a new multivariable generalization of MittagLeffler type function (1.8)

Theorem 2.3 Let $a \in \mathbb{R}_+; \alpha, \lambda, \rho_j, \gamma_j, l_j, \omega_j \in \mathbb{C}; \min\limits_{1\le j\le m} \{Re(\alpha), Re(\lambda), Re(\rho_j), Re(l_j), Re(\gamma_j)\} > 0$ and $p_j, q_j > 0;\ j = 1, 2, ..., m$ then for $x > a$, there holds the relation

$$\left\{ I_{a+}^{\alpha} \left[(t-a)^{\lambda-1} E_{(\rho_j),\ \lambda,\ (q_j)}^{(\gamma_j),(l_j),(p_j)} (\omega_1 (t-a)^{\rho_1}, ..., \omega_m (t-a)^{\rho_m}) \right] \right\} (x)$$

$$= (x-a)^{\lambda+\alpha-1} E_{(\rho_j),\lambda+\alpha,(q_j)}^{(\gamma_j),(l_j),(p_j)} (\omega_1 (x-a)^{\rho_1}, ..., \omega_m (x-a)^{\rho_m}) \tag{2.3}$$

Proof: We use the result (1.8) in L.H.S of (2.3), we find that
$\left\{ I_{a+}^{\alpha} \left[(t-a)^{\lambda-1} E_{(\rho_j),\ \lambda,\ (q_j)}^{(\gamma_j),(l_j),(p_j)} (\omega_1 (t-a)^{\rho_1}, ..., \omega_m (t-a)^{\rho_m}) \right] \right\} (x)$

$$= \left\{ I_{a+}^{\alpha} \left[\sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1} \cdots (\gamma_m)_{p_m r_m} \omega_1^{r_1} ... \omega_m^{r_m} (t-a)^{\lambda+\rho_1 r_1+\ldots+\rho_m r_m-1}}{\Gamma\left(\lambda + \sum\limits_{j=1}^{m} \rho_j r_j\right) (l_1)_{q_1 r_1} \cdots (l_m)_{q_m r_m}} \right] \right\} (x)$$

FInally, we arrive at the R. H. S. of (2.3) after using following relation [6]

$$\left\{ I_{a+}^{\alpha} \left[(t-a)^{\beta-1} \right] \right\} (x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1},\ (\alpha, \beta; Re(\alpha) > 0, Re(\beta) > 0)$$

Hence

$$\left\{ I_{a+}^{\alpha} \left[(t-a)^{\lambda-1} E_{(\rho_j),\ \lambda,\ (q_j)}^{(\gamma_j),(l_j),(p_j)} (\omega_1 (t-a)^{\rho_1}, ..., \omega_m (t-a)^{\rho_m}) \right] \right\} (x)$$

$$= (x-a)^{\lambda+\alpha-1} E_{(\rho_j),\lambda+\alpha,(q_j)}^{(\gamma_j),(l_j),(p_j)} (\omega_1 (x-a)^{\rho_1}, ..., \omega_m (x-a)^{\rho_m})$$

Theorem 2.4 Let $a \in \mathbb{R}_+; \alpha, \lambda, \rho_j, \gamma_j, l_j, \omega_j \in \mathbb{C}; \min_{1 \leq j \leq m} \{Re(\alpha), Re(\lambda), Re(\rho_j), Re(l_j), Re(\gamma_j)\} > 0$; $p_j, q_j > 0$; $j = 1, 2, ..., m$ then for $x > a$, there holds the relation

$$\left\{D_{a+}^{\alpha}\left[(t-a)^{\lambda-1} E_{(\rho_j),\ \lambda,\ (q_j)}^{(\gamma_j),(l_j),(p_j)}(\omega_1(t-a)^{\rho_1}, ..., \omega_m(t-a)^{\rho_m})\right]\right\}(x)$$

$$= (x-a)^{\lambda-\alpha-1} E_{(\rho_j),\lambda-\alpha,(q_j)}^{(\gamma_j),(l_j),(p_j)}(\omega_1(x-a)^{\rho_1}, ..., \omega_m(x-a)^{\rho_m}) \tag{2.4}$$

Proof: We use the results (1.8) and (1.14), then we find that

$$\left\{D_{a+}^{\alpha}\left[(t-a)^{\lambda-1} E_{(\rho_j),\ \lambda,\ (q_j)}^{(\gamma_j),(l_j),(p_j)}(\omega_1(t-a)^{\rho_1}, ..., \omega_m(t-a)^{\rho_m})\right]\right\}(x)$$

$$= \left(\frac{d}{dx}\right)^n \left\{I_{a+}^{n-\alpha}\left[(t-a)^{\lambda-1} E_{(\rho_j),\ \lambda,\ (q_j)}^{(\gamma_j),(l_j),(p_j)}(\omega_1(t-a)^{\rho_1}, ..., \omega_m(t-a)^{\rho_m})\right]\right\}(x)$$

Now, applying theorem 2.3 with α replaced $n - \alpha$. We get the desire result.

In this section, the image of $E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}(\omega_1 z^{\sigma_1}, ..., \omega_m z^{\sigma_m})$ under Beta and Laplace transforms are proved in the following theorems:

Theorem 2.5 (Beta Transform)

$$B\left\{E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}(\omega_1 z^{\sigma_1}, ..., \omega_m z^{\sigma_m}); a, b\right\} = \frac{\Gamma(b)\Gamma(l_1)...\Gamma(l_m)}{\Gamma(\gamma_1)...\Gamma(\gamma_m)}$$

$$\times {}_{2m+1}\psi_{m+2}\left[\begin{array}{c|c} (\gamma_1, p_1), \cdots, (\gamma_m, p_m), \left(a, \sum_{j=1}^{m}\sigma_j\right), \overbrace{(1,1), ..., (1,1)}^{(m\, times)} & \omega_1 \\ & \vdots \\ (l_1, q_1), \cdots, (l_m, q_m), \left(\lambda, \sum_{j=1}^{m}\rho_j\right), \left(a+b, \sum_{j=1}^{m}\sigma_j\right) & \omega_m \end{array}\right] \tag{2.5}$$

where conditions (1.9) are satisfied and $Re(a) > 0, Re(b) > 0$.

Remark 1: If we take $m = 1, \gamma_1 = \gamma, l_1 = \delta, p_1 = q, q_1 = p, \rho_1 = \alpha, \lambda = \beta, \omega_1 = x$ and $\sigma_1 = \sigma$ in equation (2.5), we get known result due to Salim and Faraj [5].

Remark 2: If we take $m = 1, \gamma_1 = \gamma, l_1 = q_1 = 1, p_1 = q, \rho_1 = \alpha, \lambda = \beta, \omega_1 = x$ and $\sigma_1 = \sigma$ in equation (2.5), we get known result due to Shukla and Prajapati [8].

Theorem 2.6 (Laplace Transform)

$$L\left\{z^{a-1} E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}(\omega_1 z^{\sigma_1}, ..., \omega_m z^{\sigma_m}); s\right\} = \frac{\Gamma(l_1)...\Gamma(l_m)}{\Gamma(\gamma_1)...\Gamma(\gamma_m)} s^{-a}$$

$$\times {}_{2m+1}\psi_{m+1}\left[\begin{array}{c|c} (\gamma_1, p_1), \cdots, (\gamma_m, p_m), \left(a, \sum_{j=1}^{m}\sigma_j\right), \overbrace{(1,1) ..., (1,1)}^{(m\, times)} & \left(\frac{\omega_1}{s^{\sigma_1}}\right) \\ & \vdots \\ (l_1, q_1), \cdots, (l_m, q_m), \left(\lambda, \sum_{j=1}^{m}\rho_j\right) & \left(\frac{\omega_m}{s^{\sigma_m}}\right) \end{array}\right] \tag{2.6}$$

3. COMPOSITIONS OF FRACTIONAL CALCULUS OPERATORS WITH INTEGRAL OPERATOR WITH A NEW MULTIVARIABLE GENERALIZATION OF MITTAG-LEFFLER TYPE FUNCTION IN THE KERNEL

The integral operator defined by

$$\left(E^{(\gamma_j),(l_j),(p_j)}_{(\rho_j),\lambda,(q_j):(\omega_j);a+}\varphi\right)(x)=\left(E^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)}_{(\rho_1,\ldots,\rho_m),\quad\lambda,\quad(q_1,\ldots,q_m):(\omega_1,\ldots,\omega_m);a+}\varphi\right)(x)$$

$$=\int_a^x (x-t)^{\lambda-1}\,E^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)}_{(\rho_1,\ldots,\rho_m),\quad\lambda,\quad(q_1,\ldots,q_m)}\left[\omega_1(x-t)^{\rho_1},\ldots,\omega_m(x-t)^{\rho_m}\right]\varphi(t)\,dt \tag{3.1}$$

this contains a new multivariable generalization of MittagLeffler type function (1.8) in its kernel is investigated.

Consider the integral operator defined in (3.1) containing the genearlized Mittag-Leffler function $E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j}(.)$ in the kernel. First we will prove that the operator $E^{(\gamma_j),(l_j),(p_j)}_{(\rho_j),\lambda,(q_j),(\omega_j);a+}$ is bounded on $L(a,b)$.

Theorem 3.1 Let $\alpha,\lambda,\rho_j,\gamma_j,l_j,\omega_j\in\mathbb{C}$; $\min\limits_{1\le j\le m}\{Re(\alpha),Re(\lambda),Re(\rho_j),Re(\gamma_j),Re\,(l_j)\}>0$ and$p_j,q_j>0$; $j=1,2,..,m$, then the relation

$$I^{\alpha}_{a+}E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j,\omega_j;a+}\varphi=E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda+\alpha,q_j,\omega_j;a+}\varphi=E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j,\omega_j;a+}I^{\alpha}_{a+}\varphi \tag{3.2}$$

holds for any summable function $L(a,b)$.

Proof: Using equation (3.1) and (1.13) and applying the Dirichlet formula, we obtain for $x>a$

$$\left(I^{\alpha}_{a+}E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j,\omega_j;a+}\varphi\right)(x)$$

$$=\frac{1}{\Gamma(\alpha)}\int_a^x(x-u)^{\alpha-1}\left\{\int_a^u(u-t)^{\lambda-1}\,E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j}\left[\omega_1(u-t)^{\rho_1},\ldots,\omega_m(u-t)^{\rho_m}\right]\ \varphi(t)\ dt\right\}du$$

$$=\int_a^x\left\{\frac{1}{\Gamma(\alpha)}\int_t^x(x-u)^{\alpha-1}(u-t)^{\lambda-1}\,E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j}\left[\omega_1(u-t)^{\rho_1},\ldots,\omega_m(u-t)^{\rho_m}\right]\ du\right\}\varphi(t)\ dt$$

Let $u-t=\tau$ then we obtained

$$=\int_a^x\left\{\frac{1}{\Gamma(\alpha)}\int_0^{x-t}(x-t-\tau)^{\alpha-1}\tau^{\lambda-1}\,E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j}\left[\omega_1\tau^{\rho_1},\ldots,\omega_m\tau^{\rho_m}\right]d\tau\right\}\varphi(t)\ dt$$

$$=\int_a^x I^{\alpha}_{a+}\left\{\tau^{\lambda-1}E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j}\left[\omega_1\tau^{\rho_1},\ldots,\omega_m\tau^{\rho_m}\right]\right\}(x-t)\,\varphi(t)\ dt$$

Using Theorem 2.3, we get

$$=\int_a^x\left\{\tau^{\lambda+\alpha-1}E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda+\alpha,q_j}\left[\omega_1\tau^{\rho_1},\ldots,\omega_m\tau^{\rho_m}\right]\right\}\varphi(t)\ dt$$

$$=\int_a^x\left\{(x-t)^{\lambda+\alpha-1}\,E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda+\alpha,q_j}\left[\omega_1(x-t)^{\rho_1},\ldots,\omega_m(x-t)^{\rho_m}\right]\right\}\varphi(t)\ dt$$

$$=\left(E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda+\alpha,q_j,\omega_j;a+}\varphi\right)(x)$$

Similarly, we can prove remaining part of (3.2).

Theorem 3.2 If the conditions of Theorem 3.1 are satisfied, then

$$D^{\alpha}_{a+}E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda,q_j,\omega_j;a+}\varphi=E^{\gamma_j,l_j,p_j}_{\rho_j,\lambda-\alpha,q_j,\omega_j;a+}\varphi \tag{3.3}$$

Proof: The proof of Theorem 3.2 can easily prove with the help of formula

$$D_{a+}^{\alpha}\varphi = D_{a}^{n}\ I_{a+}^{n-\alpha}\varphi$$

$$\left(D_{a+}^{\alpha}E_{\rho_j,\lambda,q_j,\omega_j;a+}^{\gamma_j,l_j,p_j}\varphi\right)(x) = \left(\frac{d}{dx}\right)^{n}\left[I_{a+}^{n-\alpha}E_{\rho_j,\lambda,q_j,\omega_j;a+}^{\gamma_j,l_j,p_j}\varphi\right](x)$$
$$= \left(\frac{d}{dx}\right)^{n}\left[E_{\rho_j,\lambda+n-\alpha,q_j,\omega_j;a+}^{\gamma_j,l_j,p_j}\varphi\right](x)$$

Using equation (3.1) in the above equation, we get

$$= \left(\frac{d}{dx}\right)^{n}\int_a^x (x-t)^{\lambda+n-\alpha-1}E_{\rho_j,\lambda+n-\alpha,q_j}^{\gamma_j,l_j,p_j}\left[\omega_1(x-t)^{\rho_1},...,\omega_m(x-t)^{\rho_m}\right]\varphi(t)\ dt$$

We apply the Leibnitz rule, we get

$$= \left(\frac{d}{dx}\right)^{n-1}\int_a^x \frac{\partial}{\partial x}\left\{(x-t)^{\lambda+n-\alpha-1}E_{\rho_j,\lambda+n-\alpha,q_j}^{\gamma_j,l_j,p_j}\left[\omega_1(x-t)^{\rho_1},...,\omega_m(x-t)^{\rho_m}\right]\right\}\varphi(t)\ dt$$
$$+\lim_{t\to x}(x-t)^{\lambda+n-\alpha-1}E_{\rho_j,\lambda+n-\alpha,q_j}^{\gamma_j,l_j,p_j}\left[\omega_1(x-t)^{\rho_1},...,\omega_m(x-t)^{\rho_m}\right]$$

$$= \left(\tfrac{d}{dx}\right)^{n-1}\int_a^x\left\{(x-t)^{\lambda+n-\alpha-2}\sum_{r_1,...,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}....(\gamma_m)_{p_mr_m}}{\Gamma\left(\lambda+n-\alpha-1+\sum\limits_{j=1}^{m}\rho_jr_j\right)}\frac{(\omega_1(x-t)^{\rho_1})^{r_1}...(\omega_m(x-t)^{\rho_m})^{r_m}}{(l_1)_{q_1r_1},...,(l_m)_{q_mr_m}}\right\}\phi(t)\ dt$$

Repeating this process $(n-1)$ times, we get

$$= \int_a^x\left\{(x-t)^{\lambda-\alpha-1}E_{\rho_j,\lambda-\alpha,q_j}^{\gamma_j,l_j,p_j}\left[\omega_1(x-t)^{\rho_1},...,\omega_m(x-t)^{\rho_m}\right]\right\}\varphi(t)\ dt$$
$$= \left(E_{\rho_j,\lambda-\alpha,q_j,\omega_j;a+}^{\gamma_j,l_j,p_j}\varphi\right)(x)$$

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References

[1] A.A. Kilbas, M.Saigo, R.K. Saxena *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Int. Trans. and Spec. Func. **15** (2004), 31-49.
[2] G.M. Mittag-Leffler, *Sur la nouvelle fonction.* C.R. Acad. Sci. Paris, **137** (1903), 554-558.
[3] T.R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Math. J., **19** (1971), 7-15.
[4] T.O. Salim, *Some properties relating to the generalized Mittag-Leffler function*, Adv. Appl. Math. Anal., **4** (2009), 21-30.
[5] T.O. Salim, A.W. Faraj, *A generalization of Mittag-Leffler function and integral operator associated with fractional calculus*, J. of Fract. Calc. and Appl., **3** (2012), 1-13.
[6] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Yverdon (Switzerland): Gordon and Breach Science Publishers; 1993.
[7] R.K. Saxena, S.L. Kalla, Ravi Saxena *Multivariate analogue of generalized Mittag-Leffler function, Int. Trans. and Special Functions*, J. of Int. Trans. and Special Functions, **7** (2011), 533-548.
[8] A.K. Shukla, J.C. Prajapati *On a generalization of Mittag-Leffler function and its properties*, Math. Anal. Appl., **336** (2007), 797-811.

[9] I.N. Sneddon, *The Use of Integral Transforms*, New Delhi: Tata McGraw Hill; 1979.
[10] A. Wiman *Uber den fundamental satz in der theory der functionen*, Acta Math., **29** (1905), 191-201.

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