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A STUDY ON A FUNCTIONAL INEQUALITY AND ITS APPLICATIONS

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Abstract. In 2009, W. T. Sulaiman presented a new method in order to check whether a function is (p, q)-Hölder. In this paper, we generalize this method.

1. Introduction

For any $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a non-negative function f on an interval $I \subseteq [0, \infty)$ is called (p, q)-Hölder on I if

$$f(xy) \leq (f(x^p))^{1/p} (f(x^q))^{1/q}$$

for all $x, y \in I$. Many examples were presented in [1].

In 2009, W. T. Sulaiman [2] proved that for any non-negative function f on an interval $I \subseteq [0, \infty)$ and for any $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, if

$$f''(x) + \frac{f'(x)}{x} - \frac{(f'(x))^2}{f(x)} \geq 0$$

for all $x \in I - \{0\}$, then f is (p, q)-Hölder on I.

For any $p_1, p_2, ..., p_n > 1$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, a non-negative function f on an interval $I \subseteq [0, \infty)$ is called $(p_1, p_2, ..., p_n)$-Hölder on I if

$$f(x_1 x_2 ... x_n) \leq \prod_{i=1}^{n} (f(x_i^p))^{1/p_i}$$

for all $x_1, x_2, ..., x_n \in I$.

In this paper, we generalize the result in 2009 using the $(p_1, p_2, ..., p_n)$-Hölder definition.

2. Main Results

Theorem 2.1. *Let f be a non-negative function f on an interval $I \subseteq [0, \infty)$ and let $p_1, p_2, ..., p_n > 1$ be such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Assume that*

$$f''(x) + \frac{f'(x)}{x} - \frac{(f'(x))^2}{f(x)} \geq 0$$

for all $x \in I - \{0\}$. Then f is $(p_1, p_2, ..., p_n)$-Hölder on I.

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Proof. It is sufficient to show that

$$f\left(\prod_{i=1}^{n} x_i^{1/p_i}\right) \le \prod_{i=1}^{n} f^{1/p_i}(x_i)$$

for all $x_1, x_2, ..., x_n \in I$.

For fixed $x_2, ..., x_n \in I$, we define

$$F(x) = f^{1/p_1}(x)\prod_{i=2}^{n} f^{1/p_i}(x_i) - f\left(x^{1/p_1}\prod_{i=2}^{n} x_i^{1/p_i}\right)$$

for all $x \in I$. It follows that

$$\begin{aligned} F'(x) &= \frac{1}{p_1}\left[\frac{f'(x)}{f^{\sum_{i=2}^{n} 1/p_i}(x)}\prod_{i=2}^{n} f^{1/p_i}(x_i) - \frac{\prod_{i=2}^{n} x_i^{1/p_i}}{x^{\sum_{i=2}^{n} 1/p_i}} f'\left(x^{1/p_1}\prod_{i=2}^{n} x_i^{1/p_i}\right)\right] \\ &= 0 \end{aligned}$$

if $x = x_2 = ... = x_n$.

For any $x_2, ..., x_n \in I$, we have

$$\begin{aligned} F''(x) = &\frac{1}{p_1}\left[\frac{f''(x)}{f^{\sum_{i=2}^{n} 1/p_i}(x)}\prod_{i=2}^{n} f^{1/p_i}(x_i) - \frac{\left(\sum_{i=2}^{n} 1/p_i\right)(f'(x))^2}{f(x)}\frac{\prod_{i=2}^{n} f^{1/p_i}(x_i)}{f^{\sum_{i=2}^{n} 1/p_i}(x)}\right] \\ &- \frac{1}{p_1}\left[\frac{1}{p_1}\left(\frac{\prod_{i=2}^{n} x_i^{1/p_i}}{x^{\sum_{i=2}^{n} 1/p_i}}\right)^2 f''\left(x^{1/p_1}\prod_{i=2}^{n} x_i^{1/p_i}\right) - \frac{\prod_{i=2}^{n} x_i^{1/p_i}}{x^{\sum_{i=2}^{n} 1/p_i}}\frac{f'\left(x^{1/p_1}\prod_{i=2}^{n} x_i^{1/p_i}\right)}{x\left(\sum_{i=2}^{n} 1/p_i\right)^{-1}}\right] \end{aligned}$$

for all $x \in I - \{0\}$.

If $x = x_2 = ... = x_n$, then

$$\begin{aligned} F''(x) &= \frac{1}{p_1}\left[f''(x) - \frac{\left(\sum_{i=2}^{n} 1/p_i\right)(f'(x))^2}{f(x)}\right] - \frac{1}{p_1}\left[\frac{1}{p_1}f''(x) - \frac{f'(x)}{x\left(\sum_{i=2}^{n} 1/p_i\right)^{-1}}\right] \\ &= \frac{1}{p_1}\left[\left(1 - \frac{1}{p_1}\right)f''(x) - \frac{\left(\sum_{i=2}^{n} 1/p_i\right)(f'(x))^2}{f(x)} + \frac{\left(\sum_{i=2}^{n} 1/p_i\right)f'(x)}{x}\right] \\ &= \frac{1}{p_1}\left[\left(\sum_{i=2}^{n}\frac{1}{p_i}\right)f''(x) + \frac{\left(\sum_{i=2}^{n} 1/p_i\right)f'(x)}{x} - \frac{\left(\sum_{i=2}^{n} 1/p_i\right)(f'(x))^2}{f(x)}\right] \\ &= \frac{\left(\sum_{i=2}^{n} 1/p_i\right)}{p_1}\left[f''(x) + \frac{f'(x)}{x} - \frac{(f'(x))^2}{f(x)}\right] \\ &\ge 0. \end{aligned}$$

This implies that $F(x) \ge F(0) = 0$ for all $x \in I$.

Hence,

$$f^{1/p_1}(x)\prod_{i=2}^{n} f^{1/p_i}(x_i) \ge f\left(x^{1/p_1}\prod_{i=2}^{n} x_i^{1/p_i}\right)$$

for all $x, x_2, ..., x_n \in I$.

This proof is completed. □

3. Applications

Corollary 3.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $f(x) = e^{g(\log_e x)}$ for all $x > 0$. Then f is $(p_1, p_2, ..., p_n)$-Hölder on $\mathbb{R}^+$ for all $p_1, p_2, ..., p_n > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$.*

Proof. By the assumption, we have $g'' > 0$, so

$$f''(x) + \frac{f'(x)}{x} - \frac{(f'(x))^2}{f(x)} = \frac{e^{x \log_e x}}{x^2} g''(\log_e x) \geq 0$$

for all $x > 0$.

By Theorem 2.1, f is $(p_1, p_2, ..., p_n)$-Hölder on $\mathbb{R}^+$ for all $p_1, p_2, ..., p_n > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$. □

Corollary 3.2. *Let $Id(x) = x$ and $Exp(x) = e^x$ for all $x > 0$. Then Id and Exp are $(p_1, p_2, ..., p_n)$-Hölder on $\mathbb{R}^+$ for all $p_1, p_2, ..., p_n > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$.*

Proof. This follows from Corollary 3.1, where $g = Id$ or $g = Exp$, respectively. □

In [2], if f is a continuous function on $\mathbb{R}^+$ such that $f^{(k)} \geq 0$ for all $k \in \mathbb{N} \cup \{0\}$, then

$$f^{(k)}(x) = \int_0^\infty t^k e^{xt} d\sigma(t)$$

for all $x > 0$ and $k \in \mathbb{N} \cup \{0\}$, where σ is a bounded and non-decreasing function on $\mathbb{R}^+$ and the integral converges.

Corollary 3.3. *Let f be a continuous function on $\mathbb{R}^+$ such that $f^{(k)} \geq 0$ for all $k \in \mathbb{N} \cup \{0\}$. Then, for any $k \in \mathbb{N} \cup \{0\}$, $f^{(k)}$ is $(p_1, p_2, ..., p_n)$-Hölder on $\mathbb{R}^+$ for all $p_1, p_2, ..., p_n > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$.*

Proof. Define $F_k(x) = f^{(k)}(x)$ for all $x > 0$ and $k \in \mathbb{N} \cup \{0\}$. Then

$$F_k(x) = \int_0^\infty t^k e^{xt} d\sigma(t)$$

for all $x > 0$ and $k \in \mathbb{N} \cup \{0\}$.

For any $x > 0$ and $k \in \mathbb{N} \cup \{0\}$, we have

$$F_k'(x) = \int_0^\infty t^{k+1} e^{xt} d\sigma(t)$$

and

$$F_k''(x) = \int_0^\infty t^{k+2} e^{xt} d\sigma(t)$$

and, by the Hölder inequality,

$$\begin{aligned}(F_k'(x))^2 &= \left(\int_0^\infty t^{k+1}e^{xt}d\sigma(t)\right)^2\\ &= \left(\int_0^\infty t^{\frac{k}{2}+1}e^{\frac{xt}{2}}t^{\frac{k}{2}}e^{\frac{xt}{2}}d\sigma(t)\right)^2\\ &\le \left(\int_0^\infty t^{k+2}e^{xt}d\sigma(t)\right)\left(\int_0^\infty t^{k}e^{xt}d\sigma(t)\right)\\ &= \left(\int_0^\infty t^{k+2}e^{xt}d\sigma(t)\right)F_k(x)\end{aligned}$$

and then

$$\begin{aligned}F_k''(x)+\frac{F_k'(x)}{x}-\frac{(F_k'(x))^2}{F_k(x)} &= \int_0^\infty t^{k+2}e^{xt}d\sigma(t)+\frac{1}{x}\int_0^\infty t^{k+1}e^{xt}d\sigma(t)-\frac{(F_k'(x))^2}{F_k(x)}\\ &\ge \frac{1}{x}\int_0^\infty t^{k+1}e^{xt}d\sigma(t)\\ &\ge 0.\end{aligned}$$

By Theorem 2.1, F_k is $(p_1,p_2,...,p_n)$-Hölder on $\mathbb{R}^+$ for all $p_1,p_2,...,p_n>1$, $\sum_{i=1}^n \frac{1}{p_i}=1$.

This proof is completed. □

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