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YOUNG'S CONVOLUTION INEQUALITIES FOR WEIGHTED MIXED (QUASI-) NORM SPACES.

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Abstract. Young's Convolution Inequality is extended to several cases of discrete, semi-discrete and continuous convolution of sequences and functions that belong to weighted mixed quasi-norm spaces and amalgam spaces.

1. Introduction

Convolution relations play a central role in the study of the Wiener-type spaces. Under the assumption that $1 \leq p, q, r \leq \infty$, Young's Convolution Inequality states that for both the function and the sequence spaces,

$$||f * g||_r \leq C(p,q)||f||_p \cdot ||g||_q \tag{1.1}$$

if and only if

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{1.2}$$

Young's Inequality allows for various generalizations, particularly for mixed-norm spaces. For example,

$$||F * G||_{r_1,r_2} \leq C(p,q) \cdot ||F||_{p_1,p_2} \cdot ||G||_{q_1,q_2} \tag{1.3}$$

if and only if

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}, \quad i \in \{1,2\}, \tag{1.4}$$

where the norm$||F||_{p_1,p_2}$ is defined by

$$||F||_{p_1,p_2} = \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \Big| F(x,\omega) \Big|^{p_1} dx \Big)^{p_2/p_1} d\omega \Big)^{1/p_2}.$$

When $0 < p < 1$, the inequality (1.1) is not valid. Moreover, in this case, $f \in L^p$ does not imply local integrability of f and, therefore, $f * g$ is not even defined. However, in this case the sequence space ℓ^p enjoyes the inequality

$$||f * g||_{\ell^p} \leq ||f||_{\ell^p} \cdot ||g||_{\ell^p}, \tag{1.5}$$

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which means that is a quasi-Banach algebra under convolution. Moreover, if we (formally) define the semi-discrete convolution of a sequence $a \in \ell^p$ and a function $f \in L^p$ by $(a *' f)(x) = \sum_{k\in\mathbb{Z}^d} a(k)f(x-k)$, we have the convolution inequality

$$||a *' f||_{L^p} \leq ||a||_{\ell^p} \cdot ||f||_{L^p}. \tag{1.6}$$

In this paper we investigate the question of how the convolution inequatlities (1.5) and (1.6) can be extended to the case of weighted mixed (quasi-)norm sequence and function spaces and derive several generalized versions of the inequality (1.3). Several particular cases of our results have been used in time-frequency analysis, particulary in the theory of modulation spaces (for example, in [4, 7]) . The case $0 < p \leq 1$ is of particular interest there because then the modulation space (quasi-)norm provides a flexible lower bound in a class of uncertaincy principles [3].

2. Definitions and Preliminary Results

When working with convolution inequalities, it is convenient to use moderate weights on $\mathbb{R}^{2d}$ and $\mathbb{Z}^{2d}$ defined as follows:

Definition 1. *Let m be a weight function on $\mathbb{R}^{2d}$ (or on $\mathbb{Z}^{2d}$). We call it ν-moderate if $m(z_1+z_2) \leq C\nu(z_1)m(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$ (or for all $z_1, z_2 \in \mathbb{Z}^{2d}$).*

We will measure the decay and and integrability of functions on the $\mathbb{R}^{2d}$ plane by means of weighted mixed (quasi-)norms as follows:

Definition 2 (weighted mixed-norm spaces)**.** *Let m be a weight function on $\mathbb{R}^{2d}$ and let $0 < p, q < \infty$. Then the weighted mixed (quasi-)norm space $L^{p,q}_m(\mathbb{R}^{2d})$ consists of all Lebesgue measurable functions on $\mathbb{R}^{2d}$, such that the (quasi-)norm*

$$||F||_{L^{p,q}_m} = \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \Big|F(x,\omega)\Big|^p m(x,\omega)^p dx \Big)^{q/p} d\omega \Big)^{1/q}$$

is finite. If $p = \infty$ of $q = \infty$, then the corresponding p-norm is replaced by the essential supremum.

$L^{p,q}_m$ is a translation-invariant quasi-Banach space (Banach space if both $p \geq 1$ and $q \geq 1$). The theory of mixed-norm spaces is developed in [1].

We will measure the decay and summability of sequences by means of diiscrete mixed (quasi-)norms as follows:

Definition 3 (discrete mixed (quasi-)norm spaces)**.** *The space $\ell^{p,q}_m(\mathbb{Z}^{2d})$ consists of all sequences $a = (a_{kn})_{k,n\in\mathbb{Z}^d}$ for which the (quasi-)norm*

$$||a||_{\ell^{p,q}_m} = \Big(\sum_{n\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} |a_{kn}|^p m(k,n)^p \Big)^{q/p} \Big)^{1/q}$$

is finite. If $p = \infty$ of $q = \infty$, then the corresponding p-norm is replaced by the supremum.

The most basic tool we will use when working with sequence spaces is the following variation of Young's Inequality:

Lemma 2.1 (Young's Inequality)**.** *Assume that m is a ν-moderate weight on $\mathbb{Z}^d$, $0 < p \leq \infty$, and $r = \min\{1, p\}$. Then for all $a \in \ell^r_\nu$ and all $b \in \ell^p_m$,*

$$||a * b||_{\ell^p_m} \leq C||a||_{\ell^r_\nu}||b||_{\ell^p_m}, \tag{2.1}$$

where C is independent of p, a and b. If $m \equiv \nu \equiv 1$, then $C = 1$.

Proof. In the case $1 \leq p \leq \infty$ the result is well known. If $0 < p < 1$, we use the inclusion $\ell^p \hookrightarrow \ell^1$ and the fact that $m(n) \leq C\nu(k)m(n-k)$ for all $k, n \in \mathbb{Z}^d$ to obtain

$$\begin{aligned} ||a * b||_{\ell^p_m} &= \Big(\sum_{n\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} a(k)b(n-k)\Big|^p m(n)^p \Big)^{1/p} \\ &\leq C\Big(\sum_{n\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |a(k)b(n-k)|^p \, \nu(k)^p m(n-k)^p \Big)^{1/p} \\ &= C||a||_{\ell^r_\nu} ||b||_{\ell^p_m}. \end{aligned}$$

as desired. □

In order to describe the space of functions defined by local integrability and global summability, we define the so-called Wiener-type amalgam spacesas follows:

Definition 4. *Assume that m is a weight function on $\mathbb{Z}^{2d}$ and that $0 < p, q \leq \infty$. Denote $Q = [0,1]^{2d}$ and $\chi = \chi_Q$. The Wiener-type amalgam space $W(L^p, L^{q,r}_m)$ consists of all measurable functions f for which the (quasi-)norm*

$$||f||_{W(L^p,L^{q,r}_m)} = \Big(\sum_{n\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} ||f \cdot T_{(k,n)}\chi||^q_{L^p} m(k,n)^q \Big)^{r/q} \Big)^{1/r},$$

where $T_{(k,n)}$ is the $2d-$dimensional translation, is finite. In particular, $W(L^\infty, L^{p,q}_m)$ consists of all measurable functions F on $\mathbb{R}^{2d}$, so that the sequence of local suprema

$$a_{kn} = \operatorname*{ess\,sup}_{x,\omega\in[0,1]^d} |F(x+k, \omega+n)| = ||F \cdot T_{(k,n)}\chi||_\infty,$$

where $\chi = \chi_{[0,1]^{2d}}$, belongs to $\ell^{p,q}_m$. The (quasi-)norm on $W(L^{p,q}_m)$ is

$$||F||_{W(L^{p,q}_m)} = ||a||_{\ell^{p,q}_m}$$

We also use the notation $W(L^{p,q}_m) = W(L^\infty, L^{p,q}_m)$.

Remark 5. *We note the following inclusion relation for Wiener-type spaces: If $0 < p_1 \leq p_2 \leq \infty$, then $W(L^{p_2}, L^{q,r}_m) \subseteq W(L^{p_1}, L^{q,r}_m)$. In particular, $W(L^{q,r}_m) \subseteq W(L^1, L^{q,r}_m)$.*

Remark 6. *Given $\alpha, \beta > 0$, an equivalent quasi-norm on $W(L^{p,q}_m)$ is defined by $||F \cdot T_{(k\alpha,n\beta)}\chi_{Q_\alpha \times Q_\beta}||_{\ell^{p,q}_{\tilde{m}}}$, where $Q_\alpha = [0,\alpha]$, $Q_\beta = [0,\beta]$, and $\tilde{m}(k,n) = m(n\alpha, n\beta)$.*

In addition to the continuous convolution of functions and the discrete convolution of sequences, we will consider the semi-discrete convolution of a sequence and a function defined as follows:

Definition 7 (Semi-discrete convolution)**.** *Given the lattice parameters $\alpha, \beta > 0$, formally define the semi-discrete convolution of a complex-valued sequence $a = \big(a(k,n) : (k,n) \in \mathbb{Z}^{2d}\big)$ and a continuous function F on $\mathbb{R}^{2d}$, by*

$$(a *' F)(x,\omega) = \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} a(k,n)F(x-k\alpha, \omega-n\beta) = \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} a(k,n) \cdot T_{k\alpha,n\beta}F(x,\omega),$$

where $T_{k\alpha,n\beta}F(x,\omega)$ is the $2d$-dimensional translation of F.

3. Main Results.

Our first result is a generalization the inequality (1.3) for sequence spaces for the case $0 < p_j, q_j, r_j \leq \infty$. We establish that under a certain range of parameters the generalization is exact, while for the remaining case it is almost exact (up to the interchange of the indices and the order of summation).

Theorem 3.1. *Assume that m is a ν-moderate weight on $\mathbb{Z}^{2d}$. Also assume that*

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i} \quad \textit{for } 1 \leq r_i \leq \infty \tag{3.1}$$

and

$$p_i = q_i = r_i \ \textit{ for } \ 0 < r_i < 1.$$

(a). If either $1 \leq r_1 \leq \infty$ or $0 < r_2 \leq r_1 < 1$, then

$$||a * b||_{\ell_m^{r_1,r_2}} \leq C||a||_{\ell_m^{p_1,p_2}} ||b||_{\ell_\nu^{q_1,q_2}}. \tag{3.2}$$

(b). If $0 < r_1 \leq r_2 \leq \infty$ then

$$||a * b||_{\ell_m^{r_1,r_2}} \leq C||a||_{\ell_m^{p_1,p_2}} ||b||_{\ell_\nu^{q_1,q_1}} \tag{3.3}$$

and

$$||a * b||_{\ell_m^{r_1,r_2}} \leq C||Ua||_{\ell_{Um}^{p_2,p_1}} ||Ub||_{\ell_{U\nu}^{q_2,q_1}}, \tag{3.4}$$

where $Ua(k,n) = a(n,k)$, $Um(k,n) = m(n,k)$, $U\nu(k,n) = \nu(n,k)$and, therefore,

$$||Ua||_{\ell_m^{p_2,p_1}} = \Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{n\in\mathbb{Z}^d} |a(k,n)|^{p_2} m(k,n)^{p_2} \Big)^{p_1/p_2} \Big)^{1/p_1}.$$

Proof. (a) Since m is ν-moderate,

$$\begin{aligned} |(a*b)(l,n)| \cdot m(l,n) &\leq C \sum_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |a(l-j,n-k) \cdot b(j,k)| \cdot \nu(l-j,n-k) m(j,k) \\ &= C \sum_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \tilde{a}_{n-k}(l-j) \cdot \tilde{b}_k(j) \\ &= C \sum_{k\in\mathbb{Z}^d} (\tilde{a}_{n-k} * \tilde{b}_k)(l), \end{aligned} \tag{3.5}$$

where $\tilde{a}_k(j) = |a(j,k) \cdot m(j,k)|$ and $\tilde{b}_k(j) = |b(j,k) \cdot \nu(j,k)|$. We can therefore estimate the $\ell_m^{r_1,r_2}$-norm of $a * b$ by

$$\begin{aligned} ||a*b||_{\ell_m^{r_1,r_2}} &= C\Big(\sum_{n\in\mathbb{Z}^d} \Big(\sum_{l\in\mathbb{Z}^d} |(a*b)(l,n) \cdot m(l,n)|^{r_1} \Big)^{r_2/r_1} \Big)^{1/r_2} \\ &\leq C\Big(\sum_{n\in\mathbb{Z}^d} \Big(\sum_{l\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} (\tilde{a}_{n-k} * \tilde{b}_k)(l) \Big|^{r_1} \Big)^{r_2/r_1} \Big)^{1/r_2} \end{aligned} \tag{3.6}$$

and consider the following cases:

Case 1 ($1 \le r_1 \le \infty$, $0 < r_2 \le \infty$). We use Minkowsky's Inequality and Young's Inequality to estimate the inner double sum in (3.6) by

$$\Big(\sum_{l\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d}(\tilde{a}_{n-k}*\tilde{b}_k)(l)\Big|^{r_1}\Big)^{1/r_1} \le C\sum_{k\in\mathbb{Z}^d}||\tilde{a}_{n-k}*\tilde{b}_k)||_{r_1} \le C\sum_{k\in\mathbb{Z}^d}||\tilde{a}_{n-k}||_{p_1}\cdot||\tilde{b}_k)||_{q_1} = C\cdot(A*B)(n), \tag{3.7}$$

where we denoted $A(k) = ||\tilde{a}_k||_{p_1}$ and $B(k) = ||\tilde{b}_k)||_{q_1}$. By substituting (3.7) into (3.6) and using Young's Inequality, we obtain

$$\begin{aligned}||a*b||_{\ell_m^{r_1,r_2}} &\le C\Big(\sum_{n\in\mathbb{Z}^d}|(A*B)(n)|^{r_2}\Big)^{1/r_2}\\ &= C||A||_{p_2}\cdot||B||_{q_2} = C||a||_{\ell_m^{p_1,p_2}}\cdot||b||_{\ell_\nu^{q_1,q_2}}.\end{aligned}$$

Case 2 ($0 < r_2 < r_1 < 1$). We use Young's Inequality and interchange the order of summation to estimate the inner double sum in (3.6) by

$$\begin{aligned}\sum_{l\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d}(\tilde{a}_{n-k}*\tilde{b}_k)(l)\Big|^{r_1} &\le \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}|(\tilde{a}_{n-k}*\tilde{b}_k)(l)|^{r_1}\\ &= \sum_{k\in\mathbb{Z}^d}||\tilde{a}_{n-k}*\tilde{b}_k)||_{r_1}^{r_1}\\ &\le \sum_{k\in\mathbb{Z}^d}||\tilde{a}_{n-k}||_{r_1}^{r_1}\cdot||\tilde{b}_k||_{r_1}^{r_1} = (A*B)(n),\end{aligned} \tag{3.8}$$

where we denoted $A(k) = ||\tilde{a}_k||_{r_1}^{r_1}$ and $B(n) = ||\tilde{b}_k||_{r_1}^{r_1}$. By substituting (3.8) into (3.6) and using Young's Inequality, we obtain

$$\begin{aligned}||a*b||_{\ell_m^{r_1,r_2}} &\le C\Big(\sum_{n\in\mathbb{Z}^d}|(A*B)(n)|^{r_2/r_1}\Big)^{1/r_2}\\ &= C(||A||_{r_2/r_1}\cdot||B||_{r_2/r_1})^{1/r_1} = C||a||_{\ell_m^{p_1,p_2}}\cdot||b||_{\ell_\nu^{q_1,q_2}}.\end{aligned}$$

(b) Case 3 ($0 < r_1 < 1$ and $r_1 < r_2 \le \infty$). We again substitute (3.8) into (3.6) and this time use Minkowsky's Inequality with the exponent $r_2/r_1 \ge 1$ to obtain

$$\begin{aligned}||a*b||_{\ell_m^{r_1,r_2}} &\le C\Big(\sum_{n\in\mathbb{Z}^d}\Big(\sum_{k\in\mathbb{Z}^d}||\tilde{a}_{n-k}||_{r_1}^{r_1}\cdot||\tilde{b}_k||_{r_1}^{r_1}\Big)^{r_2/r_1}\Big)^{1/r_2}\\ &\le C\Big(\sum_{k\in\mathbb{Z}^d}||\tilde{b}_k||_{r_1}^{r_1}\Big(\sum_{n\in\mathbb{Z}^d}||\tilde{a}_{n-k}||_{r_1}^{r_2}\cdot\Big)^{r_1/r_2}\Big)^{1/r_1}\\ &= C||a||_{\ell_m^{p_1,p_2}}\cdot||b||_{\ell_\nu^{q_1,q_1}}.\end{aligned}$$

In order to derive (3.4), we redefine $\tilde{a}_j(k) = |a(j,k)\cdot m(j,k)|$ and $\tilde{b}_j(k) = |b(j,k)\cdot \nu(j,k)|$ and estimate $a*b$ by

$$\begin{aligned}|(a*b)(l,n)|\cdot m(l,n) &\le C\sum_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}|a(l-j,n-k)\cdot b(j,k)|\cdot\nu(l-j,n-k)m(j,k)\\ &= C\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\tilde{a}_{l-j}(n-k)\cdot\tilde{b}_j(k)\\ &= C\sum_{j\in\mathbb{Z}^d}(\tilde{a}_{l-j}*\tilde{b}_l)(n). \end{aligned}\tag{3.9}$$

Next, applying Minkowsky's Inequality with the exponent $r_2/r_1\ge 1$ and substituting (3.9) we obtain

$$\begin{aligned}||a*b||_{\ell_m^{r_1,r_2}} &= C\Big(\sum_{n\in\mathbb{Z}^d}\Big(\sum_{l\in\mathbb{Z}^d}|(a*b)(l,n)\cdot m(l,n)|^{r_1}\Big)^{r_2/r_1}\Big)^{1/r_2}\\ &\le C\Big(\sum_{l\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d}|(a*b)(l,n)\cdot m(l,n)|^{r_2}\Big)^{r_1/r_2}\Big)^{1/r_1}\\ &\le C\Big(\sum_{l\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d}\Big|\sum_{j\in\mathbb{Z}^d}(\tilde{a}_{l-j}*\tilde{b}_j)(n)\Big|^{r_2}\Big)^{r_1/r_2}\Big)^{1/r_1} \end{aligned}\tag{3.10}$$

and consider the following two cases.
Case 1 ($0<r_1<r_2<1$). We use Young's Inequality and change the order of summation to estimate the inner double sum in (3.9) by

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d}\Big|\sum_{j\in\mathbb{Z}^d}(\tilde{a}_{l-j}*\tilde{b}_j)(n)\Big|^{r_2} &\le \sum_{j\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|(\tilde{a}_{l-j}*\tilde{b}_j)(n)|^{r_2}\\ &\le \sum_{j\in\mathbb{Z}^d}||(\tilde{a}_{l-j}||_{p_2}^{r_2}\cdot||\tilde{b}_j||_{q_2}^{r_2} = (A*B)(l), \end{aligned}\tag{3.11}$$

where we denoted $A(j) = ||(\tilde{a}_j||_{p_2}^{r_2}$ and $B(j) = ||\tilde{b}_j||_{q_2}^{r_2}$. Substituting (3.11) into (3.10) we estimate

$$\begin{aligned}||a*b||_{\ell_m^{r_1,r_2}} &\le C\Big(\sum_{l\in\mathbb{Z}^d}\Big(|(A*B)(l)|^{r_1/r_2}\Big)^{1/r_1}\\ &= C\cdot||A*B||_{r_1/r_2}^{1/r_2}\\ &\le C\cdot||A||_{r_1/r_2}^{1/r_2}\cdot||B||_{r_1/r_2}^{1/r_2} = C||Ua||_{\ell_m^{r_2,r_1}}||Ub||_{\ell_\nu^{r_2,r_1}}. \end{aligned}$$

Case 2 ($1\le r_2\le\infty$). We use Minkowsky's Inequality to estimate the inner double sum in (3.10) by

$$\begin{aligned}\Big(\sum_{n\in\mathbb{Z}^d}\Big|\sum_{j\in\mathbb{Z}^d}(\tilde{a}_{l-j}*\tilde{b}_j)(n)\Big|^{r_2}\Big)^{1/r_2} &\le C\sum_{j\in\mathbb{Z}^d}||\tilde{a}_{l-j}*\tilde{b}_j)(n)||_{r_2}\\ &\le C\sum_{j\in\mathbb{Z}^d}||\tilde{a}_{l-j}||_{p_2}\cdot||\tilde{b}_j||_{q_2} = C\cdot(A*B)(l), \end{aligned}\tag{3.12}$$

where we denoted $A(j) = ||\tilde{a}_j||_{p_2}$ and $B(j) = ||\tilde{b}_j||_{q_2}$. Substituting (3.12) into (3.10) we obtain

$$\begin{aligned} ||a * b||_{\ell_m^{r_1,r_2}} &\leq C \cdot \Big(\sum_{l \in \mathbb{Z}^d} |(A * B)(l)|^{r_1} \Big)^{1/r_1} \\ &\leq C||A||_{p_1}||B||_{q_2} = C||Ua||_{\ell_m^{p_2,p_1}}||Ub||_{\ell_\nu^{q_2,q_1}}. \end{aligned}$$

The theorem is proved completely. □

Convolution relations for general Wiener-type amalgam spaces were first proved in [2]. We refer the reader to [6] for an introduction to the theory of $W(L^p, L^q_m)$. With the help of Theorem 3.1, we can prove the following result:

Corollary 3.1. *Denote $X = L^{\lambda_1,\lambda_2}$, $Y = L^{\mu_1,\mu_2}$, and $Z = L^{\sigma_1,\sigma_2}$. Choose the values for λ_j, μ_j and $\sigma_j, j \in \{1,2\}$ so that $X * Y \hookrightarrow Z$. Assume that m is a ν-moderate weight on $\mathbb{Z}^{2d}$, and that*

$$p_i = q_i = r_i \quad \textit{for} \quad 0 < r_i < 1$$

and

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i} \quad \textit{for} \quad r_i \geq 1.$$

(a). If either $1 \leq r_1 \leq \infty$ and $0 < r_2 \leq \infty$, or $0 < r_2 \leq r_1 < 1$, then

$$||F * G||_{W\left(Z, L_m^{r_1,r_2}\right)} \leq C||F||_{W(X, L_\nu^{p_1,p_2})}||G||_{W(Y, L_m^{q_1,q_2})}. \tag{3.13}$$

(b). If $0 < r_1 \leq r_2 \leq \infty$ then

$$||F * G||_{W\left(Z, L_m^{r_1,r_2}\right)} \leq C||F||_{W(X, L_\nu^{p_1,p_2})}||G||_{W(Y, L_m^{q_1,q_1})} \tag{3.14}$$

and

$$||F * G||_{W\left(Z, L_m^{r_1,r_2}\right)} \leq C||Ua||_{\ell_{U\nu}^{p_2,p_1}}||Ub||_{\ell_{Um}^{q_2,q_1}}, \tag{3.15}$$

where $Ua(k,n) = a(n,k)$, $Ub(k,n) = b(n,k)$, $a_{kn} = ||F \cdot T_{(k,n)}\chi_{[0,1]^{2d}}||_X$, and $b_{kn} = ||G \cdot T_{(k,n)}\chi_{[0,1]^{2d}}||_Y$.

Proof. A particular case of Corrolary 3.1 with $p_j, q_j = r_j \geq 1$ is given in [6, Section 11.8]. The proof of the general case follows the same steps, relies on Theorem 3.1, and does not contain any new ideas. □

Remark 8. *Several particular cases of Corollary 3.1 have been used extensively in time-frequency analysis, particulary in the theory of modulation spaces. We refere the reader to* [4] *and* [7]. *For a comprehensive introduction to time-frequency analysis we refer to* [5].

When $0 < p < 1$, $f * g$ is not even defined in general because, in this case, $f \in L^p$ does not imply local integrability of f . However, under additional assumptions (for example, when f and g are assumed to be band-limited), we are able to extend Young's inequality even in the case of $0 < p < 1$.

Corollary 3.2. *Suppose that $0 < \lambda < 1$, Γ, Γ' are compact subsents of $\mathbb{R}^{2d}$ and that $supp(\hat{F}) \in \Gamma$ and $supp(\hat{G}) \in \Gamma'$. Assume that m is a ν-moderate weight on $\mathbb{Z}^{2d}$, and that the exponents p_j, q_j, and r_j satisfy the conditions of Theorem 3.1.*
(a). If either $1 \leq r_1 \leq \infty$ and $0 < r_2 \leq \infty$, or $0 < r_2 \leq r_1 < 1$, then

$$||F * G||_{W\left(L^\lambda, L_m^{r_1,r_2}\right)} \leq C||F||_{W(L^\lambda, L_\nu^{p_1,p_2})}||G||_{W(L^\lambda, L_m^{q_1,q_2})}. \tag{3.16}$$

(b). If $0 < r_1 \le r_2 \le \infty$ *then*

$$||F * G||_{W\left(L^\lambda, L_m^{r_1,r_2}\right)} \le C||F||_{W(L^\lambda, L_\nu^{p_1,p_2})}||G||_{W(L^\lambda, L_m^{q_1,q_1})} \tag{3.17}$$

and

$$||F * G||_{W\left(L^\lambda, L_m^{r_1,r_2}\right)} \le C||Ua||_{\ell_{U\nu}^{p_2,p_1}}||Ub||_{\ell_{Um}^{q_2,q_1}}, \tag{3.18}$$

where $Ua(k,n) = a(n,k)$, $Ub(k,n) = b(n,k)$, $a_{kn} = ||F \cdot T_{(k,n)}\chi_{[0,1]^{2d}}||_{L^\lambda}$, *and* $b_{kn} = ||G \cdot T_{(k,n)}\chi_{[0,1]^{2d}}||_{L^\lambda}$.

Proof. The proof relies on Theorem 3.1, on a convolution inequality for band-limited functions which can be found in [7, Lemma 2.6], and follows the steps of [6, Theorem 11.8.3]. □

We next establish several versions of Young's inequality for the semi-discrete convolution $a *' F$. In view of Remark 6, for $F \in W(L_m^{q_1,q_2})$, replacing the function F with the step function defined by the sequence of the local suprema of F, yields the following corrolary to Theorem 3.1:

Corollary 3.3. *Assume that* m *is* ν*-moderate, and that the exponents* p_j, q_j, *and* r_j *satisfy the conditions of Theorem 3.1.*
(a). If either $1 \le r_1 \le \infty$ *and* $0 < r_2 \le \infty$, *or* $0 < r_2 \le r_1 < 1$, *then*

$$||a *' F||_{W(L_m^{r_1,r_2})} \le C||a||_{\ell_\nu^{p_1,p_2}}||F||_{W(L_m^{q_1,q_2})}. \tag{3.19}$$

(b). If $0 < r_1 \le r_2 \le \infty$ *then*

$$||a *' F||_{W(L_m^{r_1,r_2})} \le C||a||_{\ell_\nu^{p_1,p_2}}||F||_{W(L_m^{q_1,q_1})} \tag{3.20}$$

and

$$||a *' F||_{W(L_m^{r_1,r_2})} \le C||Ua||_{\ell_{U\nu}^{p_2,p_1}}||Ub||_{\ell_{Um}^{q_2,q_1}}, \tag{3.21}$$

where $Ua(k,n) = a(n,k)$, $Ub(k,n) = b(n,k)$, *and* $b_{kn} = ||F \cdot T_{(k,n)}\chi_{[0,1]^{2d}}||_\infty$.

If we drop the assumptionthat $F \in W(L_m^{q_1,q_2})$ and only assume that $F \in L_m^{q_1,q_2}$, we are still able to generalize Young's Inequality for the semi-direct convolution as follows:

Theorem 3.2. *Assume that* m *is* ν*-moderate,* $0 < p, q \le \infty$, *and* $\alpha, \beta > 0$.
Let $r = \min\{p, 1\}$, $t = \min\{q, 1\}$, *and* $s = \min\{p, q, 1\}$. *If* $F \in L_m^{p,q}$ *and* $a \in \ell_{\tilde\nu}^{r,s}$, *where* $\tilde\nu(k,n) = \nu(k\alpha, n\beta)$, *then* $a *' F \in L_m^{p,q}$ *and*

$$||a *' F||_{L_m^{p,q}} \le C||a||_{\ell_{\tilde\nu}^{r,s}}||F||_{L_m^{p,q}}. \tag{3.22}$$

Moreover, in the case $0 < p \le q \le \infty$, *the inequality*

$$||a *' F||_{L_m^{p,q}} \le C||Ua||_{\ell_{U\tilde\nu}^{t,r}}||UF||_{L_{Um}^{q,p}} \tag{3.23}$$

also holds.

Proof. We denote $G = a *' F$, and use a periodization idea to write

$$||G||_{L_m^{p,q}} = \Big(\int_{[0,\beta]^d} \sum_{l\in\mathbb{Z}^d} \Big(\int_{[0,\alpha]^d} \sum_{j\in\mathbb{Z}^d} |G(x + j\alpha, \omega + l\beta)|^p\, m(x+j\alpha, \omega+l\beta)^p dx\Big)^{q/p} d\omega\Big)^{1/q}. \tag{3.24}$$

$$|G(x+j\alpha,\omega+l\beta)|\cdot m(x+j\alpha,\omega+l\beta)\leq C\sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\tilde{a}_n(k)\tilde{F}_{x,\omega,l-n}(j-k)$$
$$=C\sum_{n\in\mathbb{Z}^d}(\tilde{a}_n*\tilde{F}_{x,\omega,l-n})(j), \tag{3.25}$$

where we denoted $\tilde{a}_n(k)=|a(k,n)|\cdot\tilde{\nu}(k,n)$ and $\tilde{F}_{x,\omega,n}(k)=|F(x+k\alpha,\omega+n\beta)|\cdot m(x+k\alpha,\omega+n\beta)$. We next use (3.25) to estimate the $L^{p,q}_m$-norm of G by

$$||G||_{L^{p,q}_m}\leq C\Big(\int_{[0,\beta]^d}\sum_{l\in\mathbb{Z}^d}\Big(\int_{[0,\alpha]^d}\sum_{j\in\mathbb{Z}^d}\Big|\sum_{n\in\mathbb{Z}^d}(\tilde{a}_n*\tilde{F}_{x,\omega,l-n})(j)\Big|^p dx\Big)^{q/p}d\omega\Big)^{1/q} \tag{3.26}$$

and consider the following two cases:
Case 1: $p\geq 1$. We use Minkowski's Inequality and Young's Inequality to estimate the inner integrand in (3.26) by

$$\sum_{j\in\mathbb{Z}^d}\Big|\sum_{n\in\mathbb{Z}^d}(\tilde{a}_n*\tilde{F}_{x,\omega,l-n})(j)\Big|^p\leq\Big(\sum_{n\in\mathbb{Z}^d}||\tilde{a}_n*\tilde{F}_{x,\omega,l-n}||_{\ell^p}\Big)^p$$
$$\leq\Big(\sum_{n\in\mathbb{Z}^d}||\tilde{a}_n||_{\ell^r}||\tilde{F}_{x,\omega,l-n}||_{\ell^p}\Big)^p. \tag{3.27}$$

We next use (3.27) and Minkowski's Inequality to estimate the inner integral in (3.26) by

$$\Big(\int_{[0,\alpha]^d}\sum_{j\in\mathbb{Z}^d}\Big|\sum_{n\in\mathbb{Z}^d}(\tilde{a}_n*\tilde{F}_{x,\omega,l-n})(j)\Big|^p dx\Big)^{1/p}\leq\Big(\int_{[0,\alpha]^d}\Big(\sum_{n\in\mathbb{Z}^d}||\tilde{a}_n||_{\ell^r}||\tilde{F}_{x,\omega,l-n}||_{\ell^p}\Big)^p dx\Big)^{1/p}$$
$$\leq\sum_{n\in\mathbb{Z}^d}||\tilde{a}_n||_{\ell^r}\Big(\int_{[0,\alpha]^d}||\tilde{F}_{x,\omega,l-n}||^p_{\ell^p}dx\Big)^{1/p}$$
$$=\sum_{n\in\mathbb{Z}^d}A(n)B_\omega(l-n)=(A*B_\omega)(l), \tag{3.28}$$

where we denoted $A(n)=||\tilde{a}_n||_{\ell^r}$ and $B_\omega(n)=\big(\int_{[0,\alpha]^d}||\tilde{F}_{x,\omega,n}||^p_{\ell^p}dx\big)^{1/p}$.
Substituting (3.28) into (3.26) and using Young's Inequality, we obtain

$$||G||_{L^{p,q}_m}\leq C\Big(\int_{[0,\beta]^d}||A*B_\omega||^q_{\ell^q}d\omega\Big)^{1/q}$$
$$\leq C||A||_{\ell^s}\Big(\int_{[0,\beta]^d}||B_\omega||^q_{\ell^q}d\omega\Big)^{1/q}$$
$$=C||a||_{\ell^{r,s}_{\tilde{\nu}}}\cdot||F||_{L^{p,q}_m}.$$

Case 2: $p<1$. We use the inclusion $\ell^p\hookrightarrow\ell^1$ and Young's Inequality to estimate the double sum in (3.26) by

$$\sum_{j\in\mathbb{Z}^d}\Big|\sum_{n\in\mathbb{Z}^d}(\tilde{a}_n*\tilde{F}_{x,\omega,l-n})(j)\Big|^p\leq\sum_{n\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}\big|(\tilde{a}_n*\tilde{F}_{x,\omega,l-n})(j)\big|^p$$
$$\leq\sum_{n\in\mathbb{Z}^d}||\tilde{a}_n||^p_{\ell^r}||\tilde{F}_{x,\omega l-n}||^p_{\ell^p}. \tag{3.29}$$

Using (3.29), we estimate the inner integral in (3.26) by

$$\begin{aligned}\int_{[0,\alpha]^d}\sum_{j\in\mathbb{Z}^d}\Big|\sum_{n\in\mathbb{Z}^d}(\tilde{a}_n*\tilde{F}_{x,\omega,l-n})(j)\Big|^p dx &\le \int_{[0,\alpha]^d}\sum_{n\in\mathbb{Z}^d}||\tilde{a}_n||^p_{\ell^r}||\tilde{F}_{x,\omega,l-n}||^p_{\ell^p}dx\\ &= \sum_{n\in\mathbb{Z}^d}||\tilde{a}_n||^p_{\ell^r}\int_{[0,\alpha]^d}||\tilde{F}_{x,\omega,l-n}||^p_{\ell^p}dx\\ &= \sum_{n\in\mathbb{Z}^d}A(n)B_\omega(l-n) = (A*B_\omega)(l),\end{aligned} \tag{3.30}$$

where we denoted $A(n)=||\tilde{a}_n||^p_{\ell^r}$ and $B_\omega(n)=\int_{[0,\alpha]^d}||\tilde{F}_{x,\omega,n}||^p_{\ell^p}dx$.
Substituting (3.30) into (3.26), we obtain

$$\begin{aligned}||G||_{L^{p,q}_m} &\le C\Big(\int_{[0,\beta]^d}||A*B_\omega||^{q/p}_{\ell^{p/q}}d\omega\Big)^{1/q}\\ &= C||A||^{1/p}_{\ell^{\min\{1,\frac{q}{p}\}}}\Big(\int_{[0,\beta]^d}||B_\omega||^{q/p}_{\ell^{p/q}}d\omega\Big)^{1/q}.\\ &= C||a||_{\ell^{r,s}_\nu}||F||_{L^{p,q}_m}.\end{aligned}$$

In order to derive (3.23), we use Minkowsky's Inequality with the exponent $q/p\ge 1$ and the periodization idea to estimate $||G||_{L^{p,q}_m}$ by

$$\begin{aligned}||G||_{L^{p,q}_m} &\le \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|G(x,\omega)|^q\,m(x,\omega)^q d\omega\Big)^{p/q}dx\Big)^{1/p}\\ &= \Big(\int_{[0,\alpha]^d}\sum_{j\in\mathbb{Z}^d}\Big(\int_{[0,\beta]^d}\sum_{l\in\mathbb{Z}^d}|G(x+j\alpha,\omega+l\beta)|^q\,m(x+j\alpha,\omega+l\beta)^q d\omega\Big)^{p/q}dx\Big)^{1/p}.\end{aligned} \tag{3.31}$$

Since m is ν-moderate, we can estimate the inner summand in (3.31) by

$$\begin{aligned}|G(x+j\alpha,\omega+l\beta)|\cdot m(x+j\alpha,\omega+l\beta) &\le C\sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\tilde{a}_k(n)\tilde{F}_{x,\omega,j-k}(l-n)\\ &= C\sum_{k\in\mathbb{Z}^d}(\tilde{a}_k*\tilde{F}_{x,\omega,j-k})(l),\end{aligned} \tag{3.32}$$

where we denoted $\tilde{a}_k(n)=|a(k,n)|\cdot\tilde{\nu}(k,n)$ and $\tilde{F}_{x,\omega,k}(n)=|F(x+k\alpha,\omega+n\beta)|\cdot m(x+k\alpha,\omega+n\beta)$. We next use (3.32) to estimate the $L^{p,q}_m$-norm of G by

$$||G||_{L^{p,q}_m}\le C\Big(\int_{[0,\alpha]^d}\sum_{j\in\mathbb{Z}^d}\Big(\int_{[0,\beta]^d}\sum_{l\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d}(\tilde{a}_k*\tilde{F}_{x,\omega,j-k})(l)\Big|^q d\omega\Big)^{p/q}dx\Big)^{1/p} \tag{3.33}$$

and consider the following two cases:
Case 1 ($0<p\le q<1$): We use the inclusion $\ell^q\hookrightarrow\ell^1$ and Young's Inequality to estimate the double sum in (3.33) by

$$\begin{aligned}\sum_{l\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d}(\tilde{a}_k*\tilde{F}_{x,\omega,j-k})(l)\Big|^q &\le \sum_{k\in\mathbb{Z}^d}||\tilde{a}_k*\tilde{F}_{x,\omega,j-k}||^q_{\ell^q}\\ &\le \sum_{k\in\mathbb{Z}^d}||\tilde{a}_k||^q_{\ell^q}\cdot||\tilde{F}_{x,\omega,j-k}||^q_{\ell^q}.\end{aligned} \tag{3.34}$$

Next, we use (3.34) to estimate the inner integral in (3.33) by

$$\int_{[0,\beta]^d}\sum_{l\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d}(\tilde{a}_k*\tilde{F}_{x,\omega,j-k})(l)\Big|^q d\omega \le \sum_{k\in\mathbb{Z}^d}||\tilde{a}_k||^q_{\ell^q}\int_{[0,\beta]^d}||\tilde{F}_{x,\omega,j-k}||^q_{\ell^q}d\omega$$
$$= (A*B_x)(j),$$

where $A(k) = ||\tilde{a}_k||^q_{\ell^q}$ and $B_x(k) = \int_{[0,\beta]^d}||\tilde{F}_{x,\omega,k}||^q_{\ell^q}d\omega$. Therefore, by Young's Inequality with $p/q<1$,

$$\begin{aligned}||G||_{L^{p,q}_m} &\le C\Big(\int_{[0,\alpha]^d}\sum_{j\in\mathbb{Z}^d}\big|(A*B_x)(j)\big|^{p/q}dx\Big)^{1/p}\\ &\le C\Big(\int_{[0,\alpha]^d}||A||^{p/q}_{\ell^{p/q}}\cdot||B_x||^{p/q}_{\ell^{p/q}}dx\Big)^{1/p}\\ &= C||A||^{1/q}_{\ell^{p/q}}\Big(\int_{[0,\alpha]^d}||B_x||^{p/q}_{\ell^{p/q}}dx\Big)^{1/p} = C||Ua||_{\ell^{q,p}_{U\tilde{\nu}}}\cdot||UF||L^{q,p}_{Um}.\end{aligned}$$

Case 2 ($1\le q\le\infty$): We use Minkowsky's Inequality with the exponent $q\ge 1$ to estimate the double sum in (3.33) by

$$\begin{aligned}\sum_{l\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d}(\tilde{a}_k*\tilde{F}_{x,\omega,j-k})(l)\Big|^q &\le \Big(\sum_{k\in\mathbb{Z}^d}||\tilde{a}_k*\tilde{F}_{x,\omega,j-k}||_{\ell^q}\Big)^q\\ &\le \Big(\sum_{k\in\mathbb{Z}^d}||\tilde{a}_k||_{\ell^1}\cdot||\tilde{F}_{x,\omega,j-k}||_{\ell^q}\Big)^q \qquad (3.35)\end{aligned}$$

and next use (3.35) and Minkowsky's Inequality to estimate the inner integral in (3.33) by

$$\begin{aligned}\Big(\int_{[0,\beta]^d}\sum_{l\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d}(\tilde{a}_k*\tilde{F}_{x,\omega,j-k})(l)\Big|^q d\omega\Big)^{1/q} &\le \Big(\int_{[0,\beta]^d}\Big(\sum_{k\in\mathbb{Z}^d}||\tilde{a}_k||_{\ell^1}\cdot||\tilde{F}_{x,\omega,j-k}||_{\ell^q}\Big)^q d\omega\Big)^{1/q}\\ &\le \sum_{k\in\mathbb{Z}^d}||\tilde{a}_k||_{\ell^1}\Big(\int_{[0,\beta]^d}||\tilde{F}_{x,\omega,j-k}||^q_{\ell^q}d\omega\Big)^{1/q}\\ &= (A*B_x)(j),\end{aligned}$$

where $A(k) = ||\tilde{a}_k||_{\ell^1}$ and $B_x(k) = \big(\int_{[0,\beta]^d}||\tilde{F}_{x,\omega,j-k}||^q_{\ell^q}d\omega\big)^{1/q}$. Therefore, by Young's Inequality

$$\begin{aligned}||G||_{L^{p,q}_m} &\le C\Big(\int_{[0,\alpha]^d}\sum_{j\in\mathbb{Z}^d}|(A*B_x)(j)|^p dx\Big)^{1/p}\\ &\le C||A||_{\ell^r}\Big(\int_{[0,\alpha]^d}||B_x||^p_{\ell^p}dx\Big)^{1/p}\\ &= C||Ua||_{\ell^{t,r}_{U\tilde{\nu}}}\cdot||UF||_{L^{q,p}_{Um}}. \qquad (3.36)\end{aligned}$$

The theorem is proved completely. □

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