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ON A INTEGRAL-TYPE OPERATOR FROM α-BLOCH SPACES TO $Q_k(p,q)$ SPACES

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Abstract. Let n be a positive integer, $g \in H(D)$ and φ be an analytic self-map of D. The boundedness and compactness of the integral operator $C^n_{\varphi,g}$, which is defined by

$$(C^n_{\varphi,g}f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi,\ z \in D,\ f \in H(D),$$

from the α-Bloch spaces and little α-Bloch spaces to the $Q_k(p,q)$ spaces and $Q_{k,0}(p,q)$ spaces are characterized.

1. Introduction

Let D be the open unit disc in the complex plane, and let $H(D)$ be the class of all analytic functions on D. The α-Bloch space B^α (α>0) is, by definition, the set of all function f in $H(D)$ such that

$$\|f\|_{B^\alpha} = |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha |f'(z)| < \infty. \tag{1.1}$$

Under the above norm, B^α is a Banach space. When α=1, $B^1 = B$ is the well-known Bloch space. Let B^α_0 denote the subspace of B^α, for f

$$B^\alpha_0 = \{f : (1-|z|^2)^\alpha |f'(z)| \to 0 \ as \ |z| \to 1, f \in B^\alpha\}. \tag{1.2}$$

This space is called the little α-Bloch space. Throughout this paper, the close unit ball in B^α and B^α_0 will be denoted by $\mathbb{B}_{B^\alpha}$ and $\mathbb{B}_{B^\alpha_0}$ respectively.

Let $0 < p < \infty, -2 < q < \infty$, $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function. The space $Q_k(p,q)$ consists of those $f \in H(D)$ such that (see, [21])

$$\|f\|_{k,p,q} = \{\sup_{z\in D}\int_D |f'(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\}^{\frac{1}{p}} < \infty, \tag{1.3}$$

where dA denotes the normalized Lebesgue area measure on D such that $A(D) = 1$, $g(z,a)$ is the Green function with logarithmic singularity at a, that is, $g(z,a) = \log\frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in D$. When $p = 2, q = 0$, the space $Q_k(p,q)$

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equals to Q_k, which was studied, for example, in [3, 10, 20]. When $K(x)=x^s, s\geq 0$, the space $Q_k(p,q)$ equals to $F(p,q,s)$, which is introduced by Zhao in [23]. Moreover (see, [23]), we have that, $F(p,q,s)=B^{\frac{q+2}{p}}$ and $F_0(p,q,s)=B_0^{\frac{q+2}{p}}$ for $s>1$, $F(p,q,s)\subseteq B^{\frac{q+2}{p}}$ and $F_0(p,q,s)\subseteq B_0^{\frac{q+2}{p}}$ for $0\leq s<1$. When $p\geq 1$, $Q_k(p,q)$ is a Banach space with the norm

$$\|f\|_{Q_k(p,q)}=|f(0)|+\|f\|_{k,p,q}.$$

From [21], we know that $Q_k(p,q)\subseteq B^{\frac{q+2}{p}}$, $Q_k(p,q)=B^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 K(\log\frac{1}{r})(1-r^2)^{-2}rdr<\infty.$$

Moreover, $\|f\|_{B^{\frac{q+2}{p}}}\leq C\|f\|_{Q_k(p,q)}$ for $f\in Q_k(p,q)$(see, [21, Theorem 2.1]).

We say that an $f\in H(D)$ belong to the space $Q_{k,0}(p,q)$ if

$$\lim_{|a|\to 1}\int_D |f'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)=0. \tag{1.4}$$

$Q_{k,0}(p,q)$ is the subspace of $Q_k(p,q)$. Throughout the paper, we always assume that K satisfies the following conditions:

(a) K is nondecreasing;
(b) K is two times differentiable on $(0,1)$;
(c) $\int_0^{\frac{1}{e}} K(\log\frac{1}{r})rdr<\infty$;
(d) $K(t)=K(1)>0$, $t\geq 1$;
(e) $K(2t)\approx K(t)$, $t\geq 0$.

Also, we assume that

$$\int_0^1 K(\log\frac{1}{r})(1-r^2)^q rdr<\infty, \tag{1.5}$$

otherwise $Q_k(p,q)$ consists only of constant functions(see, [21]). In order to obtain the main results in this paper, we further assume that

$$\int_0^1 \varphi_k(s)\frac{ds}{s}<\infty,$$

where

$$\varphi_k(s)=\sup_{0<t\leq 1}\frac{K(st)}{K(t)},\ 0<s\leq\infty.$$

For a subarc $I\subset\partial D$, the boundary of D, let

$$S(I)=\{r\xi\in D: 1-|I|<r<1, \xi\in I\},$$

where $|I|$ denotes the arc length of $I\subset\partial D$. If $|I|\geq 1$ then we set $S(I)=D$. A positive Borel measure μ on D is said to be a $K-$Carleson measure if

$$\sup_{I\in\partial D}\int_{S(I)} K(\frac{1-|z|}{|I|})d\mu(z)<\infty.$$

If

$$\lim_{|I|\to 0}\int_{S(I)} K(\frac{1-|z|}{|I|})d\mu(z)=0,$$

then we say μ is a vanishing $K-$Carleson measure. Clearly, if $K(t)=t^p$, $0<p<\infty$, then μ is a $K-$Carleson measure if and only if $(1-|z|^2)^p d\mu(z)$ is a $p-$Carleson measure. Note that $p=1$ give the classical Carleson measure.

Let φ be an analytic self-map of D. The composition operator C_φ is defined by

$$(C_\varphi f)(z)=f(\varphi(z)),\ f\in H(D).$$

The composition operator has been studied by many researchers on various spaces (see, e.g.,[1, 5, 13] and the references therein).

Let $g\in H(D)$ and φ be an analytic self-map of D. In [7], the authors defined the generalized composition operator as follows:

$$(C_\varphi^g f)(z)=\int_0^z f'(\varphi(\xi))g(\xi)d\xi,\ z\in D,\ f\in H(D). \tag{1.6}$$

When $g=\varphi'$, we see that this operator is essentially composition operator C_φ. Therefore, C_φ^g is a generalization of the composition operator C_φ. The boundedness and compactness of the generalized composition operator on the Zygmund space, the α-Bloch space and the little α-Bloch space was investigated in [7]. Some related results can be found, for example, in [8, 9, 18, 22].

Let n be a positive integer, $g\in H(D)$ and φ be an analytic self-map of D. Here we study the following integral-type operator

$$(C_{\varphi,g}^n f)(z)=\int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi,\ z\in D,\ f\in H(D). \tag{1.7}$$

When $n=1$, $C_{\varphi,g}^1$ is generalized composition operator C_φ^g. Operator (1.7) extends several operators which has been introduced and studied recently (see, e.g.,[7, 8, 12]). For related operators in $n-$dimensional case, see, for example, [15-17, 19]. Operator (1.7) has been studied by many researchers on various spaces (see, e.g.,[6, 11, 25] and the references therein). The purpose of this paper is to study the operator $C_{\varphi,g}^n$. The boundedness and compactness of the operator $C_{\varphi,g}^n$ from the B^α and B_0^α spaces to $Q_k(p,q)$ and $Q_{k,0}(p,q)$ spaces are completely characterized.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A\approx B$ means that there is a positive constant C such that $\frac{B}{C}\leq$A$\leq$CB.

2. Auxiliary Results

Here, we quote several lemmas which will be used in the proofs of the main results in this paper. The following lemma is obtained in [4].

Lemma 2.1. *Let $\alpha>0$. Then there are two functions $f_1, f_2\in B^\alpha$ such that*

$$|f_1'(z)|+|f_2'(z)|\geq\frac{C}{(1-|z|^2)^\alpha},\ z\in D. \tag{2.1}$$

We also need the following results of Wulan and Zhu in [20], in which Q_k space are characterized in terms of $K-$Carleson measures.

Lemma 2.2. *A positive Borel measure μ on D is a $K-$Carleson measure if and only if*

$$\sup_{a\in D}\int_D K(1-|\varphi_a(z)|^2)d\mu(z)<\infty.$$

Also, μ is a vanishing $K-$Carleson measure if and only if

$$\lim_{|a|\to 1}\int_D K(1-|\varphi_a(z)|^2)d\mu(z)=0.$$

Lemma 2.3. *[22] Let $0<p<\infty$, $-2<q<\infty$. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Then*

$$\|f\|_{k,p,q}^p \approx \sup_{z\in D}\int_D |f'(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z), \tag{2.2}$$

and $f\in Q_{k,0}(p,q)$ if and only if

$$\lim_{|a|\to 1}\int_D |f'(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z)=0. \tag{2.3}$$

By modifying the proof of Theorem 3.5 of [10], we can prove the following lemma. We omit the details.

Lemma 2.4. *Let $0<\alpha<\infty$, $0<p<\infty$, $-2<q<\infty$, $g\in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Then $C^n_{\varphi,g}: B^\alpha\to Q_{k,0}(p,q)$ is compact if and only if $C^n_{\varphi,g}: B^\alpha\to Q_{k,0}(p,q)$ is bounded and*

$$\lim_{|a|\to 1}\sup_{\|f\|_{B^\alpha}\le 1}\int_D |(C^n_{\varphi,g}f)'(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)=0. \tag{2.4}$$

Lemma 2.5. *Let $0<\alpha<\infty$, $0<p<\infty$, $-2<q<\infty$, $g\in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Then $C^n_{\varphi,g}: B^\alpha_0\to Q_k(p,q)$ (or $Q_{k,0}(p,q)$) is weakly compact if and only if it is compact.*

Proof. By a known theorem $C^n_{\varphi,g}: B^\alpha_0\to Q_k(p,q)$ (or $Q_{k,0}(p,q)$) is weakly compact if and only if $(C^n_{\varphi,g})^*$: $(Q_k(p,q))^*$ (or $(Q_{k,0}(p,q))^*$)$\to(B^\alpha_0)^*$ is weakly compact. Since $(B^\alpha_0)^*\cong A^1$ (the Bergman space) and A^1 has the Schur property, it follows that it is equivalent to $(C^n_{\varphi,g})^*$: $(Q_k(p,q))^*$ (or $(Q_{k,0}(p,q))^*$)$\to(B^\alpha_0)^*$ is compact, which is equivalent to $C^n_{\varphi,g}: B^\alpha_0\to Q_k(p,q)$ (or $Q_{k,0}(p,q)$), is compact, as claimed. □

Lemma 2.6. *Let $0<\alpha<\infty$, $0<p<\infty$, $-2<q<\infty$, $g\in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Then $C^n_{\varphi,g}$: B^α (or B^α_0)$\to Q_k(p,q)$ is compact if and only if for any bounded sequence $\{f_l\}_{l\in N}$ in B^α (or B^α_0) which converges to zero uniformly on compact subsets of D as $l\to\infty$, we have $\|C^n_{\varphi,g}f_l\|_{Q_k(p,q)}\to 0$ as $l\to\infty$.*

Proof. It can be proved by standard way (see, [1, proposition 3.11]). □

Lemma 2.7. *Let $0<\alpha<\infty$, $0<p<\infty$, $-2<q<\infty$, $g\in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Suppose that $C^n_{\varphi,g}: B^\alpha_0\to Q_k(p,q)$ is compact, then for every $a\in D$,*

$$\lim_{r\to 1}\int_{|\varphi(z)|>r}|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)=0. \tag{2.5}$$

Proof. Let $f_l(z) = \frac{z^l}{l}$, $l \in \mathbb{N}$. It is easy to see that $(f_l)_{l\in\mathbb{N}}$ is bounded sequence in B_0^α converging to zero uniformly on compact subsets of D. Hence, by Lemma 2.6, it follows that $\|C_{\varphi,g}^n f_l\|_{Q_k(p,q)} \to 0$ as $l \to \infty$. Thus, for every $\varepsilon > 0$, there is an $l_0 \in \mathbb{N}$, $l_0 > n$ such that for $l \geq l_0$

$$(\prod_{j=1}^{n-1}(l-j))^p \sup_{a\in D}\int_D |\varphi(z)|^{p(l-n)}|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon. \tag{2.6}$$

From (2.6) we have that for each $r \in (0,1)$ and $l \geq l_0$

$$r^{p(l-n)}(\prod_{j=1}^{n-1}(l-j))^p \sup_{a\in D}\int_{|\varphi(z)|>r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon. \tag{2.7}$$

Hence, for $r \in [(\Pi_{j=1}^{n-1}(l_0-j))^{\frac{-1}{l_0-n}}, 1)$, we have

$$\sup_{a\in D}\int_{|\varphi(z)|>r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon.$$

We complete the proof. □

Lemma 2.8. *Let $0 < \alpha < \infty$, $0 < p < \infty$, $-2 < q < \infty$, $g \in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Suppose that $C_{\varphi,g}^n : B^\alpha(B_0^\alpha) \to Q_k(p,q)$ is compact, then for every $a \in D$,*

$$\lim_{r\to 1}\int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) = 0.$$

Proof. We only give the proof of B_0^α and the proof for B^α is similar. For $f \in \mathbb{B}_{B_0^\alpha}$, and let $f_t(z) = f(tz)$, $0 < t < 1$. Then $\sup_{0<t<1}\|f_t\|_{B^\alpha} \leq \|f\|_{B^\alpha}$, $f_t \in B_0^\alpha$, $t \in (0,1)$, and $f_t \to f$ uniformly on compact subsets of D as $t \to 1$. Since $C_{\varphi,g}^n : B_0^\alpha \to Q_k(p,q)$ is compact, $\|C_{\varphi,g}^n f_t - C_{\varphi,g}^n f\|_{Q_k(p,q)} \to 0$ as $t \to 1$. Hence, for every given $\varepsilon > 0$, there exists a $t \in (0,1)$ such that

$$\sup_{a\in D}\int_D |f_t^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon. \tag{2.8}$$

Combination Lemma 2.7 and (2.8), we have that for such t and each $r \in [(\Pi_{j=1}^{n-1}(l_0-j))^{\frac{-1}{l_0-n}}, 1)$

$$\begin{aligned}
&\sup_{a\in D}\int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\
&\leq 2^p \sup_{a\in D}\int_{|\varphi(z)|>r} |f_t^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\
&+ 2^p \sup_{a\in D}\int_{|\varphi(z)|>r} |f_t^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\
&< 2^p\varepsilon(1+\|f_t^{(n)}\|_\infty^p)
\end{aligned} \tag{2.9}$$

From (2.9) we conclude that for every $f \in \mathbb{B}_{B_0^\alpha}$, there is a $\delta \in (0,1)$ and $\delta = \delta(f,\varepsilon)$ such that for $r \in (\delta, 1)$

$$\sup_{a\in D}\int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon.$$

We complete the proof. □

3. Main Results and Proofs

In this section, we will state main results and prove them.

Theorem 3.1. *Let $0 < \alpha < \infty$, $0 < p < \infty$, $-2 < q < \infty$, $g \in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0, \infty)$. Then the following statements are equivalent.*

(1) $C^n_{\varphi,g} : B^\alpha \to Q_k(p, q)$ is bounded.

(2) $C^n_{\varphi,g} : B^\alpha_0 \to Q_k(p, q)$ is bounded.

(3) $\sup_{a\in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} K(g(z, a))dA(z) < \infty$.

(4) $\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} dA(z)$ is a $K-$Carleson measure.

Proof. (1)⇒(2). It is obvious.

(2)⇒(3). For $f \in B^\alpha$ if we set $f_s(z) = f(sz)$, $0 < s < 1$, then we get $f_s \in B^\alpha_0$ and $\|f_s\|_{B^\alpha} \le \|f\|_{B^\alpha}$. Thus, by the assumption for all $f \in B^\alpha$, we have

$$\|C^n_{\varphi,g} f_s\|_{Q_k(p,q)} \le \|C^n_{\varphi,g}\|_{B^\alpha_0\to Q_k(p,q)}\|f_s\|_{B^\alpha} \le \|C^n_{\varphi,g}\|_{B^\alpha_0\to Q_k(p,q)}\|f\|_{B^\alpha}. \tag{3.1}$$

By using Lemma 2.1, there exist $f_1,\ f_2 \in B^\alpha$ such that

$$|f_1'(z)| + |f_2'(z)| \ge \frac{C}{(1-|z|^2)^\alpha},\ z \in D. \tag{3.2}$$

Let

$$h_1(z) = f_1(z) - \sum_{k=1}^{n-1} \frac{f_1^{(k)}(0)}{k!} z^k,\ h_2(z) = f_2(z) - \sum_{k=1}^{n-1} \frac{f_2^{(k)}(0)}{k!} z^k. \tag{3.3}$$

It is known (see [24]) that for each $f \in H(D)$ and $n \in \mathbb{N}$, we have

$$(1-|z|^2)^{\alpha+n-1}|f^{(n)}(z)| + \sum_{k=1}^{n-1} |f^{(k)}(0)| \approx (1-|z|^2)^\alpha|f'(z)|.$$

From this, (3.2), and since $h_1^{(k)}(0) = h_2^{(k)}(0) = 0$, $k = 0, 1, \cdots, n-1$, we have that there is a $\delta > 0$ such that

$$|h_1^{(n)}(z)| + |h_2^{(n)}(z)| \ge \frac{C}{(1-|z|^2)^{\alpha+n-1}} \tag{3.4}$$

for $|z| > \delta$.

Replacing f in (3.1) by h_1 and h_2 respectively and applying (3.4), using an elementary inequality, the boundedness of $C^n_{\varphi,g} : B^\alpha_0 \to Q_k(p, q)$, we obtain that

$$\begin{aligned}
&\int_{|s\varphi(z)|>\delta} \frac{|s^n g(z)|^p(1-|z|^2)^q}{(1-|s\varphi(z)|^2)^{p(\alpha+n-1)}} K(g(z, a))dA(z)\\
&\le C\int_D (|h_1^{(n)}(s\varphi(z)|^p + |h_2^{(n)}(s\varphi(z)|^p)|s^n g(z)|^p(1-|z|^2)^q K(g(z, a))dA(z)\\
&= C\int_D (|(C^n_{\varphi,g}h_{1s})'(z)|^p + |(C^n_{\varphi,g}h_{2s})'(z)|^p)(1-|z|^2)^q K(g(z, a))dA(z)\\
&\le C\|C^n_{\varphi,g}\|^p_{B^\alpha_0\to Q_k(p,q)}(\|h_1\|^p_{B^\alpha} + \|h_2\|^p_{B^\alpha}) < \infty
\end{aligned} \tag{3.5}$$

hold for all $a \in D$.

Letting $s \to 1$ in (3.5) and using the Fatou′s Lemma, we get

$$\sup_{a\in D}\int_{|\varphi(z)|>\delta}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(g(z,a))dA(z)<\infty. \tag{3.6}$$

On the other hand, for $f_0(z)=\frac{z^n}{n!}\in B_0^\alpha$, we get $C_{\varphi,g}^n f_0\in Q_k(p,q)$ which implies

$$\begin{aligned}&\sup_{a\in D}\int_{|\varphi(z)|\le\delta}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(g(z,a))dA(z)\\ &\le\frac{\|C_{\varphi,g}^n\|^p_{B_0^\alpha\to Q_k(p,q)}\|f_0\|^p_{B^\alpha}}{(1-\delta^2)^{p(\alpha+n-1)}}.\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), (3) follows.

(3)⇒(4). From properties of K and the condition (3), we obtain

$$\begin{aligned}&\sup_{a\in D}\int_D\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(1-|\varphi_a(z)|^2)dA(z)\\ &\le\sup_{a\in D}\int_D\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(2g(z,a))dA(z)\\ &\approx\sup_{a\in D}\int_D\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(g(z,a))dA(z)<\infty.\end{aligned}$$

Thus, by Lemma 2.2, $\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}dA(z)$ is a $K-$Carleson measure.

(4)⇒(1). For any $f\in B^\alpha$, we have

$$\begin{aligned}&\sup_{a\in D}\int_D|f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)dA(z)\\ &\le\|f\|^p_{B^\alpha}\sup_{a\in D}\int_D\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(1-|\varphi_a(z)|^2)dA(z).\end{aligned}$$

In addition, that $(C_{\varphi,g}^n f)(0)=0$. From Lemma 2.2 and Lemma 2.3, we have that $C_{\varphi,g}^n:B^\alpha\to Q_k(p,q)$ is bounded. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $0<\alpha<\infty$, $0<p<\infty$, $-2<q<\infty$, $g\in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Then the following statements are equivalent.*

(1) $C_{\varphi,g}^n:B^\alpha\to Q_{k,0}(p,q)$ is bounded.
(2) $C_{\varphi,g}^n:B^\alpha\to Q_{k,0}(p,q)$ is compact.
(3) $C_{\varphi,g}^n:B_0^\alpha\to Q_{k,0}(p,q)$ is weakly compact.
(4) $C_{\varphi,g}^n:B_0^\alpha\to Q_{k,0}(p,q)$ is compact.
(5) $\lim_{|a|\to1}\int_D\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(g(z,a))dA(z)=0$.
(6) $\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}dA(z)$ is a vanishing $K-$Carleson measure.

Proof. (3)⇔(4). It follows from Lemma 2.5.

(1)⇔(4). By Lemma 2.5, $C_{\varphi,g}^n:B_0^\alpha\to Q_{k,0}(p,q)$ is compact if and only if it is weakly compact, which, by Gantmacher′s theorem (see [2]), is equivalent to $(C_{\varphi,g}^n)^{**}((B_0^\alpha)^{**})\subseteq Q_{k,0}(p,q)$. Since $(B_0^\alpha)^{**}=B^\alpha$ and by a standard duality argument $(C_{\varphi,g}^n)^{**}=C_{\varphi,g}^n$ on B^α, this can be written as $C_{\varphi,g}^n(B^\alpha)\subseteq Q_{k,0}(p,q)$, which by the closed graph theorem is equivalent to $C_{\varphi,g}^n:B^\alpha\to Q_{k,0}(p,q)$ is bounded.

(2)⇒(1). It is obvious.

(4)⇒(5). Assume that $C^n_{\varphi,g}: B^\alpha_0 \to Q_{k,0}(p,q)$ is compact, from above proofs we have that $C^n_{\varphi,g}: B^\alpha \to Q_{k,0}(p,q)$ is bounded. Hence, as in the proof of Theorem 3.1, let h_1 and h_2 be as in (3.3). Then from (3.4) and an elementary inequality, we get

$$\begin{aligned}
&\int_{|\varphi(z)|>\delta} \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} K(g(z,a))dA(z) \\
&\leq C\int_D (|h_1^{(n)}(\varphi(z)|^p + |h_2^{(n)}(\varphi(z)|^p)|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) \\
&= C\int_D (|(C^n_{\varphi,g}h_1)'(z)|^p + |(C^n_{\varphi,g}h_2)'(z)|^p)(1-|z|^2)^q K(g(z,a))dA(z). \qquad (3.8)
\end{aligned}$$

For $f_0(z) = \frac{z^n}{n!} \in B^\alpha$, we get $C^n_{\varphi,g}f_0 \in Q_{k,0}(p,q)$. From this and since $C^n_{\varphi,g}h_j \in Q_{k,0}(p,q), j = 1,2$, by letting $|a| \to 1$, we get that (5) holds.

(5)⇔(6). It follows from properties of K and Lemma 2.2.

(5)⇒(2). Assume that (5) holds. Let

$$\psi_{p,q,\varphi,K}(a) = \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} K(g(z,a))dA(z).$$

By the assumption, we have that for every $\varepsilon > 0$, there is a $r \in (0,1)$ such that for $|a| > r$, $\psi_{p,q,\varphi,K}(a) < \varepsilon$. Similarly to the proof of Lemma 2.3 of [14], we see that $\psi_{p,q,\varphi,K}(a)$ is continuous on $|a| \leq r$, hence is bounded on $|a| \leq r$. Therefore $\psi_{p,q,\varphi,K}(a)$ is bounded on D. From Theorem 3.1, $C^n_{\varphi,g}: B^\alpha \to Q_k(p,q)$ is bounded. We first prove that $C^n_{\varphi,g}: B^\alpha \to Q_{k,0}(p,q)$ is bounded. For any $f \in B^\alpha$, we have

$$\begin{aligned}
&\int_D |(C^n_{\varphi,g}f)'(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) \\
&\leq \|f\|^p_{B^\alpha} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} K(g(z,a))dA(z), \qquad (3.9)
\end{aligned}$$

which together with condition (5) imply that $C^n_{\varphi,g}f \in Q_{k,0}(p,q)$, hence $C^n_{\varphi,g}: B^\alpha \to Q_{k,0}(p,q)$ is bounded. Fix $f \in \mathbb{B}_{B^\alpha}$. The righthand side of (3.9) tend to 0, as $|a| \to 1$ by condition (5). From Lemma 2.4, we see that $C^n_{\varphi,g}: B^\alpha \to Q_{k,0}(p,q)$ is compact. The proof of the theorem 3.2 is completed. □

Theorem 3.3. *Let $0 < \alpha < \infty$, $0 < p < \infty$, $-2 < q < \infty$, $g \in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Then the following statements are equivalent.*

(1) $C^n_{\varphi,g}: B^\alpha_0 \to Q_{k,0}(p,q)$ is bounded.

(2) $\lim_{|a|\to 1}\int_D |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) = 0$, and $\sup_{a\in D}\int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} K(g(z,a))dA(z) < \infty$.

Proof. (2)⇒(1). Suppose that condition (2) holds and $f \in B^\alpha_0$. From Theorem 3.1, $C^n_{\varphi,g}: B^\alpha_0 \to Q_k(p,q)$ is bounded. To prove that $C^n_{\varphi,g}: B^\alpha_0 \to Q_{k,0}(p,q)$ is bounded, it suffices to prove that $C^n_{\varphi,g}f \in Q_{k,0}(p,q)$ for any $f \in B^\alpha_0$. Since $f \in B^\alpha_0$, we have that, for every $\varepsilon > 0$, there is a $r \in (0,1)$ such that as $r < |\varphi(z)| < 1$

$$|f^{(n)}(\varphi(z))|^p(1-|\varphi(z)|^2)^{p(\alpha+n-1)} < \varepsilon.$$

Thus,

$$\sup_{a\in D}\int_{|\varphi(z)|>r}|(C^n_{\varphi,g}f)'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)$$
$$\le \varepsilon\sup_{a\in D}\int_D\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(g(z,a))dA(z)<C\varepsilon. \tag{3.10}$$

We also have

$$\lim_{|a|\to 1}\int_{|\varphi(z)|\le r}|(C^n_{\varphi,g}f)'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)$$
$$\le C\frac{\|f\|^p_{B^\alpha}}{(1-r^2)^{p(\alpha+n-1)}}\lim_{|a|\to 1}\int_{|\varphi(z)|\le r}|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)$$
$$\le C\frac{\|f\|^p_{B^\alpha}}{(1-r^2)^{p(\alpha+n-1)}}\lim_{|a|\to 1}\int_D|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)=0. \tag{3.11}$$

Combining (3.10) and (3.11), we get $C^n_{\varphi,g}f\in Q_{k,0}(p,q)$. Hence $C^n_{\varphi,g}:B^\alpha_0\to Q_{k,0}(p,q)$ is bounded.

(1)⇒(2). If $C^n_{\varphi,g}:B^\alpha_0\to Q_{k,0}(p,q)$ is bounded, then $C^n_{\varphi,g}:B^\alpha_0\to Q_k(p,q)$ is bounded too. Thus, by Theorem 3.1, we get the second condition in (2). For $f_0(z)=\frac{z^n}{n!}\in B^\alpha_0$, $n\in N$, we get $C^n_{\varphi,g}f_0\in Q_{k,0}(p,q)$, which is equivalent to the first condition in (2). This completes the proof of Theorem 3.3. □

Theorem 3.4. *Let $0<\alpha<\infty$, $0<p<\infty$, $-2<q<\infty$, $g\in H(D)$ and φ be an analytic self-map of D. Assume that K is a nonnegative nondecreasing function on $[0,\infty)$. Then the following statements are equivalent.*

(1) $C^n_{\varphi,g}:B^\alpha\to Q_k(p,q)$ is compact.

(2) $C^n_{\varphi,g}:B^\alpha_0\to Q_k(p,q)$ is compact.

(3) $C^n_{\varphi,g}:B^\alpha_0\to Q_k(p,q)$ is weakly compact.

(4) $\sup_{a\in D}\int_D|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)<\infty$, and $\lim_{r\to 1}\sup_{a\in D}\int_{|\varphi(z)|>r}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}K(g(z,a))dA(z)=0$.

Proof. (2)⇔(3). It follows from Lemma 2.5.

(1)⇒(2). It is obvious.

(2)⇒(4). Assume that $C^n_{\varphi,g}:B^\alpha_0\to Q_k(p,q)$ is compact, then $C^n_{\varphi,g}:B^\alpha_0\to Q_k(p,q)$ is bounded. By choosing $f(z)=\frac{z^n}{n!}\in B^\alpha_0, n\in\mathbb{N}$, we obtain

$$\sup_{a\in D}\int_D|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)<\infty.$$

Thus, we have that the first condition in (4) holds.

Since $C^n_{\varphi,g}:B^\alpha_0\to Q_k(p,q)$ is compact, we have that for every $\varepsilon>0$ there is a finite collection of functions $f_1,f_2,\cdots,f_k\in\mathbb{B}_{B^\alpha_0}$ such that, for each $f\in\mathbb{B}_{B^\alpha_0}$, there is a $j\in\{1,2,\cdots,k\}$, such that

$$\sup_{a\in D}\int_D|f_j^{(n)}(\varphi(z))-f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)<\varepsilon. \tag{3.12}$$

On the other hand, by Lemma 2.8, it follows that if $\eta:=\max_{1\le j\le k}\eta_j(f_j,\varepsilon)$, then for $r\in(\eta,1)$ and all $j\in\{1,2,\cdots,k\}$, we have

$$\sup_{a\in D}\int_{|\varphi(z)|>r}|f_j^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)<\varepsilon. \tag{3.13}$$

From (3.12) and (3.13), we have that for $r \in (\eta, 1)$ and every $f \in \mathbb{B}_{B_0^\alpha}$

$$\sup_{a\in D}\int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) < 2^p\varepsilon. \tag{3.14}$$

If we apply (3.14) to the delays of the functions in (3.3) which are normalized so that they belong to $\mathbb{B}_{B^\alpha}$ and then use the monotone convergence theorem, we easily get that for $r > \max\{\delta, \eta\}$ where δ is chosen as in (3.4)

$$\sup_{a\in D}\int_{|\varphi(z)|>r} \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} K(g(z,a))dA(z) < C\varepsilon, \tag{3.15}$$

from which the second condition in (4) follows, as desired.

(4)⇒(1). Assume that $\{f_l\}_{l\in N} \subset B^\alpha$, $\|f_l\|_{B^\alpha} \le 1$ and $f_l \to 0$ uniformly on compact subsets of D. By condition (4), for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that as $r \in (\delta, 1)$,

$$\begin{aligned} &\sup_{a\in D}\int_{|\varphi(z)|>r} |f_l^{(n)}(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\ &\le \|f_l\|_{B^\alpha}^p \sup_{a\in D}\int_{|\varphi(z)|>r} \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} K(g(z,a))dA(z) < \varepsilon. \end{aligned} \tag{3.16}$$

On the other hand, since $f_l^{(n)}(\varphi(z)) \to 0$ uniformly on $\{z : |\varphi(z)| \le r\}$, for the above $\varepsilon > 0$ there is an integer $N > 1$ such that as $l \ge N$,

$$\begin{aligned} &\sup_{a\in D}\int_{|\varphi(z)|\le r} |f_l^{(n)}(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\ &\le \sup_{|\varphi(z)|\le r} |f_l^{(n)}(\varphi(z))| \sup_{a\in D}\int_{|\varphi(z)|\le r} |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\ &\le \varepsilon \sup_{a\in D}\int_{|\varphi(z)|\le r} |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \le C\varepsilon. \end{aligned} \tag{3.17}$$

Since $(C^n_{\varphi,g} f_l)(0) = 0$, then we obtain $\|C^n_{\varphi,g} f_l\|_{Q_k(p,q)} \to 0$, as $l \to \infty$. By using Lemma 2.6, we get $C^n_{\varphi,g} : B^\alpha \to Q_k(p,q)$ is compact. This completes the proof of Theorem 3.4. □

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References

[1] C.C.Cowen and B.D.Maccluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Florida, 1995.

[2] N. Dunford and J. Schwartz, *Linear operators*, Vol. 1, Interscience, New York, NY, USA, 1958.

[3] M. Essén and H. Wulan, *On analytic and meromorphic functions and spaces of Q_k-type*, Illinois J. Math., **46** (2002) 1233-1258.

[4] Z. Lou, *Composition operators on Bloch type spaces*, Analysis, **23** (2003) 81–95.

[5] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc., **347** (1995) 2679-2687.

[6] S. Li, *On an integral-type operator from the Bloch space into the $Q_k(p,q)$ spaces*, Filomat, **26** 2 (2012) 125-133.

[7] S. Li and S. Stević, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl., **338** 2 (2008) 1282-1295.

[8] S. Li and S. Stević, *Products of integral-type operators and composition operators between Bloch type spaces*, J. Math. Anal. Appl., **349** 2 (2009) 596-610.

[9] S. Li, *Volterra composition operator between weighted Bergman space and Bloch-type space* , J. Korea Math. Soc., **45** (2008) 229-245.

[10] S. Li and H. Wulan, *Composition operators on Q_k spaces*, J. Math. Anal. Appl., **327** (2007) 948-958.

[11] C. Pan, *On an integral-type operator from $Q_k(p,q)$ spaces to $\alpha-$Bloch space*, Filomat, **25** 3 (2011) 163-173.

[12] A. Sharma and A.K.Sharma, *Carleson measures and a class of generalized integration operators on the Bergman space*, The Rocky Mountain J. Math., **41** 5 (2011) 1711-1724.

[13] J.H.Shapiro, *Composition operators and classical function theory*, Springer, New York, 1993.

[14] W. Smith and R. Zhao, *TComposition operators mapping into Q_p space*, Analysis, **17** (1997) 239-262.

[15] S. Stević, *On an integral-type operators from logarithmic Bloch-type and mixed-norm space to Bloch-type spaces*, Nonlinear Analysis, **71** 12 (2009) 6323-6342.

[16] S. Stević, *On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball*, J. Math. Anal. Appl., **354** 2 (2009) 426-434.

[17] S. Stević, *On a new operator from H^∞ to the Bloch-type space on the unit ball*, Util. Math., **77** (2008) 257-263.

[18] S. Stević, *Generalized composition operators between mixed norm space and some weighted spaces*, Numer. Funct. Anal. Opt., **29** (2009) 426-434.

[19] W. Yang and X. Meng, *Generalized composition operators from $F(p,q,s)$ space to Bloch-type spaces*, Appl. Math. Computation, **217** 6 (2010) 2513-2519.

[20] H. Wulan and K. Zhu, *Derivative-free characterizations of Q_k spaces*, J. Australian Math. Soc., **82** 8 (2007) 283-295.

[21] H. Wulan and J. Zhou, *Q_k Type spaces of analytic functions*, J. Funct. Spaces Appl., **4** 1 (2006) 73-84.

[22] F. Zhang and Y. Liu, *Generalized composition operators from Bloch type spaces to Q_k type spaces*, J. Funct. Spaces Appl., **8** 1 (2010) 55-66.

[23] R. Zhao, *On a general family of function spaces*, Annales Academiæ Scientiarum Fennicæ,math-Ematica dissertations, **105** (1996) 1-56.

[24] K. Zhu, *Bloch type spces of analytic functions*, The Rocky Mountain J. Math., **23** 3 (1993) 1143-1177.

[25] X. Zhu, *An Integral-type operator from H^∞ to Zygmund-type spaces*, Bull. Malays. Math. Sci. Soc., **35** (2012) 679-686.

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