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MORE ON SOME NEW FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. In 2011, W. T. Sulaiman presented new integral inequalities of the type Riemann-Liouville. In this paper, we present some generalizations of the integral inequalities.

1. Introduction

For any integrable functions f and g in $[0,1]$, Chebyshev [1] proved, in 1882, that if f and g are simultaneously increasing or decreasing for same values of x in $[0,1]$, then

$$\int_0^1 f(x)g(x)dx \geq \int_0^1 f(x)dx \int_0^1 g(x)dx$$

For any $\alpha > 0$, the Riemann-Liouville fractional integral operator $I^\alpha f$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx,$$

for all $t > 0$.

In 2009, Belarbi and Dahmani [2] proved that for each $\alpha, \beta > 0$, if f and g are functions such that

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \tag{1.1}$$

for all $x, y > 0$, then

$$\frac{t^\beta I^\alpha (fg)(t)}{\Gamma(\beta+1)} + \frac{t^\alpha I^\beta (fg)(t)}{\Gamma(\alpha+1)} \geq I^\beta f(t) I^\alpha (g)(t) + I^\alpha f(t) I^\beta (g)(t) \tag{1.2}$$

for all $t > 0$.

In 2011, Sulaiman [3] proved that for each $\alpha, \beta > 0$, if f and g are functions satisfying the inequality (1.1) and if $h \geq 0$, then

$$\frac{t^\beta I^\alpha (fgh)(t)}{\Gamma(\beta+1)} + \frac{t^\alpha I^\beta (fgh)(t)}{\Gamma(\alpha+1)} \geq I^\beta f(t) I^\alpha (gh)(t) + I^\alpha f(t) I^\beta (gh)(t) \tag{1.3}$$

for all $t > 0$. This is a generalization of the inequality (1.2).

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Moreover, Sulaiman [3] also proved that for each $\alpha, \beta > 0$, if f, g, h are functions such that

$$(f(x) - f(y))(g(x) - g(y))(h(x) - h(y)) \geq 0$$

for all $x, y > 0$, then

$$\frac{t^\beta I^\alpha (fgh)(t)}{\Gamma(\beta+1)} - \frac{t^\alpha I^\beta (fgh)(t)}{\Gamma(\alpha+1)} \geq I^\beta f(t) I^\alpha (gh)(t) - I^\alpha f(t) I^\beta (gh)(t) \tag{1.4}$$

for all $t > 0$.

In this paper, we generalize of the integral inequalities (1.2), (1.3) and (1.4).

2. Results

Theorem 2.1. *Let $\alpha, \beta > 0$ and let $f_1, f_2, ..., f_n$ be functions such that*

$$\prod_{i=1}^{n} (f_i(x) - f_i(y)) \geq 0$$

for all $x, y > 0$, then

$$\begin{aligned} &\frac{t^\beta}{\Gamma(\beta+1)} I^\alpha \left(\prod_{i=1}^{n} f_i \right)(t) + (-1)^n \frac{t^\alpha}{\Gamma(\alpha+1)} I^\beta \left(\prod_{i=1}^{n} f_i \right)(t) \\ &\qquad \geq \sum_{\Omega \subseteq \{1,2,\ldots,n\}, 0 < \#\Omega < n} (-1)^{1+\#\Omega} \left(I^\beta \left(\prod_{i \in \Omega} f_i \right)(t) \right) \left(I^\alpha \left(\prod_{i \notin \Omega} f_i \right)(t) \right) \end{aligned} \tag{2.1}$$

for all $t > 0$.

Proof. By the assumption, we have

$$\int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} \prod_{i=1}^{n} (f_i(x) - f_i(y)) dx dy \geq 0$$

for all $t > 0$.

We note that $\Gamma(\lambda) I^\lambda (1) = \int_0^t (t-z)^{\lambda-1} dz = \frac{t^\lambda}{\lambda} = \frac{t^\lambda \Gamma(\lambda)}{\Gamma(\lambda+1)}$ for all $\lambda, t > 0$. For any $t > 0$, we have

$$\begin{aligned} &\int_0^t (t-x)^{\alpha-1} \prod_{i=1}^{n} f_i(x) dx \int_0^t (t-y)^{\beta-1} dy \\ &+ (-1)^n \int_0^t (t-y)^{\beta-1} \prod_{i=1}^{n} f_i(y) dy \int_0^t (t-x)^{\alpha-1} dx \\ &\geq - \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} \sum_{\{z_1, z_2, \ldots z_n\} = \{x, y\}} (-1)^{\#\{i | z_i = y\}} \left(\prod_{i=1}^{n} f_i(z_i) \right) dx dy, \end{aligned}$$

so

$$\left(\Gamma(\alpha)I^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t)\right)\frac{t^{\beta}\Gamma(\beta)}{\Gamma(\beta+1)}+(-1)^{n}\left(\Gamma(\beta)I^{\beta}\left(\prod_{i=1}^{n} f_i\right)(t)\right)\frac{t^{\alpha}\Gamma(\alpha)}{\Gamma(\alpha+1)}$$
$$\geq -\int_0^t\int_0^t (t-x)^{\alpha-1}(t-y)^{\beta-1}\sum_{\{z_1,z_2,\dots z_n\}=\{x,y\}}(-1)^{\#\{i|z_i=y\}}\left(\prod_{i=1}^{n} f_i(z_i)\right)dxdy,$$

so

$$\frac{t^{\beta}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\beta+1)}I^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t)+(-1)^{n}\frac{t^{\alpha}\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha+1)}I^{\beta}\left(\prod_{i=1}^{n} f_i\right)(t)$$
$$\geq -\int_0^t\int_0^t (t-x)^{\alpha-1}(t-y)^{\beta-1}\sum_{\{z_1,z_2,\dots z_n\}=\{x,y\}}(-1)^{\#\{i|z_i=y\}}\left(\prod_{i=1}^{n} f_i(z_i)\right)dxdy$$
$$= -\sum_{\Omega\subseteq\{1,2,\dots,n\},0<\#\Omega<n}(-1)^{\#\Omega}\left(\Gamma(\beta)I^{\beta}\left(\prod_{i\in\Omega} f_i\right)(t)\right)\left(\Gamma(\alpha)I^{\alpha}\left(\prod_{i\notin\Omega} f_i\right)(t)\right),$$

so

$$\frac{t^{\beta}}{\Gamma(\beta+1)}I^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t)+(-1)^{n}\frac{t^{\alpha}}{\Gamma(\alpha+1)}I^{\beta}\left(\prod_{i=1}^{n} f_i\right)(t)$$
$$\geq -\sum_{\Omega\subseteq\{1,2,\dots,n\},0<\#\Omega<n}(-1)^{\#\Omega}\left(I^{\beta}\left(\prod_{i\in\Omega} f_i\right)(t)\right)\left(I^{\alpha}\left(\prod_{i\notin\Omega} f_i\right)(t)\right).$$

This is implies the inequality (2.1). □

We note on Theorem 2.1 that (i) if $n=2$ then we obtain the inequality (1.2), and (ii) if $n=3$ then we obtain the inequality (1.4).

Theorem 2.2. *Let $\alpha,\beta>0$, $f_0\geq 0$, and let $f_1,f_2,...,f_n$ be functions such that*

$$\prod_{i=1}^{n}(f_i(x)-f_i(y))\geq 0$$

for all $x,y>0$, then

$$\frac{t^{\beta}}{\Gamma(\beta+1)}I^{\alpha}\left(\prod_{i=0}^{n} f_i\right)(t)+(-1)^{n}\frac{t^{\alpha}}{\Gamma(\alpha+1)}I^{\beta}\left(\prod_{i=0}^{n} f_i\right)(t)$$
$$\geq \sum_{\Omega\subseteq\{0,1,\dots,n\},0<\#\Omega<n+1}(-1)^{1+\#(\Omega-\{0\})}\left(I^{\beta}\left(\prod_{i\in\Omega} f_i\right)(t)\right)\left(I^{\alpha}\left(\prod_{i\notin\Omega} f_i\right)(t)\right) \tag{2.2}$$

for all $t>0$.

Proof. By the assumption, we have

$$\int_0^t\int_0^t (t-x)^{\alpha-1}(t-y)^{\beta-1}(f_0(x)+f_0(y))\prod_{i=1}^{n}(f_i(x)-f_i(y))dxdy\geq 0$$

for all $t > 0$.

Similar to the proof of Theorem 2.1, we have, for any $t > 0$,

$$\int_0^t (t-x)^{\alpha-1} \prod_{i=0}^n f_i(x)dx \int_0^t (t-y)^{\beta-1} dy$$
$$+ (-1)^n \int_0^t (t-y)^{\beta-1} \prod_{i=0}^n f_i(y)dy \int_0^t (t-x)^{\alpha-1} dx$$
$$\geq - \int_0^t \int_0^t (t-x)^{\alpha-1}(t-y)^{\beta-1} \sum_{\{z_0,\ldots z_n\}=\{x,y\}} (-1)^{\#\{i>0|z_i=y\}} \left(\prod_{i=0}^n f_i(z_i)\right) dxdy,$$

so

$$\frac{t^\beta\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\beta+1)} I^\alpha \left(\prod_{i=0}^n f_i\right)(t) + (-1)^n \frac{t^\alpha\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha+1)} I^\beta \left(\prod_{i=0}^n f_i\right)(t)$$
$$\geq - \sum_{\Omega\subseteq\{0,\ldots,n\},0<\#\Omega<n+1} (-1)^{\#(\Omega-\{0\})} \left(\Gamma(\beta) I^\beta \left(\prod_{i\in\Omega} f_i\right)(t)\right)$$
$$\times \left(\Gamma(\alpha) I^\alpha \left(\prod_{i\notin\Omega} f_i\right)(t)\right).$$

This is implies the inequality (2.2). □

We note on Theorem 2.2 that if $n = 2$ then we obtain the inequality (1.3).

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