Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com
Volume 4 Issue 3(2013), Pages 21-28.

TWO-STEP ITERATION SCHEME FOR NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACE

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Abstract. In this paper, we introduced a new type of two-step iterative process to approximate the common fixed points of two nonexpansive mappings in uniformly convex Banach spaces and established weak and strong convergence results for common fixed points of nonexpansive mappings. The results obtained in this paper are generalizations and improvement of recently proved by another author.

1. Introduction :

Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $F(T) \neq \phi$ i.e., $F(T) = \{x \in K : Tx = x\}$. A mapping $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in K$. We know that a point $x \in K$ is a fixed point of T if $Tx = x$.

Several authors have been studied interative techniques for approximation fixed points of nonexpansive mappings (see [9], [10], [2], [4] and [5]) by using the Mann iteration method (see [11]) or the Ishikawa iteration method (see [8]).

The Picard and Mann [11] iteration schemes for a mapping $T : K \to K$ are defined by

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = Tx_n \end{cases} \tag{1.1}$$

and

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is in $(0, 1)$.

2000 *Mathematics Subject Classification.* 47H17, 47H05, 47H10.

Key words and phrases. Two-step iteration process, Nonexpansive mappings, Condition (A'), Opial's condition, Common fixed point.

Submitted December 31, 2012. Published May 17, 2013.

Recently, Khan et al. ([6], [7]) modified the iteration process to the case of two mappings as follows:

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n S y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, n \in \mathbb{N} \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0, 1)$.

In this paper, we have introduced a new implicit iteration scheme given below to compute the common fixed points for a pair of single valued mappings:

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n S y_n, \\ y_n = (1-\beta_n)Sx_n + \beta_n T x_n, n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Observe that in (1.4), if we set $S = I, \beta_n = 0$, then the scheme will reduce to Mann iteration process [11]:

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n x_n, n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

2. Preliminaries

Let $X = \{x \in E : \|x\| = 1\}$ and E^* be the dual of E. The space E has :
(i) Gâteaux differentiable norm if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for each $x, y \in K$;
(ii) Frèchet differentiable norm (see e.g. [12]) for each x in S, the above limit exists and is attained uniformly for y in S and in this case, it is also well-known that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x + h\|^2 \leq \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) \tag{2.1}$$

for all $x, h \in E$, where J is the Frèchet derivative of the function $\frac{1}{2}\|.\|^2$ at $x \in E$, $\langle ., .\rangle$ is the dual pairing between E and E^*, and b is an increasing function defined on $[0, \infty)$ such that $\lim_{t\to 0} \frac{b(t)}{t} = 0$;
(iii) Opial's condition [13] if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$.

Following are the definitions and lemma used to prove the results in the next section.

Definition 2.1. *A self-mapping T of a subset K of a normed linear space is said to be nonexpansive if*

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for all $x, y \in K$.

Definition 2.2. *A self-mapping T of a subset K of a normed linear space is said to be quasi-nonexpansive provided T has at least one fixed point in K, and if $p \in K$ is any fixed point of T, then*

$$\|Tx - p\| \leq \|x - p\|,$$

holds for all $x \in K$.

Definition 2.3. . *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is said to be demi-closed at 0, if $x_n \to x$ converges weakly and $x_n - Tx_n \to 0$ converges strongly, then it is implies that $x \in K$ and $Tx = x$.*

Definition 2.4. [1] . *Suppose two mappings $S, T : K \to K$, where K is a subset of a normed space E, said to be satisfy **condition** (A') if there exists a nondecreasing function $F : [0, \infty) \to [0, \infty)$ with $F(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either $\|x - Sx\| \geq f(d(x, F))$ or $\|x - Tx\| \geq f(d(x, F))$ for all $x \in K$ where $d(x, F) = inf\{\|x - p\| : p \in F = F(S) \cap F(T)\}$.*

Lemma 2.5. [3]*: Suppose that E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $limsup_{n\to\infty}\|x_n\| \leq r$, $limsup_{n\to\infty}\|y_n\| \leq r$ and $lim_{n\to\infty}\|(1 - t_n)x_n + t_n y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Convergence Results :

In this section, we have proved the approximate common fixed points of two nonexpansive mapping for weak and strong convergence results, using a new type of iteration process. In the consequence, F denotes the set of common fixed point of the mapping S and T.

Lemma 3.1. *: Let K be a nonempty convex subset of a uniformly convex Banach space E. Suppose $S, T : K \to K$ be an nonexpansive mappings and $\{x_n\}$ be the sequence as defined by (1.4), with restrictions $\sum_{n=1}^{\infty} \alpha_n < \infty$, and $\sum_{n=1}^{\infty} \beta_n < \infty$. If $F(S) \cap F(T) \neq \phi$, and*

$$\|x - Sy\| \leq \|Tx - Sy\|, \text{for all } x, y \in K, \tag{3.1}$$

then

$$\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|Tx_n - x_n\| = 0,$$

for all $p \in F(S) \cap F(T)$.

Proof. Suppose $p \in F(S) \cap F(T)$ and $F(S) \cap F(T) \neq \phi$. Since S, T are nonexpansive mappings, now using (1.4), we have

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)Sx_n + \beta_n Tx_n - p\| \\ &\leq (1 - \beta_n)\|Sx_n - p\| + \beta_n\|Tx_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &\leq \|x_n - p\|, \end{aligned} \tag{3.2}$$

and,

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)Tx_n+\alpha_n Sy_n-p\|\\ &\le (1-\alpha_n)\|Tx_n-p\|+\alpha_n\|Sy_n-p\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|x_n-p\|\\ &\le \|x_n-p\|. \end{aligned} \tag{3.3}$$

Since $\{\|x_n-p\|\}$ is a non-increasing and bounded sequence, so $\lim_{n\to\infty}\|x_n-p\|$ exist. Let $\lim_{n\to\infty}\|x_n-p\|=c$ and suppose that $c>0$,
we get

$$\begin{aligned} c &= \lim_{n\to\infty}\|x_{n+1}-p\|\\ &= \lim_{n\to\infty}\|(1-\alpha_n)Tx_n+\alpha_n Sy_n-p\|\\ &= \lim_{n\to\infty}(1-\alpha_n)\|Tx_n-p\|+\alpha_n\|Sy_n-p\|. \end{aligned} \tag{3.4}$$

Lemma 2.5 gives

$$\lim_{n\to\infty}\|Tx_n-Sy_n\|=0. \tag{3.5}$$

Now, from **(3.2)** and **(3.3)** , we have

$$\limsup_{n\to\infty}\|Sy_n-p\|\le\limsup_{n\to\infty}\|y_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|, \tag{3.6}$$

also

$$\limsup_{n\to\infty}\|Tx_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|=c. \tag{3.7}$$

It follows from **(3.1)** and **(3.5)**,

$$\begin{aligned}\|Tx_n-x_n\| &= \|Tx_n-Sy_n\|+\|Sy_n-x_n\|\\ &\le \|Tx_n-Sy_n\|+\|Tx_n-Sy_n\|\\ &\le 2\|Tx_n-Sy_n\|\\ &\to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.8}$$

Taking limsup on both sides of the above inequality, we obtain

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0,$$

and hence

$$\begin{aligned}\|Sy_n-x_n\| &\le \|Sy_n-Tx_n\|+\|Tx_n-x_n\|\\ &\to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.9}$$

Using (3.8) and (3.9), we have

$$\begin{aligned}\|Sx_n-x_n\| &\le \|Sx_n-Sy_n\|+\|Sy_n-x_n\|\\ &\le \|y_n-x_n\|+\|Sy_n-x_n\|\\ &\le (1-\beta_n)\|Sx_n-x_n\|+\beta_n\|Tx_n-x_n\|+\|Sy_n-x_n\|\\ &\le \frac{1}{\beta_n}(\beta_n\|Tx_n-x_n\|+\|Sy_n-x_n\|), \end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0.$$

This completes the proof. □

Example 3.1 : Let E be the real line with the usual norm $|.|$ and suppose $K = [0, 1]$. Define $S, T : K \to K$ by

$$Tx = \frac{2-x}{2} \text{ and } Sy = \frac{4-y}{5}$$

for all $x, y \in K$. Obviously both S and T are nonexpansive with the common fixed point $\frac{2}{3}$ for all $x, y \in K$. Now we check that our condition $\|x - Sy\| \leq \|Tx - Sx\|$ for all $x, y \in K$ is true. If $x, y \in [0, 1]$, then

$$|x - Sy| = |x - \frac{(4-y)}{5}| = |\frac{5x + y - 4}{5}|, \text{ and}$$

$$|Tx - Sy| = |\frac{2-x}{2} - \frac{4-y}{5}| = |\frac{2y - 5x + 2}{10}|.$$

Clearly, $|\frac{5x+y-4}{5}| \leq |\frac{2y-5x+2}{10}|$, so that $|x - Sy| \leq |Tx - Sx|$ for all $x, y \in K$. Now, we check that S and T are quasi-nonexpansive type mappings. In fact, if $x \in [0, 1]$ and $p \in [0, 1]$, then

$$|Tx - p| = |\frac{2-x}{2} - 0| = |\frac{2-x}{2}| = |\frac{2-x}{2}| \leq |x| = |x - 0| = |x - p|,$$

that is

$$|Tx - p| \leq |x - p|.$$

Similarly, we prove that

$$|Sx - p| \leq |x - p|.$$

Therefore, S and T are quasi-nonexpansive type mappings.

Lemma 3.2. *: Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Suppose $\{x_n\}$ be the sequence defined in Theorem (3.3) with $F \neq \phi$. Then, for any $p_1, p_2 \in F$, $\lim_{n\to\infty}\langle x_n, J(p_1 - p_2)\rangle$ exist, in particular, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_\omega(x_n)$.*

Proof. Take $x = p_1 - p_2$, with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in the inequality **(2.1)** to get:

$$\begin{aligned}\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle &\leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\ &\leq \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle + b(t\|x_n - p_1\|).\end{aligned}$$

As $sup_{n\geq1}\|x_n - p_1\| \leq M'$ for some $M' > 0$, it follows that

$$\begin{aligned}\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty}&\langle x_n - p_1, J(p_1 - p_2)\rangle \\ &\leq \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2 \\ &\leq \frac{1}{2}\|p_1 - p_2\|^2 + b(tM') + t\liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle.\end{aligned}$$

That is,

$$\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle \leq \liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{b(tM')}{tM'}M'.$$

If $t \to 0$, then $\lim_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F$, in particular, we get

$$\langle p - 1, J(p_1 - p_2)\rangle = 0$$

for all $p, q \in \omega_\omega(x_n)$. □

Theorem 3.3. *Let E be a uniformly convex Banach space satisfying Opial condition and K, T, S and $\{x_n\}$ be taken as* ***Lemma 3.1****. If $F(S) \cap F(T) \neq \phi, I - T$ and $I - S$ are demiclosded at zero, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $p \in F(S) \cap F(T)$, then as proved in **Lemma 3.1** $\lim_{n\to\infty}\|x_n - p\|$ exist. Since E is uniformly convex Banach space. Thus there exists subsequences $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z_1 \in K$. From **Lemma 3.1**, we have

$$\lim_{n\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0,$$

and

$$\lim_{n\to\infty}\|Sx_{n_k} - x_{n_k}\| = 0.$$

Since $I - T$ and $I - S$ are demiclosed at zero, therefore $Sz_1 = z_1$. Similarly $Tz_1 = z_1$. Finally, we prove that $\{x_n\}$ converges weakly to z_1. Let on contrary that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $z_2 \in K$ and $z_1 \neq z_2$. Again in the same way, we can prove that $z_2 \in F(S) \cap F(T)$. From **Lemma 3.1** the limits $\lim_{n\to\infty}\|x_n - z_1\|$ and $\lim_{n\to\infty}\|x_n - z_2\|$ exists. Suppose that $z_1 \neq z_2$, then by the Opial's condition, we get

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z_1\| &= \lim_{n_i\to\infty}\|x_{n_i} - z_1\| < \lim_{n_i\to\infty}\|x_{n_i} - z_2\| \\ &= \lim_{n\to\infty}\|x_n - z_2\| = \lim_{n_j\to\infty}\|x_{n_j} - z_2\| \\ &< \lim_{n_j\to\infty}\|x_{n_j} - z_1\| = \lim_{n\to\infty}\|x_n - z_1\|.\end{aligned}$$

This is a contradiction so $z_1 = z_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T and S. □

Theorem 3.4. *Let E be a real uniformly convex Banach space and $K, S, T, F, \{x_n\}$ be as in* ***Lemma 3.1****. Then $\{x_n\}$ converges strongly to a point of F if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Proof. Necessity is evident, let $\lim inf_{n\to\infty} d(x_n, F) = 0$. From **Lemma 3.1**, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$, so that $\lim_{n\to\infty} d(x_n, F)$ exists. Since by hypothesis, $\lim inf_{n\to\infty} d(x_n, F) = 0$, so that, we get

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

But $\{x_n\}$ is Cauchy sequence and therefore converges to p. We know that $\lim_{n\to\infty} d(x_n, F) = 0$, we obtained $d(p, F) = 0$, therefore $p \in F$.

□

Using **Theorem 3.4**, we obtain a strong convergence theorem of the iteration scheme (1.4) under the **condition** (A') as below:

Theorem 3.5. *Let E be a uniformly convex Banach space and $K, S, T, , F, \{x_n\}$ be as in **Lemma 3.1**. Let S, T satisfy the **condition** (A') and $F \neq \phi$. Then $\{x_n\}$ converges strongly to a point of F.*

Proof. We proved in **Lemma 3.1**, i.e.

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$$

Then from the definition of **condition** (A'), we obtain

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|Tx_n - x_n\| = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

In above cases, we get

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

But $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, so that we get

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

□

All the conditions of **Theorem 3.4** are satisfied, therefore by its conclusion $\{x_n\}$ converges to strongly to a fixed point of F.

The following results are immediate sequel of our strong convergence theorem.

Corollary 3.6. . *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Suppose T be a nonexpansive mapping of K. Let $\{x_n\}$ be defined by the iteration (1.4), where $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Suppose $S = T$ in the above theorem.

Corollary 3.7. . *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Suppose T be a nonexpansive mapping of K. Let $\{x_n\}$ be defined by the iteration (1.2), where $\{\alpha_n\}$ in $[0, 1]$ for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Suppose $S = I$ in the above theorem.

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