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MULTIVARIABLE CONSTRUCTION OF EXTENDED JACOBI MATRIX POLYNOMIALS

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Abstract. The main aim of this paper is to construct a multivariable extension with the help of the extended Jacobi matrix polynomials (EJMPs). Generating matrix functions and recurrence relations satisfied by these multivariable matrix polynomials are derived. Furthermore, general families of multilinear and multilateral generating matrix functions are obtained and their applications are presented.

1. Introduction

The areas of orthogonal polynomials have many applications in various branches of mathematics and other disciplines. There are many papers in the literature dealing with orthogonal polynomials (see, for example, [3, 14, 24, 29] and the references therein). Recently, matrix orthogonal polynomials have started to become the focus of interest. General theory of matrix valued orthogonal polynomials which have started with the work of M. G. Krein [25, 26] and then have studied by many authors (see [7, 8, 15, 17, 18]) plays an important part in many areas of mathematics just as their scalar counterparts. A good source for matrix polynomials is the book by Gohberg, Lancaster and Rodman [16]. In [1, 5, 2, 4, 9, 6, 10, 12, 20, 21], the classical orthogonal polynomials have been extended to the orthogonal matrix polynomials. Jódar and Cortés introduced and studied the hypergeometric matrix function $F(A, B; C; z)$ and the hypergeometric matrix differential equation in [22] and the explicit closed form general solution of it has been given in [23]. In [6, 11, 10, 12, 19, 20, 27], Jacobi, Chebyshev, Gegenbauer, Laguerre and Hermite matrix polynomials were introduced and various results were given for these matrix polynomials. Furthermore, the authors introduced new extension of Jacobi matrix polynomials in [30]. In [1], authors studied the polynomial $F_n^{(A,B)}(x; a, b, c)$ called extended Jacobi matrix polynomial (EJMP) for parameter matrices A and B whose eigenvalues, z, all satisfy $Re(z) > -1$. For any natural number $n \geq 0$, the n-th

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degree extended Jacobi matrix polynomial $F_n^{(A,B)}(x;a,b,c)$ is defined by

$$\begin{aligned} & F_n^{(A,B)}(x;a,b,c) \\ = & \frac{(c(a-b))^n}{n!} F\left(-nI, A+B+(n+1)I; A+I; \frac{x-a}{b-a}\right)(A+I)_n . \end{aligned} \tag{1}$$

Also, it is shown that these matrix polynomials have the Rodrigues formula [1]:

$$\begin{aligned} F_n^{(A,B)}(x;a,b,c) = & \frac{(-c)^n}{n!}(x-a)^{-A}(b-x)^{-B} \\ & .\frac{d^n}{dx^n}\left\{(x-a)^{A+nI}(b-x)^{B+nI}\right\} \quad , \ (c>0) \end{aligned} \tag{2}$$

where A and $B \in \mathbb{C}^{r\times r}$ satisfy

$$Re(z) > -1 \text{ for } z \in \sigma(A),\ Re(\eta) > -1 \text{ for } \eta \in \sigma(B),\ AB = BA.$$

By comparing the Rodrigues representation for Jacobi matrix polynomials and (2), we have

$$F_n^{(A,B)}(x;a,b,c) = \{c(a-b)\}^n P_n^{(A,B)}\left(\frac{2(x-a)}{a-b}+1\right) \tag{3}$$

or, equivalently,

$$P_n^{(A,B)}(x) = \{c(a-b)\}^{-n} F_n^{(A,B)}\left(\frac{1}{2}\{a+b+(a-b)x\}; a,b,c\right). \tag{4}$$

The EJMPs $F_n^{(A,B)}(x;a,b,c)$ are orthogonal over the interval (a,b) with respect to the weight function $\omega(x;A,B) = (x-a)^A(b-x)^B$ [1]. In fact, it is hold that

$$\begin{aligned} & \int_a^b (x-a)^A(b-x)^B F_n^{(A,B)}(x;a,b,c) F_m^{(A,B)}(x;a,b,c)\,dx \\ = & \begin{cases} \frac{c^{2n}}{n!}(b-a)^{A+B+(2n+1)I}\Gamma(A+B+(2n+1)I) \\ \quad .\Gamma^{-1}(A+B+(n+1)I)\Gamma(B+(n+1)I) & , \quad m=n \\ \quad .\Gamma(A+(n+1)I)\Gamma^{-1}(A+B+2(n+1)I) \\ \mathbf{0} & , \quad m\neq n \end{cases} \\ & (m,n\in\mathbb{N}_0 := \mathbb{N}\cup\{0\} = \{0,1,2,...\}) \end{aligned} \tag{5}$$

where A and $B \in \mathbb{C}^{r\times r}$ satisfy

$$Re(z) > -1 \text{ for } z \in \sigma(A),\ Re(\eta) > -1 \text{ for } \eta \in \sigma(B),\ AB = BA.$$

In this paper, we construct a multivariable extension of extended Jacobi matrix polynomials and show that these matrix polynomials are orthogonal with respect to weight matrix function. Generating matrix functions are obtained for the multivariable extended Jacobi matrix polynomials (MEJMPs) and with the help of generating matrix function, several recurrence formulas are given for these polynomials. Furthermore, multilinear and multilateral generating matrix functions are derived for MEJMPs and some applications of the results obtained are presented.

Throughout this paper, for a matrix $A \in \mathbb{C}^{r\times r}$, its spectrum is denoted by $\sigma(A)$. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^r$, $\|y\|_2 = \left(y^T y\right)^{1/2}$ is the Euclidean norm of y. I and $\mathbf{0}$ will denote the identity matrix and the null matrix in $\mathbb{C}^{r\times r}$, respectively. We say that a matrix A in $\mathbb{C}^{r\times r}$ is a positive stable if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A. If $A_0, A_1, ..., A_n$ are elements of $\mathbb{C}^{r\times r}$ and $A_n \neq \mathbf{0}$, then we call

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0$$

a matrix polynomial of degree n in x. From [22], one can see

$$(P)_n = P(P+I)(P+2I)...(P+(n-1)I)\ ;\ n \geq 1\ ;\ (P)_0 = I. \tag{6}$$

From the relation (6), we see that

$$\frac{(-1)^k}{(n-k)!}I = \frac{(-nI)_k}{n!}\ ;\ 0 \leq k \leq n. \tag{7}$$

The hypergeometric matrix function $F(A,B;C;z)$ has been given in the form [22]

$$F(A,B;C;z) =_{n=0}^{\infty} \frac{(A)_n(B)_n}{n!}\left[(C)_n\right]^{-1} z^n$$

for matrices A, B and C in $\mathbb{C}^{r\times r}$ such that $C + nI$ is invertible for all integer $n \geq 0$ and for $|z| < 1$. For any matrix A in $\mathbb{C}^{r\times r}$, the authors exploited the following relation due to [22]

$$(1-x)^{-A} =_{n=0}^{\infty} \frac{(A)_n}{n!}x^n\ ,\ |x| < 1. \tag{8}$$

In [13], if $f(z)$ and $g(z)$ are holomorphic functions in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{r\times r}$ for which $\sigma(A) \subset \Omega$, then

$$f(A)g(A) = g(A)f(A).$$

Hence, if $B \in \mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B) \subset \Omega$ and $AB = BA$, then

$$f(A)g(B) = g(B)f(A).$$

Furthermore, in [12], the reciprocal scalar Gamma function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z. Thus, for any $C \in \mathbb{C}^{\setminus\times\setminus}$, the Riesz-Dunford functional calculus [13] shows that $\Gamma^{-1}(C)$ is well defined and is, indeed, the inverse of $\Gamma(C)$. Hence: if $C \in \mathbb{C}^{r\times r}$ is such that $C + nI$ is invertible for every integer $n \geq 0$, then

$$(C)_n = \Gamma(C+nI)\Gamma^{-1}(C).$$

2. Multivariable extension of the extended Jacobi matrix polynomials

A systematic investigation of a multivariable extension of the extended Jacobi matrix polynomials $F_n^{(A,B)}(x;a,b,c)$ is defined by

$$F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x}) = F_{n_1,\ldots,n_s}^{(A_1,\ldots,A_s;B_1,\ldots,B_s)}(\mathbf{x}) = F_{n_1}^{(A_1,B_1)}(x_1;a_1,b_1,c_1)\ldots F_{n_s}^{(A_s,B_s)}(x_s;a_s,b_s,c_s) \tag{9}$$

where $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{A} = (A_1, ..., A_s)$, $\mathbf{B} = (B_1, ..., B_s)$, A_i and B_i are matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $Re(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $Re(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$ and $\mathbf{n} = n_1 + ... + n_s$; $n_1, ..., n_s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, ...\}$. The multivariable extended

Jacobi matrix polynomials $F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$ (MEJMPs) are orthogonal with respect to the weight matrix function

$$\begin{aligned}\omega(\mathbf{x},\mathbf{A},\mathbf{B}) &= \omega(x_1,...,x_s;A_1,...,A_s;B_1,...,B_s)\\ &= \omega_1(x_1,A_1,B_1)...\omega_s(x_s,A_s,B_s)\\ &= (x_1-a_1)^{A_1}(b_1-x_1)^{B_1}...(x_s-a_s)^{A_s}(b_s-x_s)^{B_s}\end{aligned}$$

over the domain

$$\Omega=\{(x_1,...,x_s):\ a_i<x_i<b_i\ ;\ i=1,2,...,s\}.$$

In fact, we have by (5)

$$\begin{aligned}&{}_{\Omega}\omega(\mathbf{x},\mathbf{A},\mathbf{B})\,F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})\,F_{\mathbf{m}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})\,\mathbf{dx}\\ =\ &{}_{a_1}^{b_1}(x_1-a_1)^{A_1}(b_1-x_1)^{B_1}F_{n_1}^{(A_1,B_1)}(x_1;a_1,b_1,c_1)\,F_{m_1}^{(A_1,B_1)}(x_1;a_1,b_1,c_1)\,dx_1\times...\\ &\times{}_{a_s}^{b_s}(x_s-a_s)^{A_s}(b_s-x_s)^{B_s}F_{n_s}^{(A_s,B_s)}(x_s;a_s,b_s,c_s)\,F_{m_s}^{(A_s,B_s)}(x_s;a_s,b_s,c_s)\,dx_s\\ =\ &{}_{i=1}^{s}\left\{\frac{c_i^{2n_i}}{n_i!}(b_i-a_i)^{A_i+B_i+(2n_i+1)I}\,\Gamma(A_i+B_i+(2n_i+1)I)\,\Gamma^{-1}(A_i+B_i+(n_i+1)I)\right.\\ &\left..\,\Gamma(B_i+(n_i+1)I)\,\Gamma(A_i+(n_i+1)I)\,\Gamma^{-1}(A_i+B_i+2(n_i+1)I)\right\}\delta_{m_i,n_i}\\ &(m_i,n_i\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\ ;\ i=1,2,...,s)\end{aligned}$$

where $\mathbf{dx}=dx_1...dx_s$ and these matrices are commutative.

Thus, the following result has been established:

The MEJMPs $F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$ are orthogonal with respect to the weight matrix function

$$\omega(\mathbf{x},\mathbf{A},\mathbf{B})=(x_1-a_1)^{A_1}(b_1-x_1)^{B_1}...(x_s-a_s)^{A_s}(b_s-x_s)^{B_s}$$

over the domain

$$\Omega=\{(x_1,...,x_s):\ a_i<x_i<b_i\ ;\ i=1,2,...,s\}$$

where A_i and B_i are matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $Re(z_i)>-1$ for each eigenvalue $z_i\in\sigma(A_i)$ and $Re(\eta_i)>-1$ for each eigenvalue $\eta_i\in\sigma(B_i)$ for $1\le i\le s$ and all matrices are commutative.

3. Generating Matrix Functions and Recurrence Relations for MEJMPs

In [1], it was shown that the EJMPs are generated by

$$\begin{aligned}&{}_{n=0}^{\infty}(c(a-b))^{-n}(A+B+I)_n\,F_n^{(A,B)}(x;a,b,c)\,[(A+I)_n]^{-1}t^n\\ =\ &(1-t)^{-(A+B+I)}F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};A+I;\frac{4t(x-a)}{(a-b)(1-t)^2}\right),\end{aligned}\quad(10)$$

where all eigenvalues z of the matrices A and B of the extended Jacobi matrix polynomials $F_n^{(A,B)}(x;a,b,c)$ satisfy the condition $Re(z)>-1$ and $|t|<1$. On the other hand, other generating function for EJMPs is as follows [1]:

$$\begin{aligned}&{}_{n=0}^{\infty}(c(a-b))^{-n}F_n^{(A,B)}(x;a,b,c)\,t^n\\ &=F_4\left(I+B,I+A;I+A,I+B;\frac{(x-a)t}{a-b},\frac{(x-b)t}{a-b}\right)\end{aligned}$$

where $AB = BA$ and $F_4(A,B;C,D;x,y)$ is the matrix version of the Appell's function of two variables which is defined by

$$F_4(A,B;C,D;x,y) = \sum_{n,k=0}^{\infty} (A)_{n+k}(B)_{n+k}(D)_n^{-1}(C)_k^{-1}\frac{x^k y^n}{k!n!}$$
$$\left(\sqrt{x}+\sqrt{y}<1\right)$$

where $C+nI$ and $D+nI$ are invertible for every integer $n \geq 0$ (see [5]). The other one is

$$\begin{aligned} &{}_{n=0}^{\infty}(C)_n(D)_n(I+B)_n^{-1}\left(c\left(a-b\right)\right)^{-n}F_n^{(A,B)}(x;a,b,c)(I+A)_n^{-1}t^n \\ = \;& F_4\left(C,D;I+A,I+B;\frac{(x-a)\,t}{a-b},\frac{(x-b)\,t}{a-b}\right) \end{aligned}$$

where $A+nI$ and $B+nI$ are invertible for every integer $n \geq 0$ [1].

In this section, we obtain generating matrix functions and recurrence relations for MEJMPs $F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$.

Using the above expressions, we can give the following results.

For the MEJMPs $F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$, we have

$$\begin{aligned} &{}_{n_1,\ldots,n_s=0}^{\infty}\left(\mathbf{c}\left(\mathbf{a}-\mathbf{b}\right)\right)^{-\mathbf{n}}(\mathbf{A}+\mathbf{B}+I)_{\mathbf{n}}\left[(\mathbf{A}+I)_{\mathbf{n}}\right]^{-1}F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})\,t_1^{n_1}\ldots t_s^{n_s} \\ = \;& {}_{i=1}^{s}(1-t_i)^{-(A_i+B_i+I)}F\left(\frac{A_i+B_i+I}{2},\frac{A_i+B_i+2I}{2};A_i+I;\frac{4t_i(x_i-a_i)}{(a_i-b_i)(1-t_i)^2}\right) \\ &(|t_i|<1) \end{aligned}$$

or equivalently,

$$\begin{aligned} &{}_{n_1,\ldots,n_s=0}^{\infty}\left(\mathbf{c}\left(\mathbf{a}-\mathbf{b}\right)\right)^{-\mathbf{n}}(\mathbf{A}+\mathbf{B}+I)_{\mathbf{n}}\left[(\mathbf{A}+I)_{\mathbf{n}}\right]^{-1}F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}\left((\mathbf{b}-\mathbf{a})\,\mathbf{v}+\mathbf{a}\right)t_1^{n_1}\ldots t_s^{n_s} \\ = \;& {}_{i=1}^{s}(1-t_i)^{-(A_i+B_i+I)}F\left(\frac{A_i+B_i+I}{2},\frac{A_i+B_i+2I}{2};A_i+I;\frac{-4t_iv_i}{(1-t_i)^2}\right) \\ &(|t_i|<1) \end{aligned}$$

where A_i and B_i are matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $Re(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $Re(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$, all matrices are commutative and

$$\begin{aligned} & {}_{i=1}^{s}\left\{\left(c_i\left(a_i-b_i\right)\right)^{-n_i}(A_i+B_i+I)_{n_i}\left[(A_i+I)_{n_i}\right]^{-1}\right\} \\ = \;& \left(\mathbf{c}\left(\mathbf{a}-\mathbf{b}\right)\right)^{-\mathbf{n}}(\mathbf{A}+\mathbf{B}+I)_{\mathbf{n}}\left[(\mathbf{A}+I)_{\mathbf{n}}\right]^{-1}, \\ (\mathbf{b}-\mathbf{a})\,\mathbf{v}+\mathbf{a} = \;& \left((b_1-a_1)\,v_1+a_1,\ldots,(b_s-a_s)\,v_s+a_s\right) \end{aligned}$$

where A_i+n_iI is invertible for every integer $n_i \geq 0$ for $i = 1,2,\ldots,s$.

The matrix polynomials $F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$ are generated by

$$\begin{aligned} &{}_{n_1,\ldots,n_s=0}^{\infty}\left(\mathbf{c}\left(\mathbf{a}-\mathbf{b}\right)\right)^{-\mathbf{n}}F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})\;t_1^{n_1}\ldots t_s^{n_s} \\ = \;& {}_{i=1}^{s}F_4\left(I+B_i,I+A_i;I+A_i,I+B_i;\frac{(x_i-a_i)\,t_i}{a_i-b_i},\frac{(x_i-b_i)\,t_i}{a_i-b_i}\right) \\ &\left(\sqrt{\tfrac{(x_i-a_i)t_i}{a_i-b_i}}+\sqrt{\tfrac{(x_i-b_i)t_i}{a_i-b_i}}<1\;;\;i=1,2,\ldots,s\right) \end{aligned}$$

where

$$\left(\mathbf{c}\left(\mathbf{a}-\mathbf{b}\right)\right)^{-\mathbf{n}} = {}_{i=1}^{s}\left(c_i\left(a_i-b_i\right)\right)^{-n_i}$$

and A_i and B_i are matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $Re(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $Re(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \le i \le s$, all matrices are commutative.

Let $A_i, B_i, C_i, D_i \in \mathbb{C}^{r\times r}$ for $i = 1, 2, ..., s$. For MEJMPs $F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$, we have the following generating matrix function

$$\begin{aligned} & {}_{n_1,\dots,n_s=0}^{\infty}(\mathbf{C})_{\mathbf{n}}(\mathbf{D})_{\mathbf{n}}(I+\mathbf{B})_{\mathbf{n}}^{-1}\left(\mathbf{c}\left(\mathbf{a}-\mathbf{b}\right)\right)^{-\mathbf{n}} F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})\,(I+\mathbf{A})_{\mathbf{n}}^{-1}t_1^{n_1}...t_s^{n_s} \\ = \ & {}_{i=1}^{s}F_4\left(C_i, D_i; I+A_i, I+B_i; \frac{(x_i-a_i)\,t_i}{a_i-b_i}, \frac{(x_i-b_i)\,t_i}{a_i-b_i}\right) \end{aligned}$$

where A_i+n_iI and B_i+n_iI are invertible for every integer $n_i \ge 0$ for $i = 1, 2, ..., s$, all matrices are commutative and

$$\begin{aligned} & (\mathbf{C})_{\mathbf{n}}(\mathbf{D})_{\mathbf{n}}(I+\mathbf{B})_{\mathbf{n}}^{-1}\left(\mathbf{c}\left(\mathbf{a}-\mathbf{b}\right)\right)^{-\mathbf{n}}(I+\mathbf{A})_{\mathbf{n}}^{-1} \\ = \ & {}_{i=1}^{s}(C)_{n_i}(D)_{n_i}(I+B_i)_{n_i}^{-1}\left(c_i\left(a_i-b_i\right)\right)^{-n_i}(I+A_i)_{n_i}^{-1}. \end{aligned}$$

For the MEJMPs $F_{n_1,n_2,\dots,n_s}^{(A_1,\dots,A_s;B_1,\dots,B_s)}(\mathbf{x})$, we have

$$\begin{aligned} & {}_{n=0}^{\infty}H_n^{(A_1,\dots,A_s;B_1,\dots,B_s)}(x_1,...,x_s)\,t^n \\ = \ & {}_{i=1}^{s}(1-t)^{-(A_i+B_i+I)}F\left(\frac{A_i+B_i+I}{2}, \frac{A_i+B_i+2I}{2}; A_i+I; \frac{4t(x_i-a_i)}{(a_i-b_i)\,(1-t)^2}\right) \\ & (|t|<1) \end{aligned}$$

where

$$\begin{aligned} & H_n^{(A_1,\dots,A_s;B_1,\dots,B_s)}(x_1,...,x_s) \\ = \ & {}_{n_1=0}^{n}{}_{n_2=0}^{n-n_1}\cdots{}_{n_{s-1}=0}^{n-n_1-\dots-n_{s-2}}\mathbf{K}\left(n_1,...,n_{s-1}\right)F_{n-\left(n_1+\dots+n_{s-1}\right),n_1,\dots,n_{s-1}}^{(A_1,\dots,A_s;B_1,\dots,B_s)}(x_1,...,x_s) \\ & .\mathbf{N}\left(n_1,...,n_{s-1}\right)\ ; \end{aligned}$$

$$\begin{aligned} \mathbf{K}\left(n_1,...,n_{s-1}\right) &= \frac{(A_1+B_1+I)_{n-(n_1+\dots+n_{s-1})}\,(A_2+B_2+I)_{n_1}...(A_s+B_s+I)_{n_{s-1}}}{\left(c_1\left(a_1-b_1\right)\right)^{n-(n_1+\dots+n_{s-1})}\left(c_2\left(a_2-b_2\right)\right)^{n_1}...\left(c_s\left(a_s-b_s\right)\right)^{n_{s-1}}}, \\ \mathbf{N}\left(n_1,...,n_{s-1}\right) &= \left[(A_s+I)_{n_{s-1}}\right]^{-1}\left[(A_{s-1}+I)_{n_{s-2}}\right]^{-1}\times ... \\ & \quad \times\left[(A_2+I)_{n_1}\right]^{-1}\left[(A_1+I)_{n-(n_1+\dots+n_{s-1})}\right]^{-1} \end{aligned}$$

and also A_i and B_i are matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $Re(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $Re(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$, all matrices are commutative and $A_i + kI$ is invertible for every integer $k \ge 0$ for $1 \le i \le s$.

For MEJMPs $F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$, we have hypergeometric matrix representation as follows:

$$F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x}) = {}_{i=1}^{s}\left\{\frac{\left(c_i\left(a_i-b_i\right)\right)^{n_i}}{n_i!}F\left(A_i+B_i+(n_i+1)\,I, -n_iI; A_i+I; \frac{x_i-a_i}{b_i-a_i}\right)(A_i+I)_{n_i}\right\}.$$

In order to obtain some recurrence relations, we need the following lemma.

Let a generating matrix function for $g_{n_1,\dots,n_s}(\mathbf{x},\mathbf{C})$ be

$$(1-t_1)^{-C_1}...(1-t_s)^{-C_s}\Psi\left(\frac{-4x_1t_1}{(1-t_1)^2}, ..., \frac{-4x_st_s}{(1-t_s)^2}\right) = {}_{n_1,\dots,n_s=0}^{\infty}\, g_{n_1,\dots,n_s}(\mathbf{x},\mathbf{C})\,t_1^{n_1}...t_s^{n_s}$$

where $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{C} = (C_1, ..., C_s)$ and $g_{n_1,...,n_s}(\mathbf{x}, \mathbf{C})$ is a matrix polynomial of degree n_i with respect to x_i (of total degree $n = n_1 + ... + n_s$), provided that

$$\begin{aligned} \Psi(u_1, ..., u_s) &= \Psi_1(u_1)...\Psi_s(u_s)\ ;\ u_i = \tfrac{-4x_i t_i}{(1-t_i)^2}\ ,\ i = 1, 2, ..., s \\ \Psi_i(u_i) &= {}_{n_i=0}^{\infty}\gamma_{n_i}\ u_i^{n_i}\quad ,\quad \gamma_0 \neq \mathbf{0} \end{aligned}$$

Then we have

i) $x_i \dfrac{\partial}{\partial x_i} g_{n_1,...,n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1,...,n_s}(\mathbf{x}, \mathbf{C})$

$$= -(C_i+(n_i-1)I)g_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_s}(\mathbf{x}, \mathbf{C}) - x_i \frac{\partial}{\partial x_i} g_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_s}(\mathbf{x}, \mathbf{C}),\ n_i \geq 1 \tag{11}$$

ii) $x_i \dfrac{\partial}{\partial x_i} g_{n_1,...,n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1,...,n_s}(\mathbf{x}, \mathbf{C})$

$$= -C_i{}_{k=0}^{n_i-1} g_{n_1,...,n_{i-1},k,n_{i+1},....,n_s}(\mathbf{x}, \mathbf{C}) - 2x_i{}_{k=0}^{n_i-1} \frac{\partial}{\partial x_i} g_{n_1,...,n_{i-1},k,n_{i+1},....,n_s}(\mathbf{x}, \mathbf{C}),\ n_i \geq 1 \tag{12}$$

iii) $x_i \dfrac{\partial}{\partial x_i} g_{n_1,...,n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1,...,n_s}(\mathbf{x}, \mathbf{C}) =$

$$ {}_{k=0}^{n_i-1}(-1)^{n_i-k}(C_i + 2kI)g_{n_1,...,n_{i-1},k,n_{i+1},...,n_s}(\mathbf{x}, \mathbf{C}),\ n_i \geq 1 \tag{13}$$

where $C_iC_j = C_jC_i$ for $i, j = 1, 2, ..., s$.

Choosing

$$C_i = A_i + B_i + I \quad ; \quad \gamma_{n_i} = \frac{(I + A_i + B_i)_{2n_i}}{2^{2n_i} n_i!}(I + A_i)^{-1}_{n_i} \tag{14}$$

in Lemma 3 and considering Theorem 3, we see that the matrix polynomials $g_{\mathbf{n}}$ is

$$\begin{aligned} g_{n_1,...,n_s}(\mathbf{v}) &= (\mathbf{c}\,(\mathbf{a}-\mathbf{b}))^{-\mathbf{n}}\,(\mathbf{A}+\mathbf{B}+I)_{\mathbf{n}}\,[(\mathbf{A}+I)_{\mathbf{n}}]^{-1}\,F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}((\mathbf{b}-\mathbf{a})\,\mathbf{v}+\mathbf{a}) \\ &= {}_{i=1}^{s}\Big\{(c_i\,(a_i-b_i))^{-n_i}\,(A_i+B_i+I)_{n_i}\,[(A_i+I)_{n_i}]^{-1} \\ &\qquad .\ F_{n_i}^{(A_i,B_i)}\,((b_i-a_i)\,v_i+a_i; a_i, b_i, c_i)\Big\} \end{aligned}$$

where A_i, B_i $(i = 1, 2, ..., s)$ are commutative matrices in $\mathbb{C}^{r\times r}$. With the help of Lemma 3 and also considering (11)-(13), one can easily obtain the next result.

For the matrix polynomials $F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$, we have

i) $(x_i-a_i)\left[(A_i+B_i+n_iI)\dfrac{\partial F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})}{\partial x_i} + c_i\,(a_i-b_i)\,\dfrac{\partial F^{(\mathbf{A},\mathbf{B})}_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_s}(\mathbf{x})}{\partial x_i}(A_i+n_iI)\right]$

$$= (A_i+B_i+n_iI)\left[n_iF_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x}) - c_i\,(a_i-b_i)\,F^{(\mathbf{A},\mathbf{B})}_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_s}(\mathbf{x})(A_i+n_iI)\right],$$

ii) $(x_i-a_i)\dfrac{\partial F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})}{\partial x_i} - n_iF_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})$

$$\begin{aligned} = \quad & -\{c_i\,(a_i-b_i)\}^{n_i}\,(A_i+B_i+I)^{-1}_{n_i}{}_{k=0}^{n_i-1}\Big\{(A_i+B_i+I)_k\,\{c_i\,(a_i-b_i)\}^{-k} \\ & .\left\{(A_i+B_i+I)F^{(\mathbf{A},\mathbf{B})}_{n_1,...,n_{i-1},k,n_{i+1},...,n_s}(\mathbf{x}) + 2(x_i-a_i)\frac{\partial F^{(\mathbf{A},\mathbf{B})}_{n_1,...,n_{i-1},k,n_{i+1},...,n_s}(\mathbf{x})}{\partial x_i}\right\} \\ & .\ (I+A_i)^{-1}_k(A_i+I)_{n_i}\Big\}, \end{aligned}$$

iii) $$(x_i-a_i)\frac{\partial F_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})}{\partial x_i}-n_iF_{\mathbf{n}}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})=\left\{c_i\left(b_i-a_i\right)\right\}^{n_i}\left(A_i+B_i+I\right)_{n_i}^{-1}$$
$$.\quad \mathop{}\limits_{k=0}^{n_i-1}\Big\{\left\{c_i\left(b_i-a_i\right)\right\}^{-k}\left(A_i+B_i+\left(2k+1\right)I\right)(A_i+B_i+I)_k$$
$$.\ F_{n_1,\ldots,n_{i-1},k,n_{i+1},\ldots,n_s}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})(I+A_i)_k^{-1}(A_i+I)_{n_i}\Big\},$$

where A_i and B_i are matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $Re(z_i)>-1$ for every eigenvalue $z_i\in\sigma(A_i)$ and $Re(\eta_i)>-1$ for every eigenvalue $\eta_i\in\sigma(B_i)$, these matrices are commutative and $A_i+B_i+n_iI$ is invertible for every integer $n_i\geq 0$ for $1\leq i\leq s$.

For the case of $s=1$ in (9), extended Jacobi matrix polynomials (EJMPs) satisfy following equations [1]:

$$(x-a)\left[(A+B+nI)\frac{d}{dx}F_n^{(A,B)}(x;a,b,c)+c\left(a-b\right)\frac{d}{dx}F_{n-1}^{(A,B)}(x;a,b,c)\left(A+nI\right)\right]$$
$$=(A+B+nI)\left[nF_n^{(A,B)}(x;a,b,c)-c\left(a-b\right)F_{n-1}^{(A,B)}(x;a,b,c)\left(A+nI\right)\right],$$

$$\begin{aligned}(x-a)\frac{d}{dx}&F_n^{(A,B)}(x;a,b,c)-nF_n^{(A,B)}(x;a,b,c)\\ &=\quad -\left\{c\left(a-b\right)\right\}^n\left(A+B+I\right)_n^{-1}\mathop{}\limits_{k=0}^{n-1}\left\{c\left(a-b\right)\right\}^{-k}\left(A+B+I\right)_k\\ &\qquad .\left\{(A+B+I)F_k^{(A,B)}(x;a,b,c)+2\left(x-a\right)\frac{d}{dx}F_k^{(A,B)}(x;a,b,c)\right\}\\ &\qquad .(I+A)_k^{-1}(I+A)_n,\end{aligned}$$

and

$$\begin{aligned}&(x-a)\frac{d}{dx}F_n^{(A,B)}(x;a,b,c)-nF_n^{(A,B)}(x;a,b,c)\\ =\quad&\left\{c\left(b-a\right)\right\}^n\left(A+B+I\right)_n^{-1}\mathop{}\limits_{k=0}^{n-1}\left\{c\left(b-a\right)\right\}^{-k}\left(A+B+\left(2k+1\right)I\right)\\ &.\left(A+B+I\right)_kF_k^{(A,B)}(x;a,b,c)\left(I+A\right)_k^{-1}(I+A)_n.\end{aligned}$$

where all eigenvalues z of the matrices A and B of the extended Jacobi matrix polynomials $F_n^{(A,B)}(x;a,b,c)$ satisfy the condition $Re(z)>-1$ and $A+B+nI$ is invertible for every integer $n\geq 0$.

If we take $a=1,b=-1$ and $c=-\frac{1}{2}$ in Theorem 3, we have recurrence relations for Jacobi matrix polynomials given in [30]. For the case $r=1$, if we take $A_i=\alpha_i$ and $B_i=\beta_i$ with α_i,β_i real parameters and $\alpha_i,\beta_i>-1$ in Theorem 3, we have some recurrence relations satisfied by classical extended Jacobi polynomials $F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(\mathbf{x})$ given in [3].

4. Multilinear and Multilateral Generating Matrix Functions

In recent years, by making use of the familiar group-theoretic (Lie algebraic) method a certain mixed trilateral finite-series relationships have been proved for orthogonal polynomials (see, for instance, [28]). In this section, we derive several families of multilinear and multilateral generating matrix functions for the MEJMPs without using Lie algebraic techniques but, with the help of the similar method as considered in [1, 5].

Corresponding to a non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ consisting of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, let

$$\Lambda_{\mu,\nu}(y_1, ..., y_r; z) \quad : \quad = \sum_{k=0}^{\infty} a_k \ \Omega_{\mu+\nu k}(y_1, ..., y_r) \ z^k \tag{15}$$

$$(a_k \quad \neq \quad 0 \, , \ \mu, \nu \in \mathbb{C})$$

and

$$\begin{aligned} & \Theta_{\mathbf{n},p,\mu,\nu}(\mathbf{x}; y_1, ..., y_r; \zeta) \\ : \quad = & \sum_{k=0}^{[n_1/p]} a_k \ \left\{ {}^{s}_{i=2} \left(c_i \left(a_i - b_i\right)\right)^{-n_i} \right\} \left(c_1(b_1 - a_1)\right)^{pk-n_1} F^{(\mathbf{A},\mathbf{B})}_{n_1-pk, n_2, ..., n_s}(\mathbf{x}) \\ & .\Omega_{\mu+\nu k}(y_1, ..., y_r) \ \zeta^k \end{aligned} \tag{16}$$

where $\mathbf{n} = n_1 + ... + n_s$; $n_1, ..., n_s, p \in \mathbb{N}$, $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{A} = (A_1, ..., A_s)$, $\mathbf{B} = (B_1, ..., B_s)$ and A_i and B_i are matrices in $\mathbb{C}^{r \times r}$ satisfying the spectral conditions $Re(z_i) > -1$ for every eigenvalue $z_i \in \sigma(A_i)$ and $Re(\eta_i) > -1$ for every eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$ and all matrices are commutative. Then we have

$$\begin{aligned} & \sum_{n_1, ..., n_s = 0}^{\infty} \Theta_{\mathbf{n},p,\mu,\nu}\left(\mathbf{x}; y_1, ..., y_r; \frac{\eta}{t_1^p}\right) \ t_1^{n_1} ....t_s^{n_s} \\ = \quad & {}^{s}_{i=1} F_4 \left(I + B_i, I + A_i; I + A_i, I + B_i; \frac{(x_i - a_i)\, t_i}{a_i - b_i}, \frac{(x_i - b_i)\, t_i}{a_i - b_i} \right) \\ & .\Lambda_{\mu,\nu}(y_1, ..., y_r; \eta) \\ & \left(\sqrt{\tfrac{(x_i - a_i) t_i}{a_i - b_i}} + \sqrt{\tfrac{(x_i - b_i) t_i}{a_i - b_i}} < 1 \ ; \ i = 1, 2, ..., s \right) \end{aligned} \tag{17}$$

provided that each member of (17) exists.

Proof. For convenience, let S denote the first member of the assertion (17) of Theorem 4. Then, plugging the polynomials

$$\Theta_{\mathbf{n},p,\mu,\nu}\left(\mathbf{x}; y_1, ..., y_r; \frac{\eta}{t_1^p}\right)$$

from the definition (16) into the left-hand side of (17), we obtain

$$\begin{aligned} S \quad = \quad & \sum_{n_1, ..., n_s = 0}^{\infty} \sum_{k=0}^{[n_1/p]} a_k \ \left\{ {}^{s}_{i=2} \left(c_i \left(a_i - b_i\right)\right)^{-n_i} \right\} \left(c_1(b_1 - a_1)\right)^{pk-n_1} \ F^{(\mathbf{A},\mathbf{B})}_{n_1-pk, n_2, ..., n_s}(\mathbf{x}) \\ & .\Omega_{\mu+\nu k}(y_1, ..., y_r) \eta^k \ t_1^{n_1-pk} \ \ t_2^{n_2} ... t_s^{n_s}. \end{aligned} \tag{18}$$

Upon changing the order of summation in (18), if we replace n_1 by $n_1 + pk$, we can write

$$
\begin{aligned}
S &= \sum_{n_1,\ldots,n_s=0}^{\infty} \sum_{k=0}^{\infty} a_k \left\{ {}_{i=1}^{s} \left(c_i \left(a_i - b_i \right) \right)^{-n_i} \right\} F^{(\mathbf{A},\mathbf{B})}_{n_1,n_2,\ldots,n_s}(\mathbf{x})\ \Omega_{\mu+\nu k}(y_1,\ldots,y_r\ )\ \eta^k\ t_1^{n_1}\ldots t_s^{n_s} \\
&= \left(\sum_{n_1,\ldots,n_s=0}^{\infty} \left\{ {}_{i=1}^{s} \left(c_i \left(a_i - b_i \right) \right)^{-n_i} \right\}\ F^{(\mathbf{A},\mathbf{B})}_{\mathbf{n}}(\mathbf{x}) t_1^{n_1}\ldots t_s^{n_s} \right) \left(\sum_{k=0}^{\infty} a_k\ \Omega_{\mu+\nu k}(y_1,\ldots,y_r\ )\ \eta^k \right) \\
&= {}_{i=1}^{s} F_4 \left(I + B_i, I + A_i; I + A_i, I + B_i; \frac{(x_i - a_i)\, t_i}{a_i - b_i}, \frac{(x_i - b_i)\, t_i}{a_i - b_i} \right) \\
&\quad .\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta)
\end{aligned}
$$

which completes the proof of Theorem 4. □

Corresponding to a non-vanishing function $\Omega_{\mu}(y_1,\ldots,y_r\ )$ of r complex variables $y_1,\ldots,y_r$ $(r \in \mathbb{N})$ and of complex order μ, let

$$
\begin{aligned}
\Lambda_{\mu,\nu}\left(y_1,\ldots,y_r;z\right) \quad : \quad &= \sum_{k=0}^{\infty} a_k\ \Omega_{\mu+\nu k}(y_1,\ldots,y_r\ )\ z^k \qquad (19) \\
&(a_k \quad \neq \quad 0\,,\ \ \mu,\nu \in \mathbb{C})
\end{aligned}
$$

and

$$
\begin{aligned}
&\Theta_{n,p,\mu,\nu}(\mathbf{x};y_1,\ldots,y_r;\zeta) \\
: \quad &= \sum_{k=0}^{[n/p]} a_k\ H^{(A,\mathbf{B})}_{n-pk}(\mathbf{x}) \Omega_{\mu+\nu k}(y_1,\ldots,y_r\ )\ \zeta^k \qquad (20)
\end{aligned}
$$

where $n,p\ \in\ \mathbb{N}$, $\mathbf{x} = (x_1,\ldots,x_s)$, $\mathbf{A} = (A_1,\ldots,A_s)$, $\mathbf{B} = (B_1,\ldots,B_s)$ and A_i and B_i are matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $Re(z_i) > -1$ for every eigenvalue $z_i \in \sigma(A_i)$ and $Re(\eta_i) > -1$ for every eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \le i \le s$. Then we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu} \left(\mathbf{x}; y_1,\ldots,y_r; \frac{\eta}{t^p} \right)\ t^n \\
= \quad &{}_{i=1}^{s} (1-t)^{-(A_i+B_i+I)} F \left(\frac{A_i + B_i + I}{2}, \frac{A_i + B_i + 2I}{2}; A_i + I; \frac{4t(x_i - a_i)}{(a_i - b_i)\,(1-t)^2} \right) \\
&.\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta) \qquad (21)
\end{aligned}
$$

provided that each member of (21) exists. Here

$$
\begin{aligned}
&H_k^{(A_1,\ldots,A_s;B_1,\ldots,B_s)}\,(x_1,\ldots,x_s) \\
= \quad &{}_{k_1=0}^{k}\ {}_{k_2=0}^{k-k_1} \cdots {}_{k_{s-1}=0}^{k-k_1-\ldots-k_{s-2}} \mathbf{K}\left(k_1,\ldots,k_{s-1}\right) F^{(A_1,\ldots,A_s;B_1,\ldots,B_s)}_{k-\left(k_1+\ldots+k_{s-1}\right),k_1,\ldots,k_{s-1}}\,(x_1,\ldots,x_s)\, \mathbf{N}\left(k_1,\ldots,k_{s-1}\right)\ ;
\end{aligned}
$$

and

$$
\mathbf{K}\left(k_1,\ldots,k_{s-1}\right) = \frac{(A_1 + B_1 + I)_{k-(k_1+\ldots+k_{s-1})}\ (A_2 + B_2 + I)_{k_1} \ldots (A_s + B_s + I)_{k_{s-1}}}{\left(c_1\left(a_1 - b_1\right)\right)^{k-(k_1+\ldots+k_{s-1})} \left(c_2\left(a_2 - b_2\right)\right)^{k_1} \ldots \left(c_s\left(a_s - b_s\right)\right)^{k_{s-1}}},
$$

$$
\begin{aligned}
\mathbf{N}\left(k_1,\ldots,k_{s-1}\right) \quad = \quad &\left[(A_s + I)_{k_{s-1}}\right]^{-1} \left[(A_{s-1} + I)_{k_{s-2}}\right]^{-1} \times \ldots \\
&\times \left[(A_2 + I)_{k_1}\right]^{-1} \left[(A_1 + I)_{k-(k_1+\ldots+k_{s-1})}\right]^{-1},
\end{aligned}
$$

all matrices are commutative and $A_i + nI$ is invertible for every integer $n \geq 0$ for $1 \leq i \leq s$.

Proof. For convenience, let S denote the first member of the assertion (21) of Theorem 4. Then, upon substituting for the polynomials

$$\Theta_{n,p,\mu,\nu}\left(\mathbf{x}; y_1, ..., y_r; \frac{\eta}{t^p}\right)$$

from the definition (20) into the left-hand side of (21), we obtain

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k \; H_{n-pk}^{(A,\mathbf{B})}(\mathbf{x}) \Omega_{\mu+\nu k}(y_1, ..., y_r\,)\; \eta^k t^{n-pk} \tag{22}$$

Upon changing the order of summation in (22), if we replace n by $n + pk$, we can write

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\, H_n^{(A,\mathbf{B})}(\mathbf{x})\; \Omega_{\mu+\nu k}(y_1, ..., y_r\,)\; \eta^k\; t^n \\ &= \left(\sum_{n=0}^{\infty} H_n^{(A,\mathbf{B})}(\mathbf{x})\; t^n\right)\left(\sum_{k=0}^{\infty} a_k\; \Omega_{\mu+\nu k}(y_1, ..., y_r\,)\; \eta^k\right) \\ &= {}_{i=1}^{s}(1-t)^{-(A_i+B_i+I)} F\left(\frac{A_i + B_i + I}{2}, \frac{A_i + B_i + 2I}{2}; A_i + I; \frac{4t(x_i - a_i)}{(a_i - b_i)\,(1-t)^2}\right) \\ &\quad .\Lambda_{\mu,\nu}(y_1, ..., y_r; \eta) \end{aligned}$$

which completes the proof of Theorem 4. □

5. Further Consequences

By expressing the multivariable function $\Omega_{\mu+\nu k}(y_1, ..., y_r\,)\,,(k \in \mathbb{N}_0\,,\; r \in \mathbb{N})$ in terms of a simpler function of one and more variables, we can give further applications of Theorem 4. For example, consider the case of $r = 1$ and $\Omega_{\mu+\nu k}(y\,) = L_{\mu+\nu k}^{(C,\lambda)}(y)$ in Theorem 4. Here, the Laguerre matrix polynomials $L_n^{(C,\lambda)}(y)$ are defined by [20] as follows:

$$L_n^{(C,\lambda)}(y) = \sum_{k=0}^{n} \frac{(-1)^k\,\lambda^k}{k!\;(n-k)!}\;(C+I)_n\;\left[(C+I)_k\right]^{-1}\; y^k$$

in which C is a matrix in $\mathbb{C}^{r\times r}$, $C + nI$ is invertible for every integer $n \geq 0$ and λ is a complex number with $Re\,(\lambda) > 0$. Notice that Laguerre matrix polynomials are generated by

$$\sum_{n=0}^{\infty} L_n^{(C,\lambda)}(y)\; t^n = (1-t)^{-(C+I)} \exp\left(\frac{-\lambda y t}{1-t}\right),\; |t| < 1,\; 0 < y < \infty. \tag{23}$$

Then we obtain the following result which provides a class of bilateral generating matrix functions for the MEJMPs and the Laguerre matrix polynomials.

Let $\Lambda_{\mu,\nu}(y; z) := \sum\limits_{k=0}^{\infty} a_k\; L_{\mu+\nu k}^{(C,\lambda)}(y)\; z^k$ where $(a_k \neq 0\,,\;\; \mu, \nu \in \mathbb{C})$ and

$$\begin{aligned} &\Theta_{\mathbf{n},p,\mu,\nu}(\mathbf{x}; y; \zeta) \\ : &= \sum_{k=0}^{[n_1/p]} a_k \left\{{}_{i=2}^{s}\left(c_i\,(a_i - b_i)\right)^{-n_i}\right\} (c_1(b_1 - a_1))^{pk-n_1} F_{n_1-pk,n_2,...,n_s}^{(\mathbf{A},\mathbf{B})}(\mathbf{x}) L_{\mu+\nu k}^{(C,\lambda)}(y)\; \zeta^k \end{aligned}$$

where $\mathbf{x}=(x_1,...,x_s)$, $\mathbf{A}=(A_1,...,A_s)$, $\mathbf{B}=(B_1,...,B_s)$, $\mathbf{n}=n_1+...+n_s$; $n_1,...,n_s,p\in\mathbb{N}$, $A_iB_i=B_iA_i$, $A_iA_j=A_jA_i, B_iB_j=B_jB_i$; $i,j=1,2,...,s$ and $C+nI$ is invertible for every integer $n\geq 0$. Then we have

$$\begin{aligned}
&\sum_{n_1,...,n_s=0}^{\infty}\Theta_{\mathbf{n},p,\mu,\nu}\left(\mathbf{x};y;\frac{\eta}{t_1^p}\right)\ t_1^{n_1}...t_s^{n_s} && (24)\\
=\ & {}_{i=1}^{s}F_4\left(I+B_i,I+A_i;I+A_i,I+B_i;\frac{(x_i-a_i)\,t_i}{a_i-b_i},\frac{(x_i-b_i)\,t_i}{a_i-b_i}\right)\Lambda_{\mu,\nu}(y;\eta)
\end{aligned}$$

provided that each member of (24) exists for $\sqrt{\frac{(x_i-a_i)t_i}{a_i-b_i}}+\sqrt{\frac{(x_i-b_i)t_i}{a_i-b_i}}<1$; $i=1,2,...,s$.

Using the generating matrix function (23) for the Laguerre matrix polynomials and taking $a_k=1$, $\mu=0$, $\nu=1$, we have

$$\begin{aligned}
&\sum_{n_1,...,n_s=0}^{\infty}\sum_{k=0}^{[n_1/p]}\left\{{}_{i=2}^{s}\left(c_i\left(a_i-b_i\right)\right)^{-n_i}\right\}(c_1(b_1-a_1))^{pk-n_1}F_{n_1-pk,n_2,...,n_s}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})\\
&.L_k^{(C,\lambda)}(y)\ \eta^k\ t_1^{n_1-pk}\ t_2^{n_2}...t_s^{n_s}\\
=\ & {}_{i=1}^{s}F_4\left(I+B_i,I+A_i;I+A_i,I+B_i;\frac{(x_i-a_i)\,t_i}{a_i-b_i},\frac{(x_i-b_i)\,t_i}{a_i-b_i}\right)\\
&.\,(1-\eta)^{-(C+I)}\exp\left(\frac{-\lambda y\eta}{1-\eta}\right)
\end{aligned}$$

where $|\eta|<1$, $0<y<\infty$ and $\sqrt{\frac{(x_i-a_i)t_i}{a_i-b_i}}+\sqrt{\frac{(x_i-b_i)t_i}{a_i-b_i}}<1$ for $i=1,...,s$.

Choose $r=1$ and $\Omega_{\mu+\nu k}(y)=C_{\mu+\nu k}^{D}(y)$ where D is a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition

$$\left(\frac{-z}{2}\right)\notin\sigma(D)\text{ for }\forall z\in\mathbb{Z}^+\cup\{0\}$$

in Theorem 4 where $\mu,\nu\in\mathbb{N}_0$. Here, Gegenbauer matrix polynomials are generated by [27] as follows:

$$\sum_{n=0}^{\infty}C_k^D(y)t^n=(1-2yt+t^2)^{-D}\quad,\quad |y|<1 \tag{25}$$

Then we obtain a class of bilateral generating matrix functions for the MEJMPs and Gegenbauer matrix polynomials.

Let $\Lambda_{\mu,\nu}(y;z):=\sum_{k=0}^{\infty}a_kC_{\mu+\nu k}^{D}(y)z^k$ where $(a_k\neq 0\,,\ \mu,\nu\in\mathbb{C})$ and

$$\begin{aligned}
&\Theta_{\mathbf{n},p,\mu,\nu}(\mathbf{x};y;\zeta)\\
:\ =\ &\sum_{k=0}^{[n_1/p]}a_k\left\{{}_{i=2}^{s}\left(c_i\left(a_i-b_i\right)\right)^{-n_i}\right\}(c_1(b_1-a_1))^{pk-n_1}F_{n_1-pk,n_2,...,n_s}^{(\mathbf{A},\mathbf{B})}(\mathbf{x})C_{\mu+\nu k}^{D}(y)\ \zeta^k
\end{aligned}$$

where $\mathbf{x}=(x_1,...,x_s)$, $\mathbf{A}=(A_1,...,A_s)$, $\mathbf{B}=(B_1,...,B_s)$, $\mathbf{n}=n_1+...+n_s$; $n_1,...,n_s,p\in\mathbb{N}$ and D is a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition

$$\left(\frac{-z}{2}\right)\notin\sigma(D)\text{ for }\forall z\in\mathbb{Z}^+\cup\{0\}\ ,$$

A_i and B_i are matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $Re(z_i) > -1$ for every eigenvalue $z_i \in \sigma(A_i)$ and $Re(\eta_i) > -1$ for every eigenvalue $\eta_i \in \sigma(B_i)$, $A_iB_i = B_iA_i$, $A_iA_j = A_jA_i$, $B_iB_j = B_jB_i$; $i, j = 1, 2, ..., s$. Then we have

$$\sum_{n_1,...,n_s=0}^{\infty} \Theta_{\mathbf{n},p,\mu,\nu}\left(\mathbf{x}; y; \frac{\eta}{t_1^p}\right)\ t_1^{n_1}...t_s^{n_s} \tag{26}$$
$$= {}_{i=1}^{s}F_4\left(I + B_i, I + A_i; I + A_i, I + B_i; \frac{(x_i - a_i)\,t_i}{a_i - b_i}, \frac{(x_i - b_i)\,t_i}{a_i - b_i}\right)\Lambda_{\mu,\nu}(y;\eta)$$

provided that each member of (26) exists for $\sqrt{\frac{(x_i-a_i)t_i}{a_i-b_i}} + \sqrt{\frac{(x_i-b_i)t_i}{a_i-b_i}} < 1$; $i = 1, 2, ..., s$.

Using (25) and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n_1,...,n_s=0}^{\infty}\sum_{k=0}^{[n_1/p]}\left\{{}_{i=2}^{s}\left(c_i\left(a_i - b_i\right)\right)^{-n_i}\right\}\left(c_1(b_1 - a_1)\right)^{pk-n_1} F^{(\mathbf{A},\mathbf{B})}_{n_1-pk,n_2,...,n_s}(\mathbf{x})$$
$$.C_k^D(y)\ \eta^k\ t_1^{n_1-pk}\ t_2^{n_2}...t_s^{n_s}$$
$$= {}_{i=1}^{s}F_4\left(I + B_i, I + A_i; I + A_i, I + B_i; \frac{(x_i - a_i)\,t_i}{a_i - b_i}, \frac{(x_i - b_i)\,t_i}{a_i - b_i}\right)$$
$$.(1 - 2y\eta + \eta^2)^{-D}.$$

Setting $r = s$ and $\Omega_{\mu+\nu k}(y_1, ..., y_s) = H^{(C,\mathbf{D})}_{\mu+\nu k}(y_1, ..., y_s)$ in Theorem 4, we obtain result which provides a class of bilinear generating matrix functions for the MEJMPs.

Let $\Lambda_{\mu,\nu}(y_1, ..., y_s; z) := \sum\limits_{k=0}^{\infty} a_k\ H^{(C,\mathbf{D})}_{\mu+\nu k}(\mathbf{y})\ z^k$ where $(a_k \neq 0\,,\ \ \mu, \nu \in \mathbb{C})$ and

$$\Theta_{n,p,\mu,\nu}(\mathbf{x}; \mathbf{y}; \zeta) := \sum_{k=0}^{[n/p]} a_k\ H^{(A,\mathbf{B})}_{n-pk}(\mathbf{x}) H^{(C,\mathbf{D})}_{\mu+\nu k}(\mathbf{y})\ \zeta^k \tag{27}$$

where $n, p \in \mathbb{N}$, $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{y} = (y_1, ..., y_s)$, $\mathbf{A} = (A_1, ..., A_s)$, $\mathbf{B}= (B_1, ..., B_s)$, $\mathbf{C} = (C_1, ..., C_s)$, $\mathbf{D}= (D_1, ..., D_s)$ and A_i, B_i, C_i and D_i $(i = 1, 2, ..., s)$ are matrices in $\mathbb{C}^{r\times r}$ whose eigenvalues, z, all satisfy $Re(z) > -1$. Then we have

$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}\left(\mathbf{x}; \mathbf{y}; \frac{\eta}{t^p}\right)\ t^n$$
$$= {}_{i=1}^{s}(1 - t)^{-(A_i+B_i+I)} F\left(\frac{A_i + B_i + I}{2}, \frac{A_i + B_i + 2I}{2}; A_i + I; \frac{4t(x_i - a_i)}{(a_i - b_i)\,(1 - t)^2}\right)$$
$$.\Lambda_{\mu,\nu}(y_1, ..., y_s; \eta) \tag{28}$$

provided that each member of (28) exists.

Taking $a_k = 1$, $\mu = 0$, $\nu = 1$ and using Theorem 3, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} H_{n-pk}^{(A,\mathbf{B})}(\mathbf{x}) H_{k}^{(C,\mathbf{D})}(\mathbf{y})\ \eta^{k}\ t^{n-pk} \\
= \quad & {}_{i=1}^{s}(1-t)^{-(A_i+B_i+I)} F\left(\frac{A_i+B_i+I}{2}, \frac{A_i+B_i+2I}{2}; A_i+I; \frac{4t(x_i-a_i)}{(a_i-b_i)(1-t)^2}\right) \\
& \cdot{}_{i=1}^{s}(1-\eta)^{-(C_i+D_i+I)} F\left(\frac{C_i+D_i+I}{2}, \frac{C_i+D_i+2I}{2}; C_i+I; \frac{4\eta(y_i-a_i)}{(a_i-b_i)(1-\eta)^2}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& H_{k}^{(C_1,\dots,C_s;D_1,\dots,D_s)}(y_1,\dots,y_s) \\
= \quad & {}_{k_1=0}^{k}\,{}_{k_2=0}^{k-k_1}\cdots{}_{k_{s-1}=0}^{k-k_1-\dots-k_{s-2}} \mathbf{K}(k_1,\dots,k_{s-1})\, F_{k-\left(k_1+\dots+k_{s-1}\right),k_1,\dots,k_{s-1}}^{(C_1,\dots,C_s;D_1,\dots,D_s)}(y_1,\dots,y_s)\, \mathbf{N}(k_1,\dots,k_{s-1})\ ;
\end{aligned}
$$

and

$$
\mathbf{K}(k_1,\dots,k_{s-1}) = \frac{(C_1+D_1+I)_{k-(k_1+\dots+k_{s-1})}\,(C_2+D_2+I)_{k_1}\dots(C_s+D_s+I)_{k_{s-1}}}{(c_1(a_1-b_1))^{k-(k_1+\dots+k_{s-1})}\,(c_2(a_2-b_2))^{k_1}\dots(c_s(a_s-b_s))^{k_{s-1}}},
$$

$$
\begin{aligned}
\mathbf{N}(k_1,\dots,k_{s-1}) \quad = \quad & \left[(C_s+I)_{k_{s-1}}\right]^{-1}\left[(C_{s-1}+I)_{k_{s-2}}\right]^{-1}\times\dots \\
& \times\left[(C_2+I)_{k_1}\right]^{-1}\left[(C_1+I)_{k-(k_1+\dots+k_{s-1})}\right]^{-1}
\end{aligned}
$$

and $C_iD_i = D_iC_i,\ C_iC_j = C_jC_i,\ D_iD_j = D_jD_i\ ;\ i,j = 1,2,\dots,s$ and $C_i + nI$ is invertible for every integer $n \geq 0$ for $1 \leq i \leq s$.

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\nu k}(y_1,\dots,y_r)$, $(r \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorem 4 and Theorem 4 can be applied in order to derive various families of multilinear and multilateral generating matrix functions for the MEJMPs.

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