Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com
Volume 4 Issue 3(2013), Pages 1-5.

BOUNDS FOR k-GAMMA AND k-BETA FUNCTIONS

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Abstract. Some power product bounds are given for the functions $\Gamma_k(x)$ and $B_k(x,y)$. The bounds for $\Gamma_k(x)$ are k-generalization of bounds given by Starc (2002) for the function $\Gamma(x)$ and the bounds for $B_k(x,y)$ are new.

1. Introduction

In [1] have been introduced the k-Pochhammer symbol by

$$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k), \quad x \in C, \quad k \in R, \quad n \in N^+, \tag{1.1}$$

the $\Gamma_k(x)$ function by

$$\Gamma_k(x) = lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k>0, \quad x \in C/kZ^- \tag{1.2}$$

and the $B_k(x,y)$ function by

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \quad k>0, \quad x \in C/kZ^-, \quad y \in C/kZ^-. \tag{1.3}$$

It is obvious that when $k \to 1$ the above functions tend to Pochhammer symbol, $\Gamma(x)$ and $B(x,y)$ respectively, so we call them k-generalization of the corresponding functions.

The introduction of $\Gamma_k(x)$ and $B_k(x,y)$ have been done because of their connection with the k-Pochhammer symbol, which appears in a variety of contexts, see [1]. There is also a relation between $\Gamma_k(x)$ and measure theory , see [2].

In [1] there have been proved the basic identities of $\Gamma_k(x)$ and $B_k(x,y)$, for $k>0$, $\Re(x)>0$ and $\Re(y)>0$:

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt \tag{1.4}$$

$$\Gamma_k(x+k) = x\Gamma_k(x) \tag{1.5}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} \tag{1.6}$$

$$\Gamma_k(k) = 1 \tag{1.7}$$

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt \tag{1.8}$$

2000 *Mathematics Subject Classification.* 33C10.
Key words and phrases. $k-$Gamma function, $k-$Beta function, bounds.

Submitted June 19, 2012. Published May, 18 2013.

$$B_k(x,y) = \int_0^\infty t^{x-1}(1+t^k)^{-\frac{x+y}{k}}\,dt \tag{1.9}$$

Another function related with $\Gamma_k(x)$ function is the $\psi_k(x)$ function defined [1] by:

$$\psi_k(x) = \frac{\Gamma'_k(x)}{\Gamma_k(x)} \tag{1.10}$$

which has the series representation [1], [3]:

$$\psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(x+nk)} \tag{1.11}$$

where $\gamma = 0.57721...$ is Euler's constant.

During the recent years there exist papers (see for example [2]- [7]) in which have been proved bounds, inequalities and monotonicity properties for the functions $\Gamma_k(x)$ and $B_k(x,y)$ and for functions involving them. The most of these results are k-generalization of known results concerning the functions $\Gamma(x)$ and $B(x,y)$.

In this paper we give bounds for $\Gamma_k(x)$ and $B_k(x,y)$ as power products. The bounds concerning $\Gamma_k(x)$ are k-generalization of corresponding bounds for $\Gamma(x)$ given in [8] and we use the analogous method. The bounds concerning the $B_k(x,y)$ function seem to be new. We have to mention that in [7] the author gave bounds for $\Gamma_k(x)$ analogous to those in [8], but due to some mistakes, the k-generalization does not hold.

2. Main results.

Theorem 2.1. *The function $\Gamma_k(x)$ satisfies the inequalities:*

$$\Big(\frac{x+kn}{k+kn}\Big)^{\frac{x}{k}+n} \frac{k^n n!}{(x)_{n,k}} e^{\frac{1}{k}[H(n)-1+\ln k-\gamma](x-k)} \leq \Gamma_k(x) \leq$$
$$\Big(\frac{x+kn}{k+kn}\Big)^{\frac{x}{k}+n} \frac{k^{n+1}(n+1)!}{(x)_{n+1,k}} e^{\frac{1}{k}[H(n+1)-1+\ln k-\gamma](x-k)} \tag{2.1}$$

for $x \geq k$, $k > 0$, $n = 1, 2, ...,$ where $H(n) = \sum_{p=1}^{n} \frac{1}{p}$, $\gamma = 0.57721...$ is Euler's constant and $(x)_{n,k}$ is the k-Pochhammer symbol defined by (1.1). The equalities hold for $x = k$.

Proof. From the series representation (1.10) for $\psi_k(x)$ we obtain:

$$\psi_k(k) = \frac{\ln k - \gamma}{k} \tag{2.2}$$

and

$$\psi'_k(x) = \sum_{i=0}^{\infty} \frac{1}{(x+ik)^2} = \sum_{i=0}^{n-1} \frac{1}{(x+ik)^2} + \sum_{i=0}^{\infty} \frac{1}{(x+nk+ik)^2}. \tag{2.3}$$

Now we apply the following proposition: "Let $g(t)$ be a positive strictly decreasing function of t for $t > 0$, with $lim_{t\to\infty} g(t) = 0$, then the following inequalities

$$\int_0^\infty g(t)dt < \sum_{i=0}^{\infty} g(i) < g(0) + \int_0^\infty g(t)dt \tag{2.4}$$

hold", for $g(t)=\frac{1}{x+nk+kt}$. Since $g(0)=\frac{1}{x+nk}$ and $\int_0^\infty g(t)dt=\frac{1}{k}\frac{1}{x+nk}$ the inequalities (2.4) become:

$$\sum_{i=0}^{n-1}\frac{1}{(x+ik)^2}+\frac{1}{k}\frac{1}{x+nk}\le\psi_k'(x)\le\sum_{i=0}^{n}\frac{1}{(x+ik)^2}+\frac{1}{k}\frac{1}{x+nk}. \tag{2.5}$$

We integrate the inequalities (2.5) from k to x and obtain:

$$\frac{1}{k}\sum_{i=0}^{n-1}\frac{1}{1+i}-\sum_{i=0}^{n-1}\frac{1}{x+ik}+\frac{1}{k}\ln\Big[\frac{x+kn}{k+nk}\Big]\le\psi_k(x)-\psi_k(k)\le$$
$$\frac{1}{k}\sum_{i=0}^{n}\frac{1}{1+i}-\sum_{i=0}^{n-1}\frac{1}{x+ik}+\frac{1}{k}\ln\Big[\frac{x+kn}{k+nk}\Big]. \tag{2.6}$$

Since $H(n)=\sum_{p=1}^{n}\frac{1}{p}$, the inequalities (2.6) takes the form:

$$\frac{1}{k}H(n)-\sum_{i=0}^{n-1}\frac{1}{x+ik}+\frac{1}{k}\ln\Big[\frac{x+kn}{k+nk}\Big]\le\psi_k(x)-\psi_k(k)\le$$
$$\frac{1}{k}H(n+1)-\sum_{i=0}^{n-1}\frac{1}{x+ik}+\frac{1}{k}\ln\Big[\frac{x+kn}{k+nk}\Big]. \tag{2.7}$$

Now, we integrate the inequalities (2.7) from k to x and obtain

$$\frac{1}{k}[H(n)-1](x-k)-\sum_{i=0}^{n-1}\ln\Big[\frac{x+ki}{k+ik}\Big]+\frac{1}{k}(x+kn)\ln\Big[\frac{x+kn}{k+nk}\Big]$$
$$\le\ln\Gamma_k(x)-\psi_k(k)(x-k)\le$$
$$\frac{1}{k}[H(n+1)-1](x-k)-\sum_{i=0}^{n}\ln\Big[\frac{x+ki}{k+ik}\Big]+\frac{1}{k}(x+kn)\ln\Big[\frac{x+kn}{k+nk}\Big]. \tag{2.8}$$

Taking in account that $\psi_k(k)$ is given by (2.2) the inequalities (2.8) take the form

$$\frac{1}{k}\Big[H(n)-1+\ln k-\gamma\Big](x-k)-\sum_{i=0}^{n-1}\ln\Big[\frac{x+ki}{k+ik}\Big]+\frac{1}{k}(x+kn)\ln\Big[\frac{x+kn}{k+nk}\Big]$$
$$\le\ln\Gamma_k(x)\le$$
$$\frac{1}{k}\Big[H(n+1)-1+\ln k-\gamma\Big](x-k)-\sum_{i=0}^{n}\ln\Big[\frac{x+ki}{k+ik}\Big]+\frac{1}{k}(x+kn)\ln\Big[\frac{x+kn}{k+nk}\Big]. \tag{2.9}$$

Finally from (2.9) we obtain the desired inequalities (2.1). □

Corollary 2.2. *The function $\Gamma_k(x)$ satisfies the inequalities*

$$\Big(\frac{x+k}{2k}\Big)^{\frac{x}{k}+1}\frac{k}{x}e^{\frac{(x-k)}{k}[\ln k-\gamma]}\le\Gamma_k(x)\le\Big(\frac{x+k}{2k}\Big)^{\frac{x}{k}+1}\frac{k}{x}e^{\frac{(x-k)}{k}[\frac{1}{2}+\ln k-\gamma]} \tag{2.10}$$

for $x\ge k$, $k>0$ and $\gamma=0.57721...$ is Euler's constant. The equalities hold for $x=k$.

Proof. The inequalities (2.10) become from the theorem 2.1 for $n=1$. □

Remark. *Theorem 2.1 and corollary 2.2 are k-generalization of the main theorem and corollary 1 of* [8] *correspondingly.*

Remark. *The main theorem of* [7] *is not the k-generalization of the corresponding theorem of* [8], *due to some mistakes.*

Corollary 2.3. *For $x = \frac{3k}{2}$ the inequalities (2.10) become:*

$$\frac{5^{5/2}}{4!}e^{\frac{-\gamma}{2}} < \sqrt{\pi} < \frac{5^{5/2}4}{5!}e^{\frac{1}{2}[\frac{1}{2}-\gamma]}. \tag{2.11}$$

Proof. For the proof we used the fact (see [3]) that $\Gamma_k(\frac{3k}{2}) = k^{1/2}\sqrt{\pi}$. □

Remark. *We note that the inequalities (2.11) are independent of k and are exactly the same as those given in* [8] *for $n = 1$.*

Corollary 2.4. *The function $\Gamma_k(x)$ satisfies the inequalities*

$$\left(\frac{x}{k}\right)^{\frac{x}{k}} \frac{k^n n!}{(n+1)^{\frac{x}{k}}} e^{\frac{1}{k}[H(n)-1+\ln k-\gamma](x-(n+1)k)} \leq \Gamma_k(x) \leq$$
$$\left(\frac{x}{k}\right)^{\frac{x}{k}} \frac{k^{n+1}(n+1)!}{x(n+1)^{\frac{x}{k}}} e^{\frac{1}{k}[H(n+1)-1+\ln k-\gamma](x-(n+1)k)}, \tag{2.12}$$

for $x \geq (n+1)k$, $k > 0$, $n = 1, 2, ...,$ where $H(n) = \sum_{p=1}^{n} \frac{1}{p}$ and $\gamma = 0.57721...$ is Euler's constant. The equalities hold for $x = (n+1)k$, $n \geq 1$.

Proof. Using (1.6) in (2.1) we obtain:

$$\left(\frac{x+kn}{k+kn}\right)^{\frac{x}{k}+n} k^n n! \frac{\Gamma_k(x)}{\Gamma_k(x+kn)} e^{\frac{1}{k}[H(n)-1+\ln k-\gamma](x-k)} \leq \Gamma_k(x) \leq$$
$$\left(\frac{x+kn}{k+kn}\right)^{\frac{x}{k}+n} k^{n+1}(n+1)! \frac{\Gamma_k(x)}{\Gamma_k(x+k(n+1))} e^{\frac{1}{k}[H(n+1)-1+\ln k-\gamma](x-k)} \tag{2.13}$$

which become

$$\left(\frac{x+kn}{k+kn}\right)^{\frac{x}{k}+n} k^n n! e^{\frac{1}{k}[H(n)-1+\ln k-\gamma](x-k)} \leq \Gamma_k(x+kn) \leq$$
$$\left(\frac{x+kn}{k+kn}\right)^{\frac{x}{k}+n} \frac{k^{n+1}(n+1)!}{x+kn} e^{\frac{1}{k}[H(n+1)-1+\ln k-\gamma](x-k)} \tag{2.14}$$

because of (1.5). By putting $x + kn = y$ in (2.14)we obtain the desired inequalities for $y \geq (n+1)k$. The equalities in (2.12) hold for $y = (n+1)k$, since it is known [3] that $\Gamma_k((n+1)k) = k^n(n+1)!$. □

Theorem 2.5. *The function $B_k(x,y)$ satisfies the inequalities:*

$$\frac{x^{\frac{x}{k}} y^{\frac{y}{k}}}{(x+y)^{\frac{x+y}{k}}} \frac{k^{n-1} n!(x+y)}{n+1} e^{-\frac{x+y}{k(n+1)}-(n+1)[H(n)-1+\ln k-\gamma-\frac{1}{n+1}]} \leq$$
$$B_k(x,y) \leq$$
$$\frac{x^{\frac{x}{k}} y^{\frac{y}{k}}}{(x+y)^{\frac{x+y}{k}}} \frac{k^{n+2}(n+1)(n+1)!(x+y)}{xy} e^{\frac{x+y}{k(n+1)}-(n+1)[H(n+1)-1+\ln k-\gamma+\frac{1}{n+1}]} \tag{2.15}$$

for $x, y \geq (n+1)k$, $k > 0$, where $H(n) = \sum_{p=1}^{n} \frac{1}{p}$ and $\gamma = 0.57721...$ is Euler's constant.

Proof. The inequalities (2.15) are obtained by using the definition (1.3) for $B_k(x,y)$ and the inequalities (2.12) for $\Gamma_k(x), \Gamma_k(y)$ and $\frac{1}{\Gamma_k(x+y)}$. □

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