Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com
Volume 4 Issue 2(2013), Pages 13-18.

SOME FIXED POINT THEOREMS IN ORDERED PARTIAL METRIC SPACES

HÜSEYİN IŞIK, DURAN TÜRKOĞLU

Abstract. The purpose of this paper is to present a fixed point theorem using a contractive condition of rational type in the context of ordered partial metric spaces.

1. Introduction and Preliminaries

In [1], Harjani et al. proved the following fixed point theorem.

Theorem 1.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \to X$ be a continuous and nondecreasing mapping such that*

$$d(Fx, Fy) \leq \alpha \cdot \frac{d(x, Fx) \cdot d(y, Fy)}{d(x, y)} + \beta \cdot d(x, y), \quad x, y \in X, \quad x \geq y, \quad x \neq y$$

with $\alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Fx_0$, then F has a unique fixed point.

The aim of this paper is to give a version of Theorem 1.1 in ordered partial metric spaces.

Partial metric space is a generalized metric space in which each object does not necessarily have to have a zero distance from itself [2]. A motivation behind introducing the concept of a partial metric space was to give a modified version of a Banach contraction mapping principle, more suitable to solve certain problems arising in computer science [2]. Valero [3], Oltra and Valero [4] and Altun et al. [5] gave some further generalization of the results in [2]. Romaguera [6] proved the Caristi type fixed point theorem on a partial metric space. First, we start by recalling some definitions and properties of partial metric spaces. For more details, on fixed point results on such spaces, we refer the reader to [7]-...-[14].

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

$(p1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
$(p2)$ $p(x, x) \leq p(x, y)$,
$(p3)$ $p(x, y) = p(y, x)$,

2000 *Mathematics Subject Classification.* 54H25,47H10.

Key words and phrases. Fixed point; contractive condition of rational type; partial metric space; ordered set.

Submitted February 5, 2013. Published June 17, 2013.

$(p4)$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Remark. *It is clear that if $p(x,y) = 0$, then from $(p1)$, $(p2)$, and $(p3)$, $x = y$. But if $x = y$, $p(x,y)$ may not be 0.*

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y), \quad (1.1)$$

is a metric on X.

Now, we give an example of partial metric spaces as follows.

Consider $X = \mathbb{R}^+$ with $p(x,y) = \max\{x,y\}$. Then $(\mathbb{R}^+, p)$ is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $p^s(x,y) = |x - y|$.

Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$;

(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$;

A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is, $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

If $F : X \to X$ is continuous at $x_0 \in X$, then for any sequences $\{x_n\}$ in X such that $x_n \to x_0$ as $n \to +\infty$ in (X,p), we have $Fx_n \to Fx_0$ as $n \to +\infty$ in (X,p).

Lemma 1.2. *Let (X,p) be a partial metric space;*

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),

(b) a partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete; furthermore, $\lim_{n\to+\infty} p^s(x_n,x) = 0$ if and only if

$$p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

2. Main Results

Let $(X,\leq)$ be a partially ordered set and $F : X \to X$. We say that F is a nondecreasing mapping if for $x,y \in X$, $x \leq y \Rightarrow Fx \leq Fy$.

In the sequel, we prove the following theorem which is a version of Theorem 1.1 in the context of ordered partial metric spaces.

Theorem 2.1. *Let $(X,\leq)$ be a partially ordered set and p be a partial metric on X such that (X,p) is a complete partial metric space. Let $F : X \to X$ be a continuous and nondecreasing mapping such that*

$$p(Fx,Fy) \leq \alpha \cdot \frac{p(x,Fx) \cdot p(y,Fy)}{p(x,y)} + \beta \cdot p(x,y), \quad x,y \in X, \quad x \geq y, \quad x \neq y \quad (2.1)$$

with $\alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Fx_0$, then F has a fixed point.

Proof. If $Fx_0 = x_0$, then the proof is finished. Suppose that $x_0 < Fx_0$. Since F is a nondecreasing mapping, we have that

$$x_0 < Fx_0 \leq F^2x_0 \leq \cdots \leq F^nx_0 \leq F^{n+1}x_0 \leq \cdots \quad (2.2)$$

Put $x_{n+1} = Fx_n$. If there exists $n \geq 1$ such that $x_{n+1} = x_n$, then from $x_{n+1} = Fx_n = x_n$, x_n is a fixed point and the proof is finished. Suppose that $x_{n+1} \neq x_n$ for $n \geq 1$.

Then, from (2.1) and as the elements x_n and x_{n-1} are comparable, we get, for $n \geq 1$,

$$\begin{aligned} p(x_{n+1}, x_n) &= p(Fx_n, Fx_{n-1}) \\ &\leq \alpha \frac{p(x_n, Fx_n) \cdot p(x_{n-1}, Fx_{n-1})}{p(x_n, x_{n-1})} + \beta p(x_n, x_{n-1}) \\ &= \alpha \frac{p(x_n, x_{n+1}) \cdot p(x_{n-1}, x_n)}{p(x_n, x_{n-1})} + \beta p(x_n, x_{n-1}) \\ &= \alpha \cdot p(x_n, x_{n+1}) + \beta \cdot p(x_n, x_{n-1}). \end{aligned} \tag{2.3}$$

The last inequality gives us

$$p(x_{n+1}, x_n) \leq \frac{\beta}{1-\alpha} p(x_n, x_{n-1}). \tag{2.4}$$

Again, using induction

$$p(x_{n+1}, x_n) \leq \left(\frac{\beta}{1-\alpha}\right)^n p(x_1, x_0). \tag{2.5}$$

Put $t = \beta/(1-\alpha) < 1$. Moreover, by $(p4)$, we have, for $m \geq n$,

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_n) - p(x_{m-1}, x_{m-1}) \\ &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_n) \\ &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + p(x_{m-2}, x_n) - p(x_{m-2}, x_{m-2}) \\ &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + p(x_{m-2}, x_n) \\ &\ \vdots \\ &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \cdots + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) \\ &\leq \left(t^{m-1} + t^{m-2} + \cdots + t^{n+1} + t^n\right) \cdot p(x_1, x_0) \\ &\leq \left(\frac{t^n}{1-t}\right) \cdot p(x_1, x_0). \end{aligned} \tag{2.6}$$

Letting $m, n \to +\infty$ in (2.6), we get

$$\lim_{m,n \to +\infty} p(x_m, x_n) = 0. \tag{2.7}$$

By (1.1), we have

$$p^s(x_m, x_n) \leq 2p(x_m, x_n). \tag{2.8}$$

Taking $m, n \to +\infty$ in (2.8) and using (2.7), we get that

$$\lim_{m,n \to +\infty} p^s(x_m, x_n) = 0. \tag{2.9}$$

Then $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Since (X, p) is complete, from Lemma 1.2, (X, p^s) is a complete metric space. Then, there exists $u \in X$ such that

$$\lim_{n \to +\infty} p^s(x_n, u) = 0. \tag{2.10}$$

On the other hand, we have

$$p^s(x_n, u) = 2p(x_n, u) - p(x_n, x_n) - p(u, u).$$

Letting $n \to +\infty$ in the above equation, using (2.7) and (2.10), we get

$$\lim_{n \to +\infty} p(x_n, u) = \frac{1}{2} p(u, u). \tag{2.11}$$

Furthermore, by $(p2)$, we have $p(u,u) \leq p(u,x_n)$ for all $n \in \mathbb{N}$. On letting $n \to +\infty$, we get that

$$p(u,u) \leq \lim_{n\to+\infty} p(u,x_n). \tag{2.12}$$

Using (2.11) and (2.12), we have

$$\lim_{n\to+\infty} p(u,x_n) = p(u,u) = 0. \tag{2.13}$$

Then, since $x_n \to u$ as $n \to +\infty$ in (X,p), and F is continuous, we get $Fx_n \to Fu$ as $n \to +\infty$ in (X,p). Therefore, we get, by $(p4)$,

$$\begin{aligned} p(u,Fu) &\leq p(u,x_{n+1}) + p(x_{n+1},Fu) - p(x_{n+1},x_{n+1}) \\ &\leq p(u,x_{n+1}) + p(x_{n+1},Fu) \\ &= p(u,x_{n+1}) + p(Fx_n,Fu). \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get that $p(u,Fu) = 0$. By $(p1)$ and $(p2)$, we have $u = Fu$. □

In what follows, we prove that Theorem 2.1 is still valid for F not necessarily continuous, assuming the following hypothesis in X :

if (x_n) is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$ for all $n \in \mathbb{N}$. (2.14)

Theorem 2.2. *Let $(X,\leq)$ be a partially ordered set and p be a partial metric on X such that (X,p) is a complete partial metric space. Assume that X satisfies* (2.14). *Let $F : X \to X$ be a nondecreasing mapping such that*

$$p(Fx,Fy) \leq \alpha \cdot \frac{p(x,Fx)\cdot p(y,Fy)}{p(x,y)} + \beta \cdot p(x,y), \quad x,y \in X, \quad x \geq y, \quad x \neq y \tag{2.15}$$

with $\alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Fx_0$, then F has a fixed point.

Proof. Following the proof of Theorem 2.1, we only have to check that $Fu = u$.

As $\{x_n\}$ is a nondecreasing sequence in X and $x_n \to u$, then, by (2.14), $u = \sup\{x_n\}$. Particularly, $x_n \leq u$ for all $n \in \mathbb{N}$.

Since F is a nondecreasing mapping, then $Fx_n \leq Fu$, for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \leq Fu$ for all $n \in \mathbb{N}$. Moreover, as $x_0 < x_1 \leq Fu$ and $u = \sup\{x_n\}$, we get $u \leq Fu$.

Suppose that $u < Fu$. Using a similar argument that in the proof of Theorem 2.1 for $x_0 \leq Fx_0$, we obtain that $\{F^n u\}$ is a nondecreasing sequence and $\lim_{n\to+\infty} F^n u = y$ for certain $y \in X$.

Again, using (2.14), we have that $y = \sup\{F^n u\}$. Moreover, from $x_0 \leq u$, we get $x_n = F^n x_0 \leq F^n u$ for $n \geq 1$ and $x_n < F^n u$ for $n \geq 1$ because $x_n \leq u < Fu \leq F^n u$ for $n \geq 1$. As x_n and $F^n u$ are comparable and distinct for $n \geq 1$, applying the contractive condition we get

$$\begin{aligned} p\left(x_{n+1},F^{n+1}u\right) &= p(Fx_n, F(F^n u)) \\ &\leq \alpha \frac{p(x_n,Fx_n)\cdot p\left(F^n u, F^{n+1}u\right)}{p(x_n,F^n u)} + \beta p(x_n,F^n u) \\ &= \alpha \frac{p(x_n,x_{n+1})\cdot p\left(F^n u, F^{n+1}u\right)}{p(x_n,F^n u)} + \beta p(x_n,F^n u). \end{aligned} \tag{2.16}$$

Making $n \to +\infty$ in the last inequality and using (2.7), we obtain

$$p(u, y) \leq \beta p(u, y). \tag{2.17}$$

As $\beta < 1$, $p(u, y) = 0$, thus, $u = y$.

Particularly, $u = y = \sup\{F^n u\}$ and, consequently, $Fu \leq u$ and this is a contradiction. Hence, we conclude that $u = Fu$. □

If $\beta = 0$ in Theorem 2.1 (or Theorem 2.2), we obtain the following fixed point theorem in ordered partial complete metric spaces.

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is a complete partial metric space. Let $F : X \to X$ be a nondecreasing mapping such that there exists $\alpha \in [0, 1)$ satisfying*

$$p(Fx, Fy) \leq \alpha \cdot \frac{p(x, Fx) \cdot p(y, Fy)}{p(x, y)}, \quad x, y \in X, \quad x \geq y, \quad x \neq y. \tag{2.18}$$

Suppose also that either F is continuous or X satisfies condition (2.14). *If there exists $x_0 \in X$ with $x_0 \leq Fx_0$, then F has a fixed point.*

References

[1] J. Harjani, B. López, K. Sadarangani, *A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space*, Abstr. Appl. Anal. (2010), Art. ID 190701.

[2] S.G. Matthews, *Partial metric topology*, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., **728** (1994), 183–197.

[3] O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topol. **6** (2005) 229–240.

[4] S. Oltra, O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Istit. Mat. Univ. Trieste **36** (2004) 17–26.

[5] I. Altun, F. Sola, H. Simsek,*Generalized contractions on partial metric spaces*, Topology Appl. **157** (2010) 2778–2785.

[6] S. Romaguera, *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl. (2010) Article ID 493298, 6 pages.

[7] H. Aydi, *Some coupled fixed point results on partial metric spaces*, Int. J. Math. Math. Sci. (2011) Article ID 647091, 11 pages.

[8] I. Altun, A. Erduran, *Fixed point theorems for monotone mappings on partial metric spaces*, Fixed Point Theory Appl. (2010) Article ID 508730, 10 pages.

[9] I. Altun, H. Simsek, *Some fixed point theorems on dualistic partial metric spaces*, J. Adv. Math. Stud. **1** (2008) 1–8.

[10] R. Heckmann, *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures **7** (1999) 71–83.

[11] Th. Abdeljawad, E. Karapınar, K. Taş, *Existence and uniqueness of a common fixed point on partial metric spaces*, Appl. Math. Lett. **24** (2011) 1900–1904.

[12] E. Karapınar, *Generalizations of Caristi Kirk's Theorem on partial metric spaces*, Fixed Point Theory Appl. (2011) 2011:4, 7 pp.

[13] E. Karapınar, İ Erhan, *Fixed point theorems for operators on partial metric spaces*, Appl. Math. Lett. **24** (2011) 1894–1899.

[14] H. Aydi, *Common fixed point results for mappings satisfying (ψ, φ)-weak contractions in ordered partial metric space*, Int. J. Math. Stat. **12 (2)** (2012).

Hüseyİn IŞIK, Department of Mathematics, Faculty of Science, University of Gazi, 06500-Teknikokullar, Ankara, TURKEY., Department of Mathematics, Muş Alparslan University, Muş 49100, TURKEY

E-mail address: huseyinisik@gazi.edu.tr

Duran TÜRKOĞLU, Department of Mathematics, Faculty of Science, University of Gazi, 06500-Teknikokullar, Ankara, TURKEY., Department of Mathematics, Faculty of Science and Arts, University of Amasya, 05100, Amasya, TURKEY
E-mail address: dturkoglu@gazi.edu.tr