

## BOUNDS FOR THE DEVIATION OF A FUNCTION FROM A GENERALISED CHORD GENERATED BY ITS EXTREMITIES WITH APPLICATIONS

SEVER S. DRAGOMIR

ABSTRACT. Bounds for the deviation of a real-valued function  $f$  defined on a compact interval  $[a, b]$  to the generalised chord

$$\frac{v(b) - v(t)}{v(b) - v(a)} \cdot f(a) + \frac{v(t) - v(a)}{v(b) - v(a)} \cdot f(b),$$

where  $v : [a, b] \rightarrow \mathbb{R}$  and  $v(a) \neq v(b)$ , that connects its end points  $(a, f(a))$  and  $(b, f(b))$  are given. Applications for normalised positive linear functionals are provided as well.

### 1. INTRODUCTION

Consider a function  $f : [a, b] \rightarrow \mathbb{R}$  and assume that it is bounded on  $[a, b]$ . Denote by  $\Phi_f(t)$  the error in approximating the function  $f$  by its (straight line) chord  $d_f$  which connects the points  $(a, f(a))$  and  $(b, f(b))$ , i.e.,

$$\Phi_f(t) := \frac{b-t}{b-a} \cdot f(a) + \frac{t-a}{b-a} f(b) - f(t), \quad t \in [a, b]. \quad (1.1)$$

In the recent paper [3], sharp error estimates for  $\Phi_f(t)$  under various assumptions on the function  $f$  have been derived. We recall here some of them.

If there exist the constants  $-\infty < m < M < \infty$  such that  $m \leq f(t) \leq M$  for each  $t \in [a, b]$ , then  $|\Phi_f(t)| \leq M - m$ . The multiplication constant 1 in front of  $(M - m)$  cannot be replaced by a smaller quantity. If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex function on  $[a, b]$ , then

$$\begin{aligned} 0 \leq \Phi_f(t) &\leq \frac{1}{b-a} (t-a)(b-t) [f'_-(b) - f'_+(a)] \\ &\leq \frac{1}{4} (b-a) [f'_-(b) - f'_+(a)], \end{aligned} \quad (1.2)$$

for any  $t \in [a, b]$ . In the case where the lateral derivatives  $f'_-(b)$  and  $f'_+(a)$  are finite, then the second inequality and the constant  $\frac{1}{4}$  are sharp.

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2000 *Mathematics Subject Classification.* Primary 26D15.

*Key words and phrases.* Bounded variation, Stieltjes integral, Monotonicity, Normalised positive functionals.

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Submitted 11. 11. 2012. Published 10. 01. 2013.

If  $f : [a, b] \rightarrow \mathbb{R}$  is a function of bounded variation, then

$$|\Phi_f(t)| \leq \frac{b-t}{b-a} \cdot \bigvee_a^t(f) + \frac{t-a}{b-a} \bigvee_t^b(f) \quad (1.3)$$

$$\leq \begin{cases} \left[ \frac{1}{2} + \left| t - \frac{a+b}{2} \right| \right] \bigvee_a^b(f); \\ \left[ \left( \frac{b-t}{b-a} \right)^p + \left( \frac{t-a}{b-a} \right)^p \right]^{\frac{1}{p}} \left[ \left( \bigvee_a^t(f) \right)^q + \left( \bigvee_t^b(f) \right)^q \right]^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^t(f) - \bigvee_t^b(f) \right|. \end{cases}$$

The first inequality in (1.3) is sharp. The constant  $\frac{1}{2}$  is best possible in the first and third branches.

In particular, if  $f$  is  $L$ -Lipschitzian on  $[a, b]$ , i.e.,  $|f(t) - f(s)| \leq L|t - s|$  for any  $t, s \in [a, b]$ , then

$$|\Phi_f(t)| \leq \frac{2(b-t)(t-a)}{b-a} L \leq \frac{1}{2}(b-a)L, \quad (1.4)$$

for any  $t \in [a, b]$ . The constants 2 and  $\frac{1}{2}$  are best possible.

For extensions to  $n$ -time differentiable functions see [4].

In this paper we consider a natural generalisation of the above problem by introducing the error function for the approximation of  $f(t)$  with  $\frac{v(b)-v(t)}{v(b)-v(a)} \cdot f(a) + \frac{v(t)-v(a)}{v(b)-v(a)} \cdot f(b)$ , where  $v : [a, b] \rightarrow \mathbb{R}$  is another function with the property that  $v(a) \neq v(b)$ . Error bounds for different pairs of functions  $(f, v)$  are derived. Applications in obtaining error bounds in approximating the quantity  $A(f \circ u)$  by the generalised trapezoid formula

$$\frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(a) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(b),$$

where  $A$  is a normalised linear functional are also given.

## 2. BOUNDS FOR $\Phi_{f,v}$ WHEN $f, v$ ARE OF BOUNDED VARIATION

For a function  $p : [a, b] \rightarrow \mathbb{R}$  we define the kernel  $Q_p : [a, b]^2 \rightarrow \mathbb{R}$  by

$$Q_p(t, s) := \begin{cases} p(t) - p(b) & \text{if } a \leq s \leq t \leq b, \\ p(t) - p(a) & \text{if } a \leq t < s \leq b. \end{cases} \quad (2.1)$$

With this notation we have the following representation of the function  $\Phi_{f,v}$ , where

$$\Phi_{f,v}(t) = \frac{v(t) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - v(t)}{v(b) - v(a)} \cdot f(a) - f(t)$$

with  $t \in [a, b]$ .

**Lemma 2.1.** *If  $f, v : [a, b] \rightarrow \mathbb{R}$  are bounded functions on  $[a, b]$ , then*

$$\begin{aligned}\Phi_{f,v}(t) &= \frac{1}{v(b) - v(a)} \int_a^b Q_v(t, s) df(s) \\ &= \frac{1}{v(b) - v(a)} \int_a^b Q_{-f}(t, s) dv(s)\end{aligned}\quad (2.2)$$

provided  $v(b) \neq v(a)$ , where the integrals are taken in the Riemann-Stieltjes sense.

*Proof.* We have

$$\begin{aligned}\Phi_{f,v}(t) &= \frac{[v(t) - v(b)][f(t) - f(a)] + [v(t) - v(a)][f(b) - f(t)]}{v(b) - v(a)} \\ &= \frac{[v(t) - v(b)] \int_a^t df(s) + [v(t) - v(a)] \int_t^b df(s)}{v(b) - v(a)} \\ &= \frac{1}{v(b) - v(a)} \int_a^b Q_v(t, s) df(s).\end{aligned}\quad (2.3)$$

Also, by rearranging the terms in the first equality, we also have

$$\begin{aligned}\Phi_{f,v}(t) &= \frac{[f(a) - f(t)] \int_t^b dv(s) + [f(b) - f(t)] \int_a^t dv(s)}{v(b) - v(a)} \\ &= \frac{1}{v(b) - v(a)} \int_a^b Q_{-f}(t, s) dv(s)\end{aligned}\quad (2.4)$$

and the representation (2.2) is proved.  $\square$

The following estimation result can be stated.

**Theorem 2.2.** *Assume that  $f, v : [a, b] \rightarrow \mathbb{R}$  are bounded and  $v(a) \neq v(b)$ .*

(i) *If  $f$  is of bounded variation on  $[a, b]$ , then*

$$\begin{aligned}|\Phi_{f,v}(t)| &\leq \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right| \cdot \bigvee_a^t(f) + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| \cdot \bigvee_t^b(f) \\ &\leq \begin{cases} \max \left\{ \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right|, \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| \right\} \bigvee_a^b(f); \\ \left[ \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right|^p + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right|^p \right]^{\frac{1}{p}} \left\{ \left[ \bigvee_a^t(f) \right]^q + \left[ \bigvee_t^b(f) \right]^q \right\}^{\frac{1}{q}}, \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|v(b) - v(t)| + |v(t) - v(a)|}{|v(b) - v(a)|} \left\{ \frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^t(f) - \bigvee_t^b(f) \right| \right\}. \end{cases}\end{aligned}\quad (2.5)$$

(ii) If  $v$  is of bounded variation on  $[a, b]$ , then

$$\begin{aligned}
|\Phi_{f,v}(t)| &\leq \left| \frac{f(b) - f(t)}{v(b) - v(a)} \right| \cdot \bigvee_a^t(v) + \left| \frac{f(t) - f(a)}{v(b) - v(a)} \right| \cdot \bigvee_t^b(v) \\
&\leq \begin{cases} \max \left\{ \left| \frac{f(b) - f(t)}{v(b) - v(a)} \right|, \left| \frac{f(t) - f(a)}{v(b) - v(a)} \right| \right\} \bigvee_a^b(v); \\ \left[ \left| \frac{f(b) - f(t)}{v(b) - v(a)} \right|^p + \left| \frac{f(t) - f(a)}{v(b) - v(a)} \right|^p \right]^{\frac{1}{p}} \left\{ \left[ \bigvee_a^t(f) \right]^q + \left[ \bigvee_t^b(f) \right]^q \right\}^{\frac{1}{q}}, \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|f(b) - f(t)| + |f(t) - f(a)|}{|v(b) - v(a)|} \left\{ \frac{1}{2} \bigvee_a^b(v) + \frac{1}{2} \left| \bigvee_a^t(v) - \bigvee_t^b(v) \right| \right\}. \end{cases}
\end{aligned} \tag{2.6}$$

*Proof.* Utilising the equality (2.3) and taking the modulus, we have successively:

$$\begin{aligned}
|\Phi_{f,v}(t)| &\leq \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right| \cdot \left| \int_a^t df(s) \right| + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| \cdot \left| \int_t^b df(s) \right| \\
&\leq \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right| \cdot \bigvee_a^t(f) + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| \cdot \bigvee_t^b(f) \\
&\leq \begin{cases} \max \left\{ \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right|, \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| \right\} \bigvee_a^b(f); \\ \left[ \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right|^p + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right|^p \right]^{\frac{1}{p}} \left\{ \left[ \bigvee_a^t(f) \right]^q + \left[ \bigvee_t^b(f) \right]^q \right\}^{\frac{1}{q}}, \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|v(b) - v(t)| + |v(t) - v(a)|}{|v(b) - v(a)|} \left\{ \frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^t(f) - \bigvee_t^b(f) \right| \right\}, \end{cases}
\end{aligned}$$

where for the last inequality we have used the Hölder inequality.

The inequality (2.6) goes likewise by utilising the equality (2.4).  $\square$

**Remark.** Since  $v(a) \neq v(b)$ , we can assume without loss the generality that  $v(a) < v(b)$ . Now, if we assume that

$$v(a) \leq v(t) \leq v(b) \quad \text{for any } t \in (a, b), \tag{2.7}$$

then from the first branch of (2.5) we get the inequality

$$|\Phi_{f,v}(t)| \leq \left[ \frac{1}{2} + \frac{|v(t) - \frac{v(a)+v(b)}{2}|}{v(b) - v(a)} \right] \bigvee_a^b(f), \quad t \in [a, b]. \tag{2.8}$$

The constant  $\frac{1}{2}$  is sharp in (2.8).

To prove the sharpness of the constant we take in (2.8)  $v(t) = t$  and then choose  $t = \frac{a+b}{2}$ . This produces the result:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{2} \bigvee_a^b(f), \tag{2.9}$$

which is sharp since for  $f(t) = |t - \frac{a+b}{2}|$ ,  $t \in [a, b]$  we obtain in both sides of (2.9) the same quantity  $\frac{b-a}{2}$ .

**Remark.** We also remark that, if  $v$  satisfies (2.7), then from the last inequality in (2.5) we get

$$|\Phi_{f,v}(t)| \leq \frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^t(f) - \bigvee_t^b(f) \right|, \quad t \in [a, b] \quad (2.10)$$

for which the first constant  $\frac{1}{2}$  is also best possible.

**Remark.** If  $f$  satisfies the property that  $f(a) \leq f(t) \leq f(b)$  for any  $t \in [a, b]$ , then from the first inequality in (2.6) we get

$$|\Phi_{f,v}(t)| \leq \left[ \frac{1}{2} \cdot \frac{f(b) - f(a)}{|v(b) - v(a)|} + \left| \frac{f(t) - \frac{f(a)+f(b)}{2}}{v(b) - v(a)} \right| \right] \bigvee_a^b(f), \quad t \in [a, b]. \quad (2.11)$$

With the same assumptions for  $f$  we have from the second inequality in (2.6) that

$$|\Phi_{f,v}(t)| \leq \frac{f(b) - f(a)}{|v(b) - v(a)|} \left\{ \frac{1}{2} \bigvee_a^b(v) + \frac{1}{2} \left| \bigvee_a^t(v) - \bigvee_t^b(v) \right| \right\}, \quad t \in [a, b]. \quad (2.12)$$

The first constant  $\frac{1}{2}$  in (2.12) is best possible.

Indeed, if we choose  $v(t) = t$  and then  $t = \frac{a+b}{2}$  in (2.12), we have

$$\left| \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{2} [f(b) - f(a)]. \quad (2.13)$$

Now, for  $f : [a, b] \rightarrow \mathbb{R}$ ,  $f(t) = 0$  if  $t \in [a, b]$  and  $f(b) = k > 0$ , we obtain on both sides the same quantity  $\frac{k}{2}$ .

### 3. BOUNDS FOR $\Phi_{f,v}$ WHEN $v(a) < v(t) < v(b)$ ( $f(a) < f(t) < f(b)$ )

The following result may be stated as well.

**Theorem 3.1.** Assume that  $f, v : [a, b] \rightarrow \mathbb{R}$  are bounded and  $v(a) \neq v(b)$ .

(i) If  $v(a) < v(t) < v(b)$  for any  $t \in (a, b)$ , then

$$|\Phi_{f,v}(t)| \leq \frac{1}{4} [v(b) - v(a)] \left[ \left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| + \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \right], \quad t \in [a, b]. \quad (3.1)$$

The constant  $\frac{1}{4}$  is best possible.

(ii) If  $f(a) < f(t) < f(b)$  for  $t \in (a, b)$ , then

$$|\Phi_{f,v}(t)| \leq \frac{1}{4} \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|} \left[ \left| \frac{v(t) - v(a)}{f(t) - f(a)} \right| + \left| \frac{v(b) - v(t)}{f(b) - f(t)} \right| \right], \quad t \in [a, b]. \quad (3.2)$$

*Proof.* (i) From the first equality in (2.3), we have:

$$\begin{aligned} |\Phi_{f,v}(t)| &\leq \frac{[v(b) - v(t)][v(t) - v(a)]}{|v(b) - v(a)|} \left[ \left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| + \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \right] \\ &= \frac{[v(b) - v(t)][v(t) - v(a)]}{|v(b) - v(a)|} \left[ \left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| + \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \right] \\ &\leq \frac{1}{4} [v(b) - v(a)] \left[ \left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| + \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \right] \end{aligned}$$

since, for any  $t \in (a, b)$ ,

$$[v(b) - v(t)][v(t) - v(a)] \leq \frac{1}{4} [v(b) - v(a)]^2.$$

For the best constant, choose  $v(t) = t$  and then  $t = \frac{a+b}{2}$  in (3.1) to obtain

$$\left| \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{2} \left[ \left| f\left(\frac{a+b}{2}\right) - f(a) \right| + \left| f(b) - f\left(\frac{a+b}{2}\right) \right| \right]. \quad (3.3)$$

If we consider the function  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$f(t) = \begin{cases} 0 & \text{if } t \in [a, b) \\ k & \text{if } t = b, k > 0, \end{cases}$$

then (3.3) becomes an equality with both terms  $\frac{k}{2}$ .

(ii) The proof goes likewise and the details are omitted.  $\square$

**Remark.**

(a) Under the assumptions of (i) of Theorem 3.1 and if there exist  $L_a > 0$ ,  $L_b > 0$ ,  $\alpha, \beta \geq 0$  such that

$$\left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| \leq L_a (t - a)^\alpha, \quad \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \leq L_b (b - t)^\beta, \quad t \in (a, b), \quad (3.4)$$

then we have the inequality:

$$|\Phi_{f,v}(t)| \leq \frac{1}{4} [v(b) - v(a)] \left[ L_a (t - a)^\alpha + L_b (b - t)^\beta \right], \quad t \in (a, b). \quad (3.5)$$

(aa) Under the assumptions of (ii) of Theorem 3.1 and if there exist the constants  $H_a, H_b > 0$  and  $\gamma, \delta \geq 0$  such that

$$\left| \frac{v(t) - v(a)}{f(t) - f(a)} \right| \leq H_a (t - a)^\gamma, \quad \left| \frac{v(b) - v(t)}{f(b) - f(t)} \right| \leq H_b (b - t)^\delta, \quad t \in (a, b), \quad (3.6)$$

then we have the inequality:

$$|\Phi_{f,v}(t)| \leq \frac{1}{4} \cdot \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|} \left[ H_a (t - a)^\gamma + H_b (b - t)^\delta \right], \quad t \in (a, b). \quad (3.7)$$

The following corollary provides some uniform bounds in the case where the functions are differentiable.

**Corollary 3.2.** Assume that  $f, v : [a, b] \rightarrow \mathbb{R}$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$  with  $v(a) \neq v(b)$ .

(i) If  $v(a) < v(t) < v(b)$  and  $v'(t) \neq 0$  for  $t \in (a, b)$ , then

$$|\Phi_{f,v}(t)| \leq \frac{1}{2} \cdot [v(b) - v(a)] \sup_{s \in (a,b)} \left| \frac{f'(s)}{v'(s)} \right|, \quad t \in (a, b). \quad (3.8)$$

(ii) If  $f(a) < f(t) < f(b)$  and  $f'(t) \neq 0$  for  $t \in (a, b)$ , then

$$|\Phi_{f,v}(t)| \leq \frac{1}{2} \cdot \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|} \sup_{s \in (a,b)} \left| \frac{v'(s)}{f'(s)} \right|, \quad t \in (a, b). \quad (3.9)$$

*Proof.* (i) Applying Cauchy's mean value theorem, we deduce that for any  $t \in (a, b)$  there exists an  $s$  between  $t$  and  $a$  such that

$$\frac{f(t) - f(a)}{v(t) - v(a)} = \frac{f'(s)}{v'(s)}.$$

Therefore,

$$\left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| \leq \sup_{s \in (a,b)} \left| \frac{f'(s)}{v'(s)} \right|, \quad t \in (a,b)$$

and in a similar manner,

$$\left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| \leq \sup_{s \in (a,b)} \left| \frac{f'(s)}{v'(s)} \right|, \quad t \in (a,b).$$

Utilising the inequality (2.13) we deduce (3.8).

The proof of (ii) goes likewise and we omit the details.  $\square$

#### 4. BOUNDS FOR $\Phi_{f,v}$ WHEN $f, v$ ARE LIPSCHITZIAN

We can state the following result.

**Theorem 4.1.** *Assume that  $f, v : [a, b] \rightarrow \mathbb{R}$  are bounded functions on  $[a, b]$  and  $v(a) \neq v(b)$ .*

- (i) *If there exist constants  $M_a, M_b > 0$  and  $\alpha, \beta > 0$  such that  $|f(t) - f(a)| \leq M_a(t-a)^\alpha$ ,  $|f(b) - f(t)| \leq M_b(b-t)^\beta$  for any  $t \in [a, b]$  and  $v : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable on  $[a, b]$ , then*

$$|\Phi_{f,v}(t)| \leq M_a \left| \frac{v(b) - v(t)}{f(b) - f(t)} \right| (t-a)^\alpha + M_b \left| \frac{v(t) - v(a)}{f(t) - f(a)} \right| (b-t)^\beta \quad (4.1)$$

for any  $t \in [a, b]$ .

- (ii) *If there exist constants  $N_a, N_b > 0$ ,  $\gamma, \delta > 0$  such that  $|v(t) - v(a)| \leq N_a(t-a)^\gamma$ ,  $|v(b) - v(t)| \leq N_b(b-t)^\delta$  for any  $t \in [a, b]$ , then*

$$|\Phi_{f,v}(t)| \leq N_b \left| \frac{f(t) - f(a)}{v(t) - v(a)} \right| (b-t)^\delta + N_a \left| \frac{f(b) - f(t)}{v(b) - v(t)} \right| (t-a)^\gamma \quad (4.2)$$

for any  $t \in [a, b]$ .

*Proof.* Utilising the representation (2.3) we have:

$$|\Phi_{f,v}(t)| \leq \frac{|f(t) - f(a)| |v(b) - v(t)| + |v(t) - v(a)| |f(b) - f(t)|}{|v(b) - v(a)|}$$

for any  $t \in [a, b]$ , which clearly produces the desired inequalities (4.1) and (4.2).  $\square$

We notice that, if more information is provided for  $f$  and  $v$ , then more specific bounds can be obtained. For instance, if  $f$  is as in (i) of Theorem 4.1 and  $v(a) < v(t) < v(b)$  for each  $t \in (a, b)$ , then we get from (4.1) the following inequality:

$$|\Phi_{f,v}(t)| \leq \left[ \frac{1}{2} + \left| \frac{v(t) - \frac{v(a)+v(b)}{2}}{v(b) - v(a)} \right| \right] \left[ M_a(t-a)^\alpha + M_b(b-t)^\beta \right] \quad (4.3)$$

for any  $t \in [a, b]$ .

Similarly, if  $v$  satisfies condition (ii) of Theorem 4.1 and  $f(a) < f(t) < f(b)$  for each  $t \in (a, b)$ , then

$$\begin{aligned} |\Phi_{f,v}(t)| \leq & \left[ \frac{1}{2} \cdot \frac{f(b) - f(a)}{|v(b) - v(a)|} + \left| \frac{f(t) - \frac{f(a)+f(b)}{2}}{v(b) - v(a)} \right| \right] \\ & \times \left[ N_b(b-t)^\delta + N_a(t-a)^\gamma \right] \end{aligned} \quad (4.4)$$

for any  $t \in [a, b]$ .

If  $f$  is  $M$ -Lipschitzian, then from (4.1) we get

$$\begin{aligned} |\Phi_{f,v}(t)| &\leq M \left[ \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right| (t - a) + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| (b - t) \right] \\ &\leq M \left[ \frac{1}{2} (b - a) + \left| t - \frac{a + b}{2} \right| \right] \left[ \left| \frac{v(b) - v(t)}{v(b) - v(a)} \right| + \left| \frac{v(t) - v(a)}{v(b) - v(a)} \right| \right], \end{aligned} \quad (4.5)$$

for any  $t \in [a, b]$ .

Also, if  $v$  is  $N$ -Lipschitzian, then from (4.1) we get

$$\begin{aligned} |\Phi_{f,v}(t)| &\leq N \left[ \left| \frac{f(t) - f(a)}{v(b) - v(a)} \right| (b - t) + \left| \frac{f(b) - f(t)}{v(b) - v(a)} \right| (t - a) \right] \\ &\leq N \left[ \frac{1}{2} (b - a) + \left| t - \frac{a + b}{2} \right| \right] \left[ \left| \frac{f(t) - f(a)}{v(b) - v(a)} \right| + \left| \frac{f(b) - f(t)}{v(b) - v(a)} \right| \right] \end{aligned} \quad (4.6)$$

for any  $t \in [a, b]$ .

Moreover, if  $f$  is  $M$ -Lipschitzian and  $v(a) < v(t) < v(b)$  for any  $t \in [a, b]$ , then from (4.5) we get the simpler inequality:

$$|\Phi_{f,v}(t)| \leq M \left[ \frac{1}{2} (b - a) + \left| t - \frac{a + b}{2} \right| \right] \quad (4.7)$$

for any  $t \in [a, b]$ .

If  $v$  is  $N$ -Lipschitzian and  $f(a) < f(t) < f(b)$ ,  $v(a) < v(b)$ , then from (4.6) we also have:

$$|\Phi_{f,v}(t)| \leq N \cdot \frac{f(b) - f(a)}{v(b) - v(a)} \left[ \frac{1}{2} (b - a) + \left| t - \frac{a + b}{2} \right| \right], \quad (4.8)$$

for each  $t \in [a, b]$ .

## 5. APPLICATIONS FOR POSITIVE LINEAR FUNCTIONALS

Let  $L$  be a linear class of real-valued functions  $g : E \rightarrow \mathbb{R}$  having the properties

- (L1)  $f, g \in L$  imply  $(\alpha f + \beta g) \in L$  for all  $\alpha, \beta \in \mathbb{R}$ ;
- (L2)  $\mathbf{1} \in L$ , i.e., if  $f_0(t) = 1$ ,  $t \in E$ , then  $f_0 \in L$ .

An isotonic linear functional  $A : L \rightarrow \mathbb{R}$  is a functional satisfying

- (A1)  $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$  for all  $f, g \in L$  and  $\alpha, \beta \in \mathbb{R}$ ;
- (A2) If  $f \in L$  and  $f \geq 0$ , then  $A(f) \geq 0$ ;
- (A3) The mapping  $A$  is *normalised* if  $A(\mathbf{1}) = 1$ .

For a function  $u : E \rightarrow [a, b]$ , we consider the function

$$\Phi_{f,v}(u) := \frac{v \circ u - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - v \circ u}{v(b) - v(a)} \cdot f(a) - f \circ u$$

and assume throughout this section that  $\Phi_{f,v}(u) \in L$ .

It is obvious that for a normalised linear functional  $A : L \rightarrow \mathbb{R}$  we have

$$A(\Phi_{f,v}(u)) = \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u)$$

and the inequalities in the previous section can be utilised to provide various upper bounds for the quantity

$$|A(\Phi_{f,v}(u))|.$$

For the sake of brevity we give here only some bounds that are simple and perhaps more useful for applications.



**Proposition 5.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be of bounded variation on  $[a, b]$  and  $v(a) < v(b)$ ,  $v(a) \leq v(t) \leq v(b)$  for each  $t \in [a, b]$ . If  $u \in L$  so that  $\Phi_{f,v}(u) \in L$  and  $A : L \rightarrow \mathbb{R}$  is a normalised positive linear functional on  $L$ , then:*

$$\begin{aligned} & \left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \\ & \leq \left[ \frac{1}{2} + \frac{1}{v(b) - v(a)} A \left( \left| v \circ u - \frac{v(a) + v(b)}{2} \cdot \mathbf{1} \right| \right) \right] \bigvee_a^b(f). \end{aligned} \quad (5.1)$$

*Proof.* Utilising the inequality (2.8) and the properties of the functional  $A$ , we have

$$\begin{aligned} |A(\Phi_{f,v}(u))| & \leq A(|\Phi_{f,v}(u)|) \\ & \leq A \left[ \left( \frac{1}{2} + \left| \frac{v \circ u - \frac{v(a) + v(b)}{2}}{v(b) - v(a)} \right| \right) \bigvee_a^b(f) \right] \\ & = \bigvee_a^b(f) \left[ \frac{1}{2} + \frac{1}{v(b) - v(a)} A \left( \left| v \circ u - \frac{v(a) + v(b)}{2} \cdot \mathbf{1} \right| \right) \right] \end{aligned}$$

and the inequality (5.1) is proved.  $\square$

**Proposition 5.2.** *Let  $f, v : [a, b] \rightarrow \mathbb{R}$  be bounded and  $v(a) \neq v(b)$ . Also, assume that  $u \in L$  such that  $\Phi_{f,v}(u) \in L$  and  $A : L \rightarrow \mathbb{R}$  is a normalised positive linear functional on  $L$ .*

(i) *If  $v(a) < v(t) < v(b)$  for each  $t \in [a, b]$ , then*

$$\begin{aligned} & \left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \\ & \leq \frac{1}{4} [v(b) - v(a)] \left[ A \left( \left| \frac{f - f(a) \cdot \mathbf{1}}{v - v(a) \cdot \mathbf{1}} \right| \right) + A \left( \left| \frac{f(b) \cdot \mathbf{1} - f}{v(b) \cdot \mathbf{1} - v} \right| \right) \right], \end{aligned} \quad (5.2)$$

*provided  $\frac{f - f(a) \cdot \mathbf{1}}{v - v(a) \cdot \mathbf{1}}, \frac{f(b) \cdot \mathbf{1} - f}{v(b) \cdot \mathbf{1} - v} \in L$ ;*

(ii) *If  $f(0) < f(t) < f(b)$  for  $t \in (a, b)$ , then*

$$\begin{aligned} & \left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \\ & \leq \frac{1}{4} \cdot \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|} \left[ A \left( \left| \frac{v - v(a) \cdot \mathbf{1}}{f - f(a) \cdot \mathbf{1}} \right| \right) + A \left( \left| \frac{v(b) \cdot \mathbf{1} - v}{f(b) \cdot \mathbf{1} - f} \right| \right) \right], \end{aligned} \quad (5.3)$$

*provided  $\frac{v - v(a) \cdot \mathbf{1}}{f - f(a) \cdot \mathbf{1}}, \frac{v(b) \cdot \mathbf{1} - v}{f(b) \cdot \mathbf{1} - f} \in L$ .*

Utilising Corollary 3.2 we can state the following result that can be utilised for applications.

**Proposition 5.3.** *Let  $f, v : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Also, assume that  $u \in L$  such that  $\Phi_{f,v}(u) \in L$  and  $A : L \rightarrow \mathbb{R}$  is a normalised positive functional on  $L$ .*

(i) If  $v$  is strictly monotonic on  $[a, b]$ , then

$$\left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \leq \frac{1}{2} |v(b) - v(a)| \sup_{s \in (a,b)} \left| \frac{f'(s)}{v'(s)} \right|. \quad (5.4)$$

(ii) If  $f$  is strictly monotonic on  $[a, b]$ , then

$$\left| \frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u) \right| \leq \frac{1}{2} \cdot \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|} \sup_{s \in (a,b)} \left| \frac{v'(s)}{f'(s)} \right|, \quad (5.5)$$

provided  $v(a) \neq v(b)$ .

For other inequalities for isotonic linear functionals, see the papers [1], [2], [6] and the books [5] and [7].

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SCHOOL OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE, VIC 8001, AUSTRALIA.

*E-mail address:* Sever.Dragomir@vu.edu.au

*URL:* <http://rgmia.vu.edu.au/dragomir>