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RELATIONSHIP BETWEEN CHARACTERIZATIONS OF THE Q-GAMMA FUNCTION

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Abstract. In this work we are interested by giving two new characterizations of the q-Gamma function other than given by Askey and show that there are intimately related. This study leads to discover a good candidate of the analogue of the Euler's constant.

1. Introduction

In literature the characterisations of the well known Gamma function are studied by many authors. For our study let us recall the following ones. We begin by recalling that In 1922, Bohr and Mollerup [2] proved their famous theorem:

Theorem 1. *If a function $f(x)$ satisfies the following three conditions:*
(a) $f(x+1) = xf(x)$,
(b) $f(1) = 1$,
(c) $Log f(x)$ is convex for $x > 0$,
then $f(x) = \Gamma(x)$.

A second characterization formulated and proved by Laugwitz and Rodewald [3] says that the convexity of $\ln\Gamma(x)$ can be replaced by the property, call it property (L), that the function $L(x) = \ln\Gamma(x+1)$ satisfies:

$$L(n+x) = L(n) + x\ln(n+1) + r_n(x),$$

where $r_n(x) \to 0$ as $n \to +\infty$.

In [8], the author gave a third characterization and proved how are these characterizations related:

Theorem 2. *The Gamma function is the unique positive function f on $]0, +\infty[$ satisfying the following three properties:*
(a) $f(x+1) = xf(x)$
(b)$f(1) = 1$
(c) $f(x+n) = f(n)n^x t_n(x)$, where $t_n(x) \to 1$ as $n \to \infty$.

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The q-Gamma function, a q-analogue of Euler's Gamma function, was introduced by Jackson [5]. As same as the Gamma function, the q-Gamma function played an important role in q-analysis and many interesting results are derived.

In 1978, R. Askey [1] proved that the q-Gamma function satisfies a q-analogue of the Bohr-Mollerup theorem:

Theorem 3. *The function $\Gamma_q(x)$ is uniquely determined by the conditions:*
(a) $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.
(b) $\Gamma_q(1) = 1$.
(c) $Log\Gamma_q(x)$ *is convex for positive* x.

Since the first characterization of q-Gamma function was given by Askey [1], in the present paper we only show explicitly the second one because the third can be deduced easily. Our aim is to prove that they are intimately related. The reader can found in this work a q-analogue of the well-known Euler's constant.

2. Notations and Preliminaries

We recall some usual notions and notations used in the q-theory (see [4] and [6]). Throughout this paper, we assume $q \in]0,1[$.

For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) = (1-a)(1-aq)....(1-aq^{n-1}),\ n = 1,2,..... \quad (2.1)$$

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i). \quad (2.2)$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C}, \quad (2.3)$$

and

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}. \quad (2.4)$$

Inspired by the relation

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in \mathbb{N}, \quad (2.5)$$

It's natural to generalize the notion of the q-shifted factorial in the following way: for any number α, define

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}. \quad (2.6)$$

For $a, x \in \mathbb{C}$ and $n \in \mathbb{N}$, we note

$$(x-a)_q^{(n)} = \begin{cases} 1 & if \quad n = 0 \\ \prod_{i=0}^{n-1}(x-aq^i) & if \quad n = 1,2,... \end{cases} \quad (2.7)$$

The q-derivatives $D_q f$ and $D_q^+ f$ of a function f are given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ (D_q^+ f)(x) = \frac{f(q^{-1}x) - f(x)}{(1-q)x}, \ if \ x \neq 0, \tag{2.8}$$

$(D_q f)(0) = f'(0)$ and $(D_q^+ f)(0) = q^{-1} f'(0)$ provided $f'(0)$ exists.

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by (see [5])

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.9}$$

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n, \tag{2.10}$$

provided the sums converge absolutely.

The q-Jackson integral in a generic interval $[a, b]$ is given by (see [5])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.11}$$

The q-integration by parts is given for suitable functions f and g by (see [7])

$$\int_a^b g(x) D_q f(x) d_q x = f(b) g(q^{-1} b) - f(a) g(q^{-1} a) - \int_a^b f(x) D_q^+ g(x) d_q x. \tag{2.12}$$

Finally we consider the following useful sets:
$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\} \cup \{0\}$, $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$ and $\widetilde{\mathbb{R}}_{q,+} = \mathbb{R}_{q,+} \cup \{0\}$.

3. The q-Gamma function:

The q-Gamma function $\Gamma_q(x)$ was introduced by Jackson [5] as the infinite product:

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} \ , x > 0, \tag{3.1}$$

where $q \in]0,1[$.

It's easy to see that $\Gamma_q(x)$ satisfies the following properties:

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1. \tag{3.2}$$

It follows that:

$$\Gamma_q(n+1) = [1]_q ... [n]_q = [n]_q!. \tag{3.3}$$

The first integral representation of $\Gamma_q(x)$ is given by Koelink and Koornwinder in [7]:

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q^{-qt} d_q t, \tag{3.4}$$

where E_q^t is one of the two q-analogue of the exponential function:

$$E_q^t = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{t^n}{[n]_q!} = (-(1-q)t;q)_\infty, \tag{3.5}$$

$$e_q^t = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} = \frac{1}{((1-q)t;q)_\infty}. \tag{3.6}$$

In the rest of the following section we will derive a product expression of the q-Gamma function and a q-analogue of the Euler's constant.

Definition 1. *For each positive integer n, we define the function $\Gamma_{n,q}$ on $]0,+\infty[$ by*

$$\Gamma_{n,q}(x) = \frac{[n]_q^{[x]_q}[n]_q!}{[x]_q[x+1]_q...[x+n]_q} \tag{3.7}$$

Lemma 1. *The sequence $\left(\gamma_{n,q} = \sum_{k=1}^{n} \frac{q^k}{[k]_q} - \ln[n]_q\right)_n$ converges for any $q \in]0,1[$.*

Proof.

$$\begin{aligned} \gamma_{n+1,q} - \gamma_{n,q} &= \frac{q^{n+1}}{[n+1]_q} - \ln\frac{[n+1]_q}{[n]_q} \\ &= \frac{q^{n+1}}{[n+1]_q} - \ln\frac{1-q^{n+1}}{1-q^n} \\ &= \frac{q^{n+1}}{[n+1]_q} - \ln\left(1 + \frac{q^n - q^{n+1}}{1-q^n}\right) \\ &= \frac{q^{n+1}}{[n+1]_q} - \ln\left(1 + \frac{q^n}{[n]_q}\right) \\ &= \frac{q^{n+1}}{[n+1]_q} - \frac{q^n}{[n]_q} + o(\frac{q^n}{[n]_q}). \end{aligned}$$

Using the fact that $\frac{1}{[n]_q} = \frac{1-q}{1-q^n} < 1$, the series $\sum \gamma_{n+1,q} - \gamma_{n,q}$ converges and so the sequence $(\gamma_{n,q})_n$ converges. □

Definition 2. *The limit of the sequence $(\gamma_{n,q})_n$ is denoted by γ_q and it will be called q-Euler's constant.*

The following table gives some approximative values of γ_q:

q=0.2	q=0.5	q=0.9	q=0.99	q=0.999	q=0.9999
0.0182	0.110	0.406	0.549	0.574	0.578

We note that when q tends to 1,the q-Euler constant γ_q tends to the classical Euler constant $\gamma = 0,5772...$

Lemma 2. *The sequence $(\Gamma_{n,q}(x))_n$ of functions on $]0,+\infty[$ converges for any $x > 0$.*

Proof. Taking logarithms, we have

$$\begin{aligned}
\ln\Gamma_{n,q}(x) &= [x]_q\ln[n]_q+\ln[n]_q!-\ln[x]_q-...-\ln[x+n]_q\\
&= [x]_q\ln[n]_q-\ln[x]_q+\sum_{k=1}^{n}\ln[k]_q-\sum_{k=1}^{n}\ln[x+k]_q\\
&= [x]_q\ln[n]_q-\ln[x]_q-\sum_{k=1}^{n}\ln\frac{[x+k]_q}{[k]_q}\\
&= [x]_q\ln[n]_q-\ln[x]_q-\sum_{k=1}^{n}\ln(1+q^k\frac{[x]_q}{[k]_q})\\
&= -\ln[x]_q-[x]_q\left(\sum_{k=1}^{n}\frac{q^k}{[k]_q}-\ln[n]_q\right)+\sum_{k=1}^{n}\left(q^k\frac{[x]_q}{[k]_q}-\ln(1+q^k\frac{[x]_q}{[k]_q})\right)\\
&= -\ln[x]_q-[x]_q\gamma_{n,q}+c_{n,q}(x).
\end{aligned}$$

The sequence $(\gamma_{n,q})_n$ converges (see lemma1).
Also the sequence $\left(c_{n,q}(x)=\sum_{k=1}^{n}\left(q^k\frac{[x]_q}{[k]_q}-\ln(1+q^k\frac{[x]_q}{[k]_q})\right)\right)$ converges, since for $k>x>0$, we have $\frac{[x]_q}{[k]_q}<1$ and so
$0<q^k\frac{[x]_q}{[k]_q}-\ln(1+q^k\frac{[x]_q}{[k]_q})=q^k\frac{[x]_q}{[k]_q}-\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}(q^k\frac{[x]_q}{[k]_q})^i\leq\frac{q^{2k}[x]_q^2}{2[k]_q^2}\leq q^{2k}$.
Thus, the sequence $(\ln\Gamma_{n,q}(x))_n$ converges and hence so does the sequence $(\Gamma_{n,q}(x))_n$ for $x>0$. This completes the proof of lemma 2. □

Theorem 4. *The limit function of the sequence* $\left((1-q)^{[x]_q-x}\Gamma_{n,q}(x)\right)_n$ *is the function q-Gamma:*

$$\Gamma_q(x)=\lim_{n\to+\infty}(1-q)^{[x]_q-x}\frac{[n]_q^{[x]_q}[n]_q!}{[x]_q[x+1]_q...[x+n]_q},\quad x>0. \tag{3.8}$$

Proof. Using the q-integration by parts rule, we obtain

$$\begin{aligned}
\int_0^{\frac{1}{1-q}}((1-q)qt;q)_n t^{x-1}d_qt &= \frac{1}{[x]_q}\int_0^{\frac{1}{1-q}}((1-q)qt;q)_n D_q(t^x)d_qt\\
&= -\frac{1}{[x]_q}\int_0^{\frac{1}{1-q}}D_q^+((1-q)qt;q)_n t^x d_qt\\
&= (1-q)\frac{[n]_q}{[x]_q}\int_0^{\frac{1}{1-q}}((1-q)qt;q)_{n-1}t^x d_qt\\
&= (1-q)^2\frac{[n]_q[n-1]_q}{[x]_q[x+1]_q}\int_0^{\frac{1}{1-q}}((1-q)qt;q)_{n-2}t^{x+1}d_qt.
\end{aligned}$$

By successive q-integration by parts, we have

$$\begin{aligned}
\int_0^{\frac{1}{1-q}}((1-q)qt;q)_n t^{x-1}d_qt &= (1-q)^n\frac{[n]_q!}{[x]_q[x+1]_q...[x+n-1]_q}\int_0^{\frac{1}{1-q}}\frac{1}{[x+n]_q}D_q(t^{x+n})d_qt\\
&= (1-q)^{-x}\frac{[n]_q!}{[x]_q[x+1]_q...[x+n]_q}.
\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\to+\infty}\int_0^{\frac{1}{1-q}}((1-q)qt;q)_n t^{x-1}d_qt &= \lim_{n\to+\infty}(1-q)^{[x]_q-x}\left(\frac{1-q^n}{1-q}\right)^{[x]_q}\frac{[n]_q!}{[x]_q[x+1]_q...[x+n]_q}\\ &= \lim_{n\to+\infty}(1-q)^{[x]_q-x}\frac{[n]_q^{[x]_q}[n]_q!}{[x]_q[x+1]_q...[x+n]_q}.\end{aligned}$$

On the other hand and by using the dominated convergence theorem we have

$$\begin{aligned}\lim_{n\to+\infty}\int_0^{\frac{1}{1-q}}((1-q)qt;q)_n t^{x-1}d_qt &= \int_0^{\frac{1}{1-q}}((1-q)qt;q)_\infty t^{x-1}d_qt\\ &= \int_0^{\frac{1}{1-q}}E_q(-qt)t^{x-1}d_qt\\ &= \Gamma_q(x).\end{aligned}$$

This completes the proof of the theorem. □

4. Characterization of the q-Gamma function:

Theorem 5. *There exists a unique function $f(x)>0$ on $]0,+\infty[$ that satisfies the following properties:*
a) $f(1)=1$
b) $f(x+1)=[x]_qf(x)$
c) $f(x+n)=(1-q)^{[x]_q-x}f(n)[n]_q^{[x]_q}t_n(x)$*, where* $t_n(x)\to 1$ *as* $n\to\infty$*.*

Proof. First we prove that $\Gamma_q(x)$ satisfies conditions (a), (b) and (c).
Its well-known that the function q-Gamma satisfies the conditions (a) $\Gamma_q(1)=1$ and (b) $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

As a consequence of the two properties, we get $\Gamma_q(n)=[n-1]_q!$
(c) Let $s_n(x)=\frac{\Gamma_q(x)}{(1-q)^{[x]_q-x}\Gamma_{n,q}(x)}$. Then $\Gamma_q(x)=s_n(x)(1-q)^{[x]_q-x}\Gamma_{n,q}(x)$ and $\lim_{n\to+\infty}s_n(x)=1$.
For $n\in\mathbb{N}$ and $x>0$, we apply (b) n times to get

$$\begin{aligned}\Gamma_q(x+n) &= [x+n-1]_q...[x+1]_q[x]_q\Gamma_q(x)\\ &= \frac{[x+n]_q...[x+1]_q[x]_q}{[x+n]_q}.(1-q)^{[x]_q-x}\frac{[n]_q^{[x]_q}[n]_q!}{[x]_q[x+1]_q...[x+n]_q}.s_n(x)\\ &= (1-q)^{[x]_q-x}[n]_q^{[x]_q}\Gamma_q(n)t_n(x).\end{aligned}$$

Where $t_n(x)=\frac{[n]_q}{[x+n]_q}.s_n(x)$. Thus, $\Gamma_q(x+n)=(1-q)^{[x]_q-x}\Gamma_q(n)[n]_q^{[x]_q}t_n(x)$ and $t_n(x)\to 1$ as $n\to+\infty$.

To show uniqueness, we assume $f(x)$ is a function that satisfies (a), (b) and (c). From properties (a) and (b), we have

$$f(n)=[n-1]_q!. \tag{4.1}$$

$$f(x+n) = [x+n-1]_q[x+n-2]_q...[x+1]_q[x]_q f(x). \tag{4.2}$$

Combining (4.1),(4.2) and (c) together, we have

$$f(x) = (1-q)^{[x]_q - x}\frac{[n]_q^{[x]_q}[n-1]_q!}{[x+n-1]_q[x+n-2]_q...[x+1]_q[x]_q}t_n(x) = (1-q)^{[x]_q-x}\Gamma_{n,q}(x).s_n(x),$$

where $s_n(x) = \frac{[x+n]_q}{[n]_q}t_n(x) \to 1$ as $n \to +\infty$. Therefore $f(x) = \Gamma_q(x)$ and hence f is uniquely determined. This completes the proof of the third characterization of the q-Gamma function. □

5. Relationship between characterizations

In what follows, we will adopt the terminology of the following definition.

Definition 3. *A function f is said to be a q-PG function (pre-q-gamma function), if f is positive on $]0,+\infty[$ and satisfies the functional equation $f(x+1) = [x]_q f(x)$.*

In the previous section we showed that the property $f(x+n) = (1-q)^{[x]_q-x}f(n)[n]_q^{[x]_q}t_n(x)$ characterizes the q-gamma function. In this section we will give three properties which are equivalent to one another for a q-PG function and characterize the q-gamma function.

Theorem 6. *For a q-PG function f, the following properties are equivalent:*
(C) $\ln f$ is convex on $]0,+\infty[$,
(L)$L(n+x) = ([x]_q - x)\ln(1-q) + L(n) + x\ln(n+1) + r_n(x)$,
where $L(x) = \ln f(x+1)$ and $r_n(x) \to 0$ as $n \to \infty$,
(P) $f(x+n) = (1-q)^{[x]_q-x}f(n)[n]_q^{[x]_q}t_n(x)$,
where $t_n(x) \to 1$ as $n \to \infty$.
A q-PG function f satisfying these properties is equal to $c\Gamma_q(x)$, for some constant c.

Proof. (a) $(P) \Leftrightarrow (L)$. We have

$$\begin{aligned}
(P) \quad &\Leftrightarrow \quad f(x+(n+1)) = (1-q)^{[x]_q-x}f(n+1)[n+1]_q^{[x]_q}t_{n+1}(x), t_{n+1}(x) \to 1 \\
&\Leftrightarrow \quad \ln f(x+(n+1)) = ([x]_q - x)\ln(1-q) + \ln f(n+1) + [x]_q \ln[n+1]_q + \ln t_{n+1}(x) \\
&\Leftrightarrow \quad L(x+n) = ([x]_q - x)\ln(1-q) + L(n) + [x]_q\ln[n+1]_q + r_n(x), r_n(x) \to 0 \\
&\Leftrightarrow \quad (L).
\end{aligned}$$

(b) $(C) \implies (P)$. Let $m < x \leq m+1$, where $m = 0, 1, 2, ...$ For any natural n, $n+m-1 < n+m < n+x \leq n+m+1$. The convexity of $\ln f$ gives us (we write $L_m = \ln f(n+m)$)

$$\frac{L_m - L_{m-1}}{n+m-(n+m-1)} \leq \frac{\ln f(n+x) - \ln f(n+m)}{(n+x)-(n+m)} \leq \frac{L_{m+1} - L_m}{(n+m+1)-(n+m)}$$

$$\Leftrightarrow (x-m)\ln[n+m-1]_q \leq \ln\frac{f(n+x)}{f(n+m)} \leq (x-m)\ln[n+m]_q$$

$$\Leftrightarrow [n+m-1]_q^{x-m} \leq \frac{f(n+x)}{[n+m-1]_q[n+m-2]_q...[n]_q f(n)} \leq [n+m]_q^{x-m}$$

$$\Leftrightarrow [n+m-1]_q^x T_m \leq \frac{f(n+x)}{f(n)} \leq [n+m-1]_q^x T_m\frac{[n+m-1]_q^m}{[n+m]_q^m},$$

where $T_m = \dfrac{[n+m-1]_q[n+m-2]_q...[n]_q}{[n+m-1]_q^m}$.
Therefore, we have

$$\lim_{n\to+\infty} \frac{f(n+x)}{f(n)} = \frac{1}{(1-q)^x},$$

by the squeezing theorem. If we let

$$t_n(x) = \frac{f(n+x)}{(1-q)^{[x]_q-x} f(n)[n]_q^{[x]_q} f(n)},$$

then

$$f(n+x) = (1-q)^{[x]_q-x} f(n)[n]_q^{[x]_q} t_n(x),$$

where $t_n(x) \to 1$ as $n \to \infty$. This proves that f satisfies (P).
(c) $(P) \Longrightarrow (C)$. From the uniqueness part of the proof of the Theorem 5 we have $f(x) = f(1)\lim_{n\to+\infty}\Gamma_{n,q}(x)$.
Using the fact that the limit function of a convergent sequence of convex functions is convex, it suffices to show that $\ln\left((1-q)^{[x]_q-x}\Gamma_{n,q}(x)\right)$ is convex.
Now

$$\ln\left((1-q)^{[x]_q-x}\Gamma_{n,q}(x)\right) = ([x]_q - x)\ln(1-q) + [x]_q \ln[n]_q + \ln([n]_q!) - \ln[x]_q - ... - \ln[x+n]_q.$$

Therefore, we have

$$\left(\ln\left((1-q)^{[x]_q-x}\Gamma_{n,q}(x)\right)\right)'$$

$$= \left(-\frac{\ln q}{1-q}q^x - 1\right)\ln(1-q) + \left(-\frac{\ln q}{1-q}q^x \ln[n]_q\right) + \frac{\ln q}{1-q}\frac{q^x}{[x]_q} + ... + \frac{\ln q}{1-q}\frac{q^{x+n}}{[x+n]_q}\ .$$

And so

$$\left(\ln\left((1-q)^{[x]_q-x}\Gamma_{n,q}(x)\right)\right)''$$

$$= -\frac{(\ln q)^2}{1-q}q^x(\ln(1-q) + \ln\frac{1-q^n}{1-q}) + \frac{(\ln q)^2}{1-q}\left[\frac{q^x[x]_q + \frac{q^{2x}}{1-q}}{[x]_q^2} + ... + \frac{q^{x+n}[x+n]_q + \frac{q^{2(x+n)}}{1-q}}{[x+n]_q^2}\right]$$

$$= -\frac{(\ln q)^2}{1-q}q^x(\ln(1-q^n)) + \frac{(\ln q)^2}{1-q}\left[\frac{q^x[x]_q + \frac{q^{2x}}{1-q}}{[x]_q^2} + ... + \frac{q^{x+n}[x+n]_q + \frac{q^{2(x+n)}}{1-q}}{[x+n]_q^2}\right].$$

Then
$\left(\ln\left((1-q)^{[x]_q-x}\Gamma_{n,q}(x)\right)\right)'' > 0.$
This completes the proof.

□

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