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ON A TWO-VARIABLE ANALOGUE OF THE BESSEL FUNCTIONS

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Abstract. The main object of this paper to construct a two-variable analogue $J_{k,s}(x,y)$ of first kind of the Bessel functions $J_k(x)$ and to give various properties such as orthogonality, recurrence relations, differential equation and generating function for these functions. Furthermore, some special cases of the results presented in this study are also indicated.

1. Introduction

The Bessel functions $J_k(x)$ are among the most important special functions with very diverse applications to physics, engineering and mathematical analysis. The importance of the Bessel functions has been further stressed by their various generalizations: with more indices as well as with more variables. Among the first kind of generalizations we mean Wright's 2, 3 and 4 index Bessel type functions, in the sense of [9]. Nowadays, new classes of multiindex functions are also introduced, see e.g. [8, 11]. For the works considering the second kind of generalizations we refer to [1, 2, 3, 4, 5, 7].

The Bessel functions of the first kind of order k are defined by

$$J_k(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2p+k} \tag{1.1}$$

or, equivalently, by

$$J_k(x) = \frac{(x/2)^k}{\Gamma(k+1)}\ {}_0F_1\left(-;k+1;-\frac{x^2}{4}\right) \tag{1.2}$$

where ${}_0F_1$ denotes the familiar hypergeometric function which corresponds to the special case $r+1=s=1$ of the generalized hypergeometric function ${}_rF_s$ with r numerator and s denominator parameters and $J_k(x)$ is one solution of Bessel differential equation:

$$x^2y'' + xy' + \left(x^2-k^2\right)y = 0. \tag{1.3}$$

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If k is an integer, it is hold that

$$J_k(x) = (-1)^k J_{-k}(x). \tag{1.4}$$

If k is not an integer, $J_k(x)$ and $J_{-k}(x)$ are two linearly independent solutions of (1.3). It is well known that the set of functions $\{f_n(x)\} = \{\sqrt{x} J_k(\alpha_n x)\}$, $n = 1, 2, ...$ forms orthogonal set over the interval $(0, 1)$. Actually, it is hold that

$$(f_n, f_m) = \int_0^1 x J_k(\alpha_n x) J_k(\alpha_m x) dx = 0 \; ; \quad n \neq m \tag{1.5}$$

where $\alpha_1, ..., \alpha_n$ are positive and real roots of $J_k(\alpha) = 0$. On the other hand, the norms of functions $f_n(x)$ are

$$\|f_n\| = \frac{1}{\sqrt{2}} J_{k+1}(\alpha_n) \;, \; n = 1, 2, ... \tag{1.6}$$

where $J_k(\alpha_n) = 0$. For orthogonality properties and zeros of the Bessel functions, we refer [6, 16].

Furthermore, the Bessel functions $J_k(x)$ satisfy the following recurrence formulas for p real parameter [12, 14]:

$$\frac{d}{dx}\left(x^k J_k(px)\right) = p x^k J_{k-1}(px) \tag{1.7}$$

and

$$\frac{d}{dx}\left(x^{-k} J_k(px)\right) = -p x^{-k} J_{k+1}(px) \tag{1.8}$$

and these functions are generated by

$$\sum_{k=-\infty}^{\infty} J_k(x) t^k = \exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right) \tag{1.9}$$

for $t \neq 0$, $t \in \mathbb{C}$ and for all finite x (see [12, 15]).

In the area of orthogonal polynomials, analogues in severable variables of some orthogonal polynomials seem to be highly nontrivial generalizations of the one-variable case. In fact, two-variable analogues of the Jacobi polynomials in several different ways were introduced in [10] (see also [13]). By the motivation of such analogues, we define a two-variable analogue of the Bessel functions.

We organize the paper as follows:

In section 2, we define a two-variable analogue $J_{k,s}(x, y)$ of first kind of the Bessel functions $J_k(x)$ and we show that they are orthogonal. We give a generating function and differential equation satisfied by $J_{k,s}(x, y)$. In section 3, for these functions, various recurrence formulas are derived. Furthermore, some special cases of the results presented in previous sections are also indicated.

2. A two-variable analogue of the Bessel functions and their properties

We define a two-variable analogue of the first kind of the Bessel functions as follows:

$$J_{k,s}(x, y) = J_k(x) J_s(y\rho(x))$$

where $\rho(x) > 0$, $x > 0$. The following results can be easily proved by using (1.2) and (1.4).

Theorem 2.1. *For the functions* $J_{k,s}(x,y)$*, we have the representations*

$$J_{k,s}(x,y)=\frac{(x/2)^k\left(\frac{y\rho(x)}{2}\right)^s}{\Gamma(k+1)\Gamma(s+1)}\ {}_0F_1\left(-;k+1;-\frac{x^2}{4}\right)\ {}_0F_1\left(-;s+1;-\frac{y^2\rho^2(x)}{4}\right)$$

or

$$J_{k,s}(x,y)=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}}{m!\ n!\Gamma(k+m+1)\Gamma(s+n+1)}\left(\frac{x}{2}\right)^{2m+k}\left(\frac{y\rho(x)}{2}\right)^{2n+s}. \tag{2.1}$$

Theorem 2.2. *Let* k *and* s *be integer, then* $J_{k,s}(x,y)$ *satisfies*

$$\begin{aligned}J_{-k,s}(x,y)&=(-1)^kJ_{k,s}(x,y),\\ J_{k,-s}(x,y)&=(-1)^sJ_{k,s}(x,y),\\ J_{-k,-s}(x,y)&=(-1)^kJ_{k,-s}(x,y)=(-1)^sJ_{-k,s}(x,y)=(-1)^{k+s}J_{k,s}(x,y).\end{aligned}$$

Theorem 2.3. *Let* $\alpha_1,...,\alpha_n$ *and* $\beta_1,...,\beta_m$ *be positive and real roots of* $J_k(\alpha)=0$ *and* $J_s(\beta)=0$*, respectively. The set of functions*

$$f_{n,m}(x,y)=\sqrt{xy}\rho(x)J_k(\alpha_nx)J_s(y\rho(x)\beta_m)\quad;\quad m,n=1,2,...$$

is orthogonal over the domain

$$D=\left\{(x,y):0<x<1\ ,\ 0<y<\frac{1}{\rho(x)}\right\}.$$

Proof. By the change of variable $u=y\rho(x)$ and then by (1.5), we get

$$\begin{aligned}&(f_{n,m}(x,y),f_{r,p}(x,y)) &(2.2)\\ =&\iint_D f_{n,m}(x,y)f_{r,p}(x,y)dydx\\ =&\iint_D \sqrt{xy}\rho(x)J_k(\alpha_nx)J_s(y\rho(x)\beta_m)\sqrt{xy}\rho(x)J_k(\alpha_rx)J_s(y\rho(x)\beta_p)dydx\\ =&\int_0^1\int_0^1 xuJ_k(\alpha_nx)J_s(\beta_mu)J_k(\alpha_rx)J_s(\beta_pu)dudx\\ =&\left(\int_0^1 xJ_k(\alpha_nx)J_k(\alpha_rx)dx\right)\left(\int_0^1 uJ_s(\beta_mu)J_s(\beta_pu)du\right)\\ =&\ 0\ ,\quad (n,m)\neq(r,p)\end{aligned}$$

which completes the proof. □

Theorem 2.4. *Let* $J_k(\alpha_n)=0$ *and* $J_s(\beta_m)=0$*. The norms of the functions* $f_{n,m}(x,y)$ *are determined by*

$$\begin{aligned}\|f_{n,m}(x,y)\|&=\frac{1}{\sqrt2}J_{k+1}(\alpha_n)\frac{1}{\sqrt2}J_{s+1}(\beta_m)\\ &=\frac12J_{k+1}(\alpha_n)J_{s+1}(\beta_m).\end{aligned}$$

Proof. It is enough to use (1.6). □

Theorem 2.5. *Two-variable analogue $J_{k,s}(x,y)$ of the first kind of the Bessel functions is one solution of differential equation of second order*

$$\begin{aligned} & x^2W_{xx}+\left(y^2+\frac{x^2y^2\left(\rho'(x)\right)^2}{\rho^2(x)}\right)W_{yy}-\frac{2yx^2\rho'(x)}{\rho(x)}W_{xy}+xW_x \\ & +\left\{-\frac{yx^2\rho''(x)}{\rho(x)}+\frac{2yx^2}{\rho^2(x)}\left(\rho'(x)\right)^2-\frac{xy\rho'(x)}{\rho(x)}+y\right\}W_y \\ & +(x^2+y^2\rho^2(x)-k^2-s^2)W=0. \end{aligned}$$

Proof. The function $W_{k,s}(u,v)=J_k(u)J_s(v)$ satisfies the differential equation of second order

$$u^2W_{uu}+v^2W_{vv}+uW_u+vW_v+(u^2+v^2-k^2-s^2)W=0.$$

By applying the change of variable

$$\left\{\begin{array}{c} u=x \\ v=y\rho(x) \end{array}\right. ,$$

we find differential equation of second order satisfied by $J_{k,s}(x,y)=J_k(x)J_s(y\rho(x))$:

$$\begin{aligned} & x^2W_{xx}+\left(y^2+\frac{x^2y^2\left(\rho'(x)\right)^2}{\rho^2(x)}\right)W_{yy}-\frac{2yx^2\rho'(x)}{\rho(x)}W_{xy}+xW_x \\ & +\left\{-\frac{yx^2\rho''(x)}{\rho(x)}+\frac{2yx^2}{\rho^2(x)}\left(\rho'(x)\right)^2-\frac{xy\rho'(x)}{\rho(x)}+y\right\}W_y \\ & +(x^2+y^2\rho^2(x)-k^2-s^2)W=0. \end{aligned}$$

□

Theorem 2.6. *The functions $J_{k,s}(x,y)$ are generated by*

$$\sum_{k,s=-\infty}^{\infty}J_{k,s}(x,y)t^kw^s=\exp\left[\frac{1}{2}\left(x\left(t-\frac{1}{t}\right)+y\rho(x)\left(w-\frac{1}{w}\right)\right)\right]$$

for $t\neq 0$, $w\neq 0$ and $t,w\in\mathbb{C}$.

Proof. It is enough to use (1.9). □

3. Recurrence Relations for the functions $J_{k,s}(x,y)$

In this section, we obtain various recurrence relations for the functions $J_{k,s}(x,y)$. For p and r real parameters, let $J_{k,s}(x,y;p,r)$ be shown by

$$J_{k,s}(x,y;p,r)=J_k(px)J_s(ry\rho(x)). \tag{3.1}$$

Theorem 3.1. *For the functions $J_{k,s}(x,y;p,r)$, we have*

$$\begin{aligned} & \frac{\partial}{\partial x}\left[x^k(\rho(x))^sJ_{k,s}(x,y;p,r)\right] \\ = \; & x^k(\rho(x))^s\left[pJ_{k-1,s}(x,y;p,r)+ry\rho'(x)J_{k,s-1}(x,y;p,r)\right]. \end{aligned}$$

Proof. By multiplying both members of (3.1) by $x^k(\rho(x))^s$ and differentiating with respect to x, and then using (1.7), we get

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[x^k(\rho(x))^s J_{k,s}(x,y;p,r)\right] \\
=\;& \frac{\partial}{\partial x}\left[x^k(\rho(x))^s J_k(px)J_s(ry\rho(x))\right] \\
=\;& (\rho(x))^s J_s(ry\rho(x))px^k J_{k-1}(px) + x^k J_k(px)\frac{\partial}{\partial x}\left[(\rho(x))^s J_s(ry\rho(x))\right]. \quad (3.2)
\end{aligned}$$

Furthermore, we can write

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[(\rho(x))^s J_s(ry\rho(x))\right] \\
=\;& \frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+s+1)}\left(\frac{ry\rho(x)}{2}\right)^{2n+s}(\rho(x))^s\right] \\
=\;& \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+s)}(\rho(x))^{2n+2s-1}\left(\frac{ry}{2}\right)^{2n+s-1} yr\rho'(x) \\
=\;& ry(\rho(x))^s\rho'(x)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+s)}\left(\frac{ry\rho(x)}{2}\right)^{2n+s-1} \\
=\;& ry(\rho(x))^s\rho'(x)J_{s-1}(ry\rho(x)). \quad (3.3)
\end{aligned}$$

Substituting (3.3) in (3.2), we complete the proof. □

Theorem 3.2. *The functions $J_{k,s}(x,y;p,r)$ satisfy the following differential formula:*

$$\frac{\partial}{\partial y}\left[y^s J_{k,s}(x,y;p,r)\right] = r\rho(x)y^s J_{k,s-1}(x,y;p,r).$$

Proof. By multiplying both members of (3.1) by y^s and then differentiating with respect to y, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial y}\left[y^s J_{k,s}(x,y;p,r)\right] = \frac{\partial}{\partial y}\left[y^s J_k(px)J_s(ry\rho(x))\right] \\
=\;& J_k(px)\frac{\partial}{\partial y}\left[y^s J_s(ry\rho(x))\right] \\
=\;& J_k(px)\frac{\partial}{\partial y}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+s+1)}\left(\frac{ry\rho(x)}{2}\right)^{2n+s} y^s\right] \\
=\;& J_k(px)\sum_{n=0}^{\infty}\frac{(-1)^n(2n+2s)}{n!\,(n+s)\Gamma(n+s)}\left(\frac{r\rho(x)}{2}\right)^{2n+s} y^{2n+2s-1} \\
=\;& r\rho(x)y^s J_k(px)J_{s-1}(ry\rho(x)) \\
=\;& r\rho(x)y^s J_{k,s-1}(x,y;p,r)
\end{aligned}$$

which completes the proof. □

As a result of Theorem 8, by induction, we obtain the following corollary.

Corollary 3.3. *The functions $J_{k,s}(x,y;p,r)$ satisfy the following equality*

$$\left(\frac{1}{y}\frac{\partial}{\partial y}\right)^m\left[y^s J_{k,s}(x,y;p,r)\right] = (r\rho(x))^m y^{s-m} J_{k,s-m}(x,y;p,r)$$

for $m = 1, 2, ...$

Theorem 3.4. *For* $J_{k,s}(x, y; p, r)$*, we have*

$$\begin{aligned} &\frac{\partial}{\partial x}\left[J_{k,s}(x, y; p, r)x^{-k}(\rho(x))^{-s}\right] \\ = \; & -x^{-k}(\rho(x))^{-s}\left\{pJ_{k+1,s}(x, y; p, r) + yr\rho'(x)J_{k,s+1}(x, y; p, r)\right\}. \end{aligned}$$

Proof. From (1.8), we get

$$\begin{aligned} &\frac{\partial}{\partial x}\left[J_{k,s}(x, y; p, r)x^{-k}(\rho(x))^{-s}\right] \\ = \; & \frac{\partial}{\partial x}\left[J_k(px)J_s(ry\rho(x))x^{-k}(\rho(x))^{-s}\right] \\ = \; & -px^{-k}(\rho(x))^{-s}J_{k+1}(px)J_s(ry\rho(x)) + J_k(px)x^{-k}\frac{\partial}{\partial x}\left[J_s(ry\rho(x))(\rho(x))^{-s}\right]. \end{aligned} \tag{3.4}$$

Moreover, by (1.1), we can write

$$\begin{aligned} &\frac{\partial}{\partial x}\left[J_s(ry\rho(x))(\rho(x))^{-s}\right] \\ = \; & \frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\;\Gamma(n+s+1)}\left(\frac{ry\rho(x)}{2}\right)^{2n+s}(\rho(x))^{-s}\right] \\ = \; & -\sum_{n=0}^{\infty}\frac{(-1)^n yr}{n!\;\Gamma(n+s+2)}\left(\frac{yr}{2}\right)^{2n+s+1}(\rho(x))^{2n+s+1}(\rho(x))^{-s}\rho'(x) \\ = \; & -ry(\rho(x))^{-s}\rho'(x)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\;\Gamma(n+s+2)}\left(\frac{ry\rho(x)}{2}\right)^{2n+s+1} \\ = \; & -ry(\rho(x))^{-s}\rho'(x)J_{s+1}(ry\rho(x)). \end{aligned} \tag{3.5}$$

If we replace (3.5) in (3.4), we complete the proof. □

Similar to Theorem 8, we can give the following relation.

Theorem 3.5. *For* $J_{k,s}(x, y; p, r)$*, we have the differential formula*

$$\frac{\partial}{\partial y}\left[J_{k,s}(x, y; p, r)y^{-s}\right] = -r\rho(x)y^{-s}J_{k,s+1}(x, y; p, r).$$

By induction, we have the next result:

Corollary 3.6. *The functions* $J_{k,s}(x, y; p, r)$ *satisfy the relation*

$$\left(\frac{1}{y}\frac{\partial}{\partial y}\right)^m\left[J_{k,s}(x, y; p, r)y^{-s}\right] = (-1)^m\left(r\rho(x)\right)^m y^{-s-m}J_{k,s+m}(x, y; p, r)$$

for $m = 1, 2, ...$

Theorem 3.7. *The functions* $J_{k,s}(x, y; p, r)$ *hold that*

$$\begin{aligned} \frac{\partial}{\partial x}J_{k,s}(x, y; p, r) \; = \; & pJ_{k-1,s}(x, y; p, r) + yr\rho'(x)J_{k,s-1}(x, y; p, r) \\ & -\left(\frac{k}{x} + \frac{s\rho'(x)}{\rho(x)}\right)J_{k,s}(x, y; p, r). \end{aligned}$$

Proof. By multiplying (3.1) by $x^k(\rho(x))^s$ and then differentiating with respect to x, we get

$$\begin{aligned} &\frac{\partial}{\partial x}\left[J_{k,s}(x,y;p,r)x^k(\rho(x))^s\right] \\ = \;& J_{k,s}(x,y;p,r)\left\{kx^{k-1}(\rho(x))^s + x^k s(\rho(x))^{s-1}\rho'(x)\right\} \\ &+x^k(\rho(x))^s\frac{\partial}{\partial x}J_{k,s}(x,y;p,r) \end{aligned} \tag{3.6}$$

and by Theorem 7, we can write

$$\begin{aligned} &\frac{\partial}{\partial x}\left[J_{k,s}(x,y;p,r)x^k(\rho(x))^s\right] \\ = \;& x^k(\rho(x))^s\left\{pJ_{k-1,s}(x,y;p,r) + yr\rho'(x)J_{k,s-1}(x,y;p,r)\right\}. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), it is obvious that

$$\begin{aligned} &\left(\frac{k}{x}+\frac{s\rho'(x)}{\rho(x)}\right)J_{k,s}(x,y;p,r)+\frac{\partial}{\partial x}J_{k,s}(x,y;p,r) \\ = \;& pJ_{k-1,s}(x,y;p,r)+yr\rho'(x)J_{k,s-1}(x,y;p,r). \end{aligned}$$

□

In a similar manner, using Theorem 10, we can give as follows:

Theorem 3.8. *For $J_{k,s}(x,y;p,r)$, it is hold that*

$$\begin{aligned} \frac{\partial}{\partial x}J_{k,s}(x,y;p,r) &= \left(\frac{k}{x}+\frac{s\rho'(x)}{\rho(x)}\right)J_{k,s}(x,y;p,r) \\ &\quad -\left\{pJ_{k+1,s}(x,y;p,r)+yr\rho'(x)J_{k,s+1}(x,y;p,r)\right\}. \end{aligned}$$

By means of Theorem 13 and 14, one can easily show as follows:

Theorem 3.9. *$J_{k,s}(x,y;p,r)$ satisfies the following relations:*

$$\begin{aligned} \frac{\partial}{\partial x}J_{k,s}(x,y;p,r) &= \frac{p}{2}\left\{J_{k-1,s}(x,y;p,r)-J_{k+1,s}(x,y;p,r)\right\} \\ &\quad +\frac{yr\rho'(x)}{2}\left\{J_{k,s-1}(x,y;p,r)-J_{k,s+1}(x,y;p,r)\right\} \end{aligned}$$

and

$$\begin{aligned} 2\left(\frac{k}{x}+\frac{s\rho'(x)}{\rho(x)}\right)J_{k,s}(x,y;p,r) &= p\left\{J_{k-1,s}(x,y;p,r)+J_{k+1,s}(x,y;p,r)\right\} \\ &\quad +yr\rho'(x)\left\{J_{k,s-1}(x,y;p,r)+J_{k,s+1}(x,y;p,r)\right\}. \end{aligned}$$

4. Some Special cases

In this section, we give some special cases of the functions $J_{k,s}(x,y)$. For the Bessel functions $J_k(x)$, it is well-known that

$$\begin{aligned} J_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\cos x, \\ J_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\sin x \end{aligned}$$

and

$$J^2_{-\frac{1}{2}}(x) + J^2_{\frac{1}{2}}(x) = \frac{2}{\pi x}.$$

By means of these equalities, we can give the following equalities.

Theorem 4.1. *For $k = \pm\frac{1}{2}, s = \pm\frac{1}{2}$ in $J_{k,s}(x,y)$, it follows*

$$\begin{aligned}
J_{\frac{1}{2},\frac{1}{2}}(x,y) &= J_{\frac{1}{2}}(x)J_{\frac{1}{2}}(y\rho(x)) = \left(\sqrt{\frac{2}{\pi x}}\sin x\right)\left(\sqrt{\frac{2}{\pi y\rho(x)}}\sin(y\rho(x))\right) \\
&= \frac{2}{\pi\sqrt{xy\rho(x)}}\sin x \sin(y\rho(x)), \\
J_{\frac{1}{2},-\frac{1}{2}}(x,y) &= J_{\frac{1}{2}}(x)J_{-\frac{1}{2}}(y\rho(x)) = \left(\sqrt{\frac{2}{\pi x}}\sin x\right)\left(\sqrt{\frac{2}{\pi y\rho(x)}}\cos(y\rho(x))\right) \\
&= \frac{2}{\pi\sqrt{xy\rho(x)}}\sin x \cos(y\rho(x)), \\
J_{-\frac{1}{2},\frac{1}{2}}(x,y) &= J_{-\frac{1}{2}}(x)J_{\frac{1}{2}}(y\rho(x)) = \left(\sqrt{\frac{2}{\pi x}}\cos x\right)\left(\sqrt{\frac{2}{\pi y\rho(x)}}\sin(y\rho(x))\right) \\
&= \frac{2}{\pi\sqrt{xy\rho(x)}}\cos x \sin(y\rho(x))
\end{aligned}$$

and

$$\begin{aligned}
J_{-\frac{1}{2},-\frac{1}{2}}(x,y) &= J_{-\frac{1}{2}}(x)J_{-\frac{1}{2}}(y\rho(x)) = \left(\sqrt{\frac{2}{\pi x}}\cos x\right)\left(\sqrt{\frac{2}{\pi y\rho(x)}}\cos(y\rho(x))\right) \\
&= \frac{2}{\pi\sqrt{xy\rho(x)}}\cos x \cos(y\rho(x)).
\end{aligned}$$

Corollary 4.2. *As a result of Theorem 16, we have*

$$\left(J_{\frac{1}{2},\frac{1}{2}}(x,y)\right)^2 + \left(J_{\frac{1}{2},-\frac{1}{2}}(x,y)\right)^2 + \left(J_{-\frac{1}{2},\frac{1}{2}}(x,y)\right)^2 + \left(J_{-\frac{1}{2},-\frac{1}{2}}(x,y)\right)^2 = \frac{4}{\pi^2 xy\rho(x)}.$$

Corollary 4.3. *Choosing $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ in Theorem 3, the set of functions*

$$f_{n,m}(x,y) = \sqrt{\frac{xy}{1-x^2}}J_k(\alpha_n x)J_s\left(\frac{\beta_m y}{\sqrt{1-x^2}}\right) \quad ; \quad m,n = 1,2,... \tag{4.1}$$

is orthogonal over the domain

$$D = \left\{(x,y) : 0 < x < 1 \quad , \quad 0 < y < \sqrt{1-x^2}\right\}.$$

The norms of these functions are

$$\|f_{n,m}\| = \frac{1}{2}J_{k+1}(\alpha_n)J_{s+1}(\beta_m) \quad ; \quad m,n = 1,2,...$$

Due to results obtained in section 3, the recurrence relations satisfied by (4.1) are given below:

i) $\frac{\partial}{\partial x}\left[x^k(1-x^2)^{-\frac{s}{2}}J_{k,s}(x,y;p,r)\right]$

$$= x^k(1-x^2)^{-\frac{s}{2}}\left[pJ_{k-1,s}(x,y;p,r)+rxy(1-x^2)^{-\frac{3}{2}}J_{k,s-1}(x,y;p,r)\right],$$

ii) $\frac{\partial}{\partial y}\left[y^sJ_{k,s}(x,y;p,r)\right]=\frac{ry^s}{\sqrt{1-x^2}}J_{k,s-1}(x,y;p,r),$

iii) $\left(\frac{1}{y}\frac{\partial}{\partial y}\right)^m\left[y^sJ_{k,s}(x,y;p,r)\right]$

$$= \frac{r^m}{(1-x^2)^{m/2}}y^{s-m}J_{k,s-m}(x,y;p,r)\ ,\ m=1,2,...,$$

iv) $\frac{\partial}{\partial x}\left[J_{k,s}(x,y;p,r)x^{-k}(1-x^2)^{\frac{s}{2}}\right]$

$$= -x^{-k}(1-x^2)^{\frac{s}{2}}\left\{pJ_{k+1,s}(x,y;p,r)+rxy\left(1-x^2\right)^{-\frac{3}{2}}J_{k,s+1}(x,y;p,r)\right\},$$

v) $\frac{\partial}{\partial y}\left[J_{k,s}(x,y;p,r)y^{-s}\right]=\frac{-ry^{-s}}{\sqrt{1-x^2}}J_{k,s+1}(x,y;p,r),$

vi) $\left(\frac{1}{y}\frac{\partial}{\partial y}\right)^m\left[y^{-s}J_{k,s}(x,y;p,r)\right]$

$$= \frac{(-1)^m r^m}{(1-x^2)^{m/2}}y^{-s-m}J_{k,s+m}(x,y;p,r)\ ,\ m=1,2,...,$$

vii) $\frac{\partial}{\partial x}J_{k,s}(x,y;p,r)+\left(\frac{k}{x}+\frac{sx}{1-x^2}\right)J_{k,s}(x,y;p,r)$

$$= \left\{pJ_{k-1,s}(x,y;p,r)+rxy\left(1-x^2\right)^{-\frac{3}{2}}J_{k,s-1}(x,y;p,r)\right\},$$

viii) $\frac{\partial}{\partial x}J_{k,s}(x,y;p,r)-\left(\frac{k}{x}+\frac{sx}{1-x^2}\right)J_{k,s}(x,y;p,r)$

$$= -pJ_{k+1,s}(x,y;p,r)-rxy\left(1-x^2\right)^{-\frac{3}{2}}J_{k,s+1}(x,y;p,r),$$

ix) $\frac{\partial}{\partial x}J_{k,s}(x,y;p,r)=\frac{p}{2}\left\{J_{k-1,s}(x,y;p,r)-J_{k+1,s}(x,y;p,r)\right\}$
$+\frac{1}{2}rxy(1-x^2)^{-\frac{3}{2}}\left\{J_{k,s-1}(x,y;p,r)-J_{k,s+1}(x,y;p,r)\right\},$

x) $2\left(\frac{k}{x}+\frac{sx}{1-x^2}\right)J_{k,s}(x,y;p,r)=p\left\{J_{k-1,s}(x,y;p,r)+J_{k+1,s}(x,y;p,r)\right\}$

$+rxy(1-x^2)^{-\frac{3}{2}}\left\{J_{k,s-1}(x,y;p,r)+J_{k,s+1}(x,y;p,r)\right\}.$

Corollary 4.4. *Choosing $\rho(x)=1$ in Theorem 3, the set of functions*

$$f_{n,m}(x,y)=\sqrt{xy}J_k(\alpha_n x)J_s\left(\beta_m y\right)\quad;\quad m,n=1,2,... \tag{4.2}$$

is orthogonal over the domain

$$D=\{(x,y):0<x<1\quad,\quad 0<y<1\}$$

and the norms of these functions are determined by

$$\|f_{n,m}\|=\frac{1}{2}J_{k+1}(\alpha_n)J_{s+1}(\beta_m)\quad;\quad m,n=1,2,...$$

Due to results obtained in section 3, the recurrence relations satisfied by (4.2) are given below:

i) $\frac{\partial}{\partial x}\left[x^kJ_{k,s}(x,y;p,r)\right]=px^kJ_{k-1,s}(x,y;p,r),$

ii) $\frac{\partial}{\partial y}\left[y^sJ_{k,s}(x,y;p,r)\right]=ry^sJ_{k,s-1}(x,y;p,r),$

iii) $\left(\frac{1}{y}\frac{\partial}{\partial y}\right)^m\left[y^sJ_{k,s}(x,y;p,r)\right]=r^my^{s-m}J_{k,s-m}(x,y;p,r)\ ,\ m=1,2,...$

iv) $\frac{\partial}{\partial x}\left[x^{-k}J_{k,s}(x,y;p,r)\right]=-px^{-k}J_{k+1,s}(x,y;p,r),$

v) $\frac{\partial}{\partial y}\left[y^{-s}J_{k,s}(x,y;p,r)\right]=-ry^{-s}J_{k,s+1}(x,y;p,r),$

vi) $\left(\frac{1}{y}\frac{\partial}{\partial y}\right)^m\left[y^{-s}J_{k,s}(x,y;p,r)\right]=(-1)^m r^my^{-s-m}J_{k,s+m}(x,y;p,r)\ ,\ m=1,2,...$

vii) $\frac{\partial}{\partial x}J_{k,s}(x,y;p,r)=pJ_{k-1,s}(x,y;p,r)-\frac{k}{x}J_{k,s}(x,y;p,r),$

viii) $\frac{\partial}{\partial x}J_{k,s}(x,y;p,r)=\frac{k}{x}J_{k,s}(x,y;p,r)-pJ_{k+1,s}(x,y;p,r),$

ix) $\frac{\partial}{\partial x}J_{k,s}(x,y;p,r)=\frac{p}{2}\left\{J_{k-1,s}(x,y;p,r)-J_{k+1,s}(x,y;p,r)\right\},$

x) $\frac{2k}{x}J_{k,s}(x,y;p,r)=p\left\{J_{k-1,s}(x,y;p,r)+J_{k+1,s}(x,y;p,r)\right\}.$

Corollary 4.5. *We have the following relation between the functions $J_{k,s}(x,y)$ and the first kind of the Bessel functions $J_k(x)$:*

$$\lim_{y\to 0}J_{k,0}(x,y)=J_k(x).$$

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