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SOME FAMILIES OF BILATERAL GENERATING RELATIONS AND OPERATIONAL METHODS

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Abstract. In this paper, the authors exploit the concept and the formalism associated with the principle of monomiality and operational techniques to derive several families of bilateral generating relations involving the product of associated Laguerre with generalized forms of Laguerre and Legendre polynomials. Certain special cases giving bilateral generating relations related to these polynomials are also discussed.

1. Introduction

The associated Laguerre polynomials $L_n^{(m)}(x)$ [1] are defined as

$$L_n^{(m)}(x) \ = \ (n+m)!\sum_{r=0}^{n}\frac{(-1)^r x^r}{(n-r)!(m+r)!r!}, \tag{1.1}$$

satisfying the properties [1]:

$$L_n^{(m)}(x) = \left(1-\frac{d}{dx}\right)L_n(x),$$
$$L_n^{(0)}(x) = L_n(x), \tag{1.2}$$

it is therefore easy to derive the generating function [3]:

$$\exp(t)L_n(x-t) \ = \ \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x). \tag{1.3}$$

Recently, an increasing interest has grown around operational techniques and special functions. The use of operational techniques combined with the principle of monomiality is a fairly useful tool for treating various families of special polynomials as well as their new and known generalizations. The idea of monomiality came from the concept of poweroid suggested by Steffensen [11]. The monomiality principle is reformulated and developed by Dattoli [3].

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According to the principle of monomiality, a family of polynomials $p_n(x)$ ($n \in \mathbb{N}$, $x \in \mathbb{C}$) is said to be "quasi-monomial", if two operators $\hat{M}$ and $\hat{P}$, called "multiplicative" and "derivative" operators repectively can be defined in such a way that

$$\hat{M}p_n(x) = p_{n+1}(x), \hat{P}p_n(x) = np_{n-1}(x). \tag{1.4}$$

The operators $\hat{M}$ and $\hat{P}$ can be recognized as raising and lowering operators acting on the polynomials $p_n(x)$. These operators satisfy the following commutation relation

$$[\hat{P}, \hat{M}] = 1. \tag{1.5}$$

and thus display a Weyl group structure.

Further, consequence of equation (1.4) is the eigen-property of $\hat{M}\hat{P}$

$$\hat{M}\hat{P}p_n(x) = np_n(x). \tag{1.6}$$

The properties of the polynomials $p_n(x)$ can be deduced from those of the $\hat{M}$ and $\hat{P}$ operators. If the operators $\hat{M}$ and $\hat{P}$ possess a differential realization then the polynomials $p_n(x)$ satisfy the differential equation (1.6). The polynomial family $p_n(x)$ can be explicitly constructed through the action of $\hat{M}^n$ on $p_0(x)$

$$p_n(x) = \hat{M}^n\{p_0(x)\}, \tag{1.7}$$

we shall always set $p_0(x) = 1$.

In view of the above identity the generating function of $p_n(x)$ can be cast in the form

$$\mathcal{G}(x,t) = \exp(t\hat{M})(1) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}. \tag{1.8}$$

It is evident from equations (1.3) and (1.7) that

$$\exp(t\hat{M})L_n(x - t\hat{M}) \ = \ \sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(m)}(x)p_m(x). \tag{1.9}$$

For our purpose, we recall some generalized forms of Laguerre and Legendre polynomials.

Recently, Dattoli and Torre [10] introduced and discussed a theory of 2-variable Laguerre polynomials (2VLP) $L_n(x,y)$. The reason of interest for this family of Laguerre polynomials is due to their mathematical importance and due to the fact that these polynomials are shown to be natural solutions of a particular set of partial differential equations which often appear in the treatment of radiation physics problems such as the electromagnetic wave propagation and beam life- time in storage rings [12].

The 2VLP $L_n(x,y)$ are quasi-monomials under the action of the operators [3]:

$$\hat{M} \ = \ y - \hat{D}_x^{-1}, \tag{1.10}$$

$$\hat{P} \ = \ -\hat{D}_x x \hat{D}_x \ = \ -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}, \tag{1.11}$$

where $\hat{D}_x$ denotes the derivative operator and $\hat{D}_x^{-1}$ its inverse and is defined in such a way that

$$\hat{D}_x^{-n}\{f(x)\} \ = \ \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1} f(\xi) d\xi, \tag{1.12}$$

so that for $f(x) = 1$, we have

$$\hat{D_x}^{-n}(1) = \frac{x^n}{n!}. \tag{1.13}$$

Further, since $L_0(x, y) = 1$, equation (1.7) yields [3]:

$$L_n(x, y) = \left(y - \hat{D_x}^{-1}\right)^n = n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)!(r!)^2}. \tag{1.14}$$

The 2VLP $L_n(x, y)$ are specified by the generating function [10]:

$$\frac{1}{(1-yt)} \exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n(x, y) t^n \ (|yt| < 1), \tag{1.15}$$

or, equivalently [3]:

$$\exp(yt) C_o(xt) = \sum_{n=0}^{\infty} L_n(x, y) t^n, \tag{1.16}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as [1]:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!(n+r)!}, \tag{1.17}$$

with the following generating function

$$\exp\left(t - \frac{x}{t}\right) = \sum_{-\infty}^{\infty} C_n(x) t^n, \tag{1.18}$$

for $t \neq 0$ and for all finite x.

It is also worth noting that the 2VLP $L_n(x, y)$ are the natural solutions of the equation

$$\frac{\partial}{\partial y} L_n(x, y) = -\Big(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\Big) L_n(x, y), \tag{1.19}$$

which is kind of heat diffusion equation.

Further, the 2-variable associated Laguerre polynomials (2VALP) $l_n^{(\nu)}(x, y)$ are quasi-monomials under the action of the operators [6]:

$$\hat{M} = y - \hat{D_{x,\nu}}^{-1}, \tag{1.20}$$

$$\hat{P} = -\Gamma(\nu + 1)\Big[\frac{\partial}{\partial x} x \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial x}\Big], \tag{1.21}$$

where the operator $\hat{D_{x,\nu}}^{-1}$ is defined in such a way that its action on the unity is specified by

$$\hat{D_{x,\nu}}^{-n}(1) = \frac{x^n}{\Gamma(n + \nu + 1)}. \tag{1.22}$$

According to equation (1.7), the 2VALP $l_n^{(\nu)}(x, y)$ are define by [6]:

$$l_n^{(\nu)}(x, y) = \left(y - \hat{D_{x,\nu}}^{-1}\right)^n = n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)! r! \Gamma(r + \nu + 1)} \tag{1.23}$$

and are specified by the generating function [6]:

$$\frac{1}{(1-yt)}E_{\nu}\Big(\frac{xt}{1-yt}\Big) \;=\; \sum_{n=0}^{\infty} l_n^{(\nu)}(x,y)t^n,\ (|yt|<1), \tag{1.24}$$

where

$$E_{\nu}(x) \;=\; \sum_{r=0}^{\infty}\frac{(-x)^r}{(\nu+r)!}.$$

The equivalent form of the above generating functions are given as [6]:

$$\exp(yt)C_{\nu}(xt) \;=\; \sum_{n=0}^{\infty} l_n^{(\nu)}(x,y)\frac{t^n}{n!}. \tag{1.25}$$

It is also worth stressing that the 2VALP $l_n^{(\nu)}(x,y)$ are the natural solutions of the equation

$$\frac{\partial}{\partial y}l_n^{(\nu)}(x,y) \;=\; \Big(-\frac{\partial}{\partial x}x\frac{\partial}{\partial x}-\nu\frac{\partial}{\partial x}\Big)l_n^{(\nu)}(x,y). \tag{1.26}$$

Furthermore, the pseudo Laguerre polynomials (PsLP) $L_n(x,y;r)$ are defined by [4]:

$$L_n(x,y;r) \;=\; \Big(y-D_x{}^{-r}\Big)^n \;=\; n!\sum_{k=0}^{n}\frac{(-1)^k y^{n-k}x^{rk}}{(n-k)!k!(rk)!}, \tag{1.27}$$

it is evident that for $y=r=1$, the PsLP $L_n(x,y;r)$ reduces to the ordinary Laguerre polynomials.

Also from equation (1.27), we conclude that the multiplicative operator for the PsLP $L_n(x,y;r)$ is given as

$$\hat{M} \;=\; y-D_x{}^{-r}. \tag{1.28}$$

The generating functions for these polynomials are given as

$$\frac{1}{(1-yt)}S_0\Big(\Big(\frac{t}{1-yt}\Big)^{\frac{1}{r}}x;r\Big) \;=\; \sum_{n=0}^{\infty}L_n(x,y;r)t^n,\ (|yt|<1), \tag{1.29}$$

where $S_0(x;r)$ denotes the 0^{th} order pseudo trigonometric function. The j^{th} order pseudo trigonometric functions $S_j(x;r)$ are defined as [4]:

$$S_j(x;r) \;=\; \sum_{k=0}^{\infty}\frac{(-1)^k(x)^{kr+j}}{(kr+j)!}. \tag{1.30}$$

The equivalent form of the above generating functions are given as [4]:

$$\exp(yt)C_0(x^r t|r) \;=\; \sum_{n=0}^{\infty}L_n(x,y;r)\frac{t^n}{n!}, \tag{1.31}$$

where $C_0(x|r)$ denotes the 0^{th} order Wright function. The n^{th} order Wright function is defined by [9]:

$$C_n(x|l) \;=\; \sum_{r=0}^{\infty}\frac{(-x)^r}{r!(n+rl)!}. \tag{1.32}$$

The generalized Laguerre-Konhouser polynomials (GLKnP)$_kL_n^{(\alpha,\beta)}(x,y)$ are defined by [2]:

$$ {}_kL_n^{(\alpha,\beta)}(x,y) \;=\; (1-\hat{D_x}^{-1}-\hat{D_y}^{-1})^n \;=\; n!\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-1)^{s+r}x^{r+\alpha}y^{ks+\beta}}{(n-s-r)!\Gamma(\alpha+r+1)s!r!\Gamma(ks+\beta+1)}, \tag{1.33} $$

which gives the multiplicative operator for the GLKP ${}_kL_n^{(\alpha,\beta)}(x,y)$ as

$$ \hat{M} \;=\; 1-\hat{D_x}^{-1}-\hat{D_y}^{-1}. \tag{1.34} $$

Next, the 2-variable Legendre polynomials (2VLeP) ${}_2L_n(x,y)$ are quasi-monomials under the action of the operators [5]:

$$ \hat{M} \;=\; y+2\hat{D_x}^{-1}\frac{\partial}{\partial y}, \tag{1.35} $$

$$ \hat{P} \;=\; \frac{\partial}{\partial y}, \tag{1.36} $$

and specified by the series

$$ {}_2L_n(x,y) \;=\; n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^{n-2r}\hat{D_x}^{-r}}{(n-2r)!r!}. \tag{1.37} $$

The generating function for 2VgLP ${}_2L_n(x,y)$ are given by [5]:

$$ \exp(yt)C_o(-xt^2) \;=\; \sum_{n=0}^{\infty}{}_2L_n(x,y)\frac{t^n}{n!} \tag{1.38} $$

and note that these polynomials are the natural solutions of the equation

$$ \frac{\partial^2}{\partial y^2}{}_2L_n(x,y) \;=\; \Big(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\Big){}_2L_n(x,y). \tag{1.39} $$

Also, the 2-variable generalized Legendre polynomials (2VGLeP) $R_n^{(\mu,\nu)}(x,y)$ are defined by [6]:

$$ R_n^{(\mu,\nu)}(x,y) \;=\; n!(\hat{D_{y,\nu}}^{-1}-\hat{D_{x,\mu}}^{-1})^n \;=\; (n!)^2\sum_{r=0}^{n}\frac{(-1)^{n-r}x^{n-r}y^r}{(n-r)!\Gamma(n-r+\mu+1)r!\Gamma(r+\nu+1)}, \tag{1.40} $$

which yields the multiplicative operator for the 2VGLeP $R_n^{(\mu,\nu)}(x,y)$ as

$$ \hat{M} \;=\; \hat{D_{y,\nu}}^{-1}-\hat{D_{x,\mu}}^{-1} \tag{1.41} $$

and specified by the generating functions

$$ C_\nu(-yt)C_\mu(xt) \;=\; \sum_{n=0}^{\infty}R_n^{(\mu,\nu)}(x,y)\frac{t^n}{(n!)^2}. \tag{1.42} $$

In the forthcoming section, we will see how the combined use of the generating function (1.3) with the monomiality principle allows the derivation of new families of bilateral generating relations involving product of associated Laguerre with Laguerre and Legendre polynomials of more general form.

2. Generating Relations for Laguerre Polynomials

We prove the following bilateral generating relations for generalized forms of Laguerre polynomials:

Theorem 2.1. *If there exist the following bilateral generating relation involving ALP* $L_n^{(m)}(x)$ *and PsLP* $L_m(y,z|l)$

$$F(x,y,z|l;t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(m)}(x) L_m(y,z|l), \tag{2.1}$$

then the following relation holds true:

$$\exp(tz)\,{}_{\psi}L_n(x,y,z;t) \;=\; \sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(m)}(x) L_m(y,z|l), \tag{2.2}$$

where ${}_{\psi}L_n(x,y,z;t) := n! \sum_{r=0}^{n} \frac{(-1)^r \psi_r(x,y,z;t)}{(r!)^2(n-r)!}$ *and* $\psi_r(x,y,z;t) := \sum_{s=0}^{r} \binom{r}{s} (x - tz)^{r-s} (y^l t)^s C_{ls}(y^l t|l)$.

Proof. For the polynomial $L_m(y,z|l)$ defined by (1.27), the multiplicative operator is given by (1.28)

In view of the relations (1.9) and (2.1), we get

$$\exp(t\hat{M}) L_n(x - t\hat{M}) \;=\; \sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(m)}(x) L_m(y,z|l), \tag{2.3}$$

which on substituting the multiplicative operator (1.28), yields

$$\exp(t(z - D_y^{-l})) L_n(x - t(z - D_y^{-l})) \;=\; \sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(m)}(x) L_m(y,z|l). \tag{2.4}$$

To explicitly evaluate the above expression we note the exponential operator

$$\exp(-tD_y^{-l}) = \sum_{n=0}^{\infty} \frac{(-t)^n D_y^{-nl}}{n!} = \sum_{n=0}^{\infty} \frac{(-t)^n y^{nl}}{(n)!(nl)!} = C_0(y^l t|l). \tag{2.5}$$

Further, applying (2.5) in the l.h.s. of equation (2.4), we find

$$\exp(tz) n! \sum_{r=0}^{n} \frac{(-1)^r}{(r!)^2(n-r)!} \sum_{s=0}^{r} \frac{r!(x - tz)^{r-s}}{(r-s)!s!} (tD_y^{-l})^s C_0(y^l t|l), \tag{2.6}$$

which on satisfying the property

$$D_x^{-s} C_n(x|l) \;=\; x^s C_{n+s}(x|l). \tag{2.7}$$

gives the generating relation (2.2). □

Corollary 2.2. *If there exist the following bilateral generating relation involving ALP* $L_n^{(m)}(x)$ *and 2VLP* $L_m(y,z)$

$$F(x,y,z;t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(m)}(x) L_m(y,z), \tag{2.8}$$

then the following relation holds true:

$$\exp(tz)\,{}_{\phi}L_n(x,y,z;t) \;=\; \sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(m)}(x) L_m(y,z), \tag{2.9}$$

where ${}_{\phi}L_n(x,y,z;t) := n!\sum_{r=0}^{n}\frac{(-1)^r\phi_r(x,y,z;t)}{(r!)^2(n-r)!}$ *and* $\phi_r(x,y,z;t) := \sum_{s=0}^{r}\binom{r}{s}(x-tz)^{r-s}(yt)^s C_s(yt)$.

Proof. Taking $l=1$ in equation (2.2), we get the generating relation (2.9).

Moreover, we have the following alternative proof of corollary (2.2):

For the polynomial $L_m(y,z)$ defined by (1.14), the multiplicative operator is given by (1.10).

In view of the relation (1.9) and (2.8), we get

$$\exp(t\hat{M})L_n(x-t\hat{M}) = \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)L_m(y,z), \tag{2.10}$$

which on substituting the multiplicative operator (1.10), yields

$$\exp(t(z-D_y^{-1}))L_n(x-t(z-D_y^{-1})) = \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)L_m(y,z). \tag{2.11}$$

To explicitly evaluate the above expression we note the exponential operator

$$\exp(-tD_y^{-1}) = \sum_{n=0}^{\infty}\frac{(-t)^n D_y^{-n}}{n!} = \sum_{n=0}^{\infty}\frac{(-t)^n y^n}{(n!)^2} = C_0(yt). \tag{2.12}$$

Further, applying (2.12) in the l.h.s. of equation (2.11), we find

$$\exp(tz)n!\sum_{r=0}^{n}\frac{(-1)^r}{(r!)^2(n-r)!}\sum_{s=0}^{r}\frac{r!(x-tz)^{r-s}}{(r-s)!s!}(tD_y^{-1})^s C_0(yt), \tag{2.13}$$

which on satisfying the property

$$D_x^{-s}C_n(x) = x^s C_{n+s}(x), \tag{2.14}$$

gives the generating relation (2.9). □

Remark. For $z=1$ equation (2.9) gives the bilateral generating relation [8]:

$$\exp(t)n!\sum_{r=0}^{n}\frac{(-1)^r\phi_r(x,y;t)}{(r!)^2(n-r)!} = \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)L_m(y), \tag{2.15}$$

where $\phi_r(x,y;t) := \sum_{s=0}^{r}\binom{r}{s}(x-t)^{r-s}(yt)^s C_s(yt)$.

Theorem 2.3. *If there exist the following bilateral generating relation involving ALP* $L_n^{(m)}(x)$ *and 2VALP* $l_m^{(\nu)}(y,z)$

$$F(x,y,z;t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)l_m^{(\nu)}(y,z), \tag{2.16}$$

then the following relation holds true:

$$\exp(tz)\,{}_{\chi}L_n(x,y,z;t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)l_m^{(\nu)}(y,z), \tag{2.17}$$

where ${}_{\chi}L_n(x,y,z;t) := n!\sum_{r=0}^{n}\frac{(-1)^r\chi_r(x,y,z;t)}{(r!)^2(n-r)!}$ *and* $\chi_r(x,y,z;t) := \sum_{s=0}^{r}\binom{r}{s}(x-tz)^{r-s}y^{s+\nu}t^s C_{s+\nu}(yt)$.

Proof. For the polynomial $l_m^{(\nu)}(y,z)$ defined by (1.23), the multiplicative operator is given by (1.20).

In view of the relation (1.9) and (2.16), we get

$$\exp(t\hat{M})L_n(x-t\hat{M}) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)l_m^{(\nu)}(y,z), \tag{2.18}$$

which on substituting the multiplicative operator (1.20), yields

$$\exp(t(z-D_{y,\nu}^{-1}))L_n(x-t(z-D_{y,\nu}^{-1})) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)l_m^{(\nu)}(y,z). \tag{2.19}$$

To explicitly evaluate the above expression we note the exponential operator

$$\exp(-tD_{y,\nu}^{-1}) = \sum_{n=0}^{\infty}\frac{(-t)^n D_{y,\nu}^{-n}}{n!} = \sum_{n=0}^{\infty}\frac{(-t)^n y^n}{(n)!(n+\nu)!} = C_\nu(yt). \tag{2.20}$$

Further, applying (2.20) in the l.h.s. of equation (2.19), we find

$$\exp(tz)n!\sum_{r=0}^{n}\frac{(-1)^r}{(r!)^2(n-r)!}\sum_{s=0}^{r}\frac{r!(x-tz)^{r-s}}{(r-s)!s!}(tD_{y,\nu}^{-1})^s C_\nu(yt), \tag{2.21}$$

which on satisfying the property

$$D_{y,\nu}^{-s}C_\nu(yt) \;=\; y^{s+\nu}C_{s+\nu}(yt). \tag{2.22}$$

gives the generating relation (2.17). □

Remark. For $\nu = 0$, equation (2.17) reduces to the bilateral generating relation (2.9).

Theorem 2.4. *If there exist the following bilateral generating relation involving ALP $L_n^{(m)}(x)$ and GLKnP ${}_kL_m^{(\alpha,\beta)}(y,z)$*

$$F(x,y,z;t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)_kL_m^{(\alpha,\beta)}(y,z), \tag{2.23}$$

then the following relation holds true:

$$\exp(t)_\xi L_n(x,y,z;t) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)_kL_m^{(\alpha,\beta)}(y,z), \tag{2.24}$$

where ${}_\xi L_n(x,y,z;t) := n!\sum_{r=0}^{n}\frac{(-1)^r\xi_r(x,y,z;t)}{(r!)^2(n-r)!}$,
$\xi_r(x,y,z;t) := \sum_{s=0}^{r}\binom{r}{s}(x-t)^{r-s}t^s\theta_s(y,z;t)$
and $\theta_s(y,z;t) := \sum_{p=0}^{s}\binom{s}{p}y^{s-p}z^{kp}C_{s-p}(yt)C_{pk}(z^kt)$.

Proof. For the polynomial ${}_kL_m^{(\alpha,\beta)}(y,z)$ defined by (1.33), the multiplicative operator is given by (1.34).

In view of the relation (1.9) and (2.23), we get

$$\exp(t\hat{M})L_n(x-t\hat{M}) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)_kL_m^{(\alpha,\beta)}(y,z), \tag{2.25}$$

which on substituting the multiplicative operator (1.34), yields

$$\exp(t(1-D_x^{-1}-D_y^{-k}))L_n(x-t(1-D_x^{-1}-D_y^{-k})) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)_kL_m^{(\alpha,\beta)}(y,z). \tag{2.26}$$

Further, applying the exponential operator (2.5) and (2.12) in the l.h.s. of equation (2.26), we find

$$\exp(t)n!\sum_{r=0}^{n}\frac{(-1)^r}{(r!)^2(n-r)!}\sum_{s=0}^{r}\frac{r!(x-t)^{r-s}t^s}{(r-s)!s!}\sum_{p=0}^{s}\frac{s!(D_y^{-1})^{s-p}(D_z^{-k})^p}{(s-p)!p!}C_0(yt)C_0(z^kt|k), \tag{2.27}$$

which on satisfying the property (2.7) and (2.14), gives the generating relation (2.24). □

3. Generating Relations for Legendre Polynomials

We prove the following bilateral generating relations for generalized forms of Legendre polynomials:

Theorem 3.1. *If there exist the following bilateral generating relation involving ALP* $L_n^{(m)}(x)$ *and 2VLeP* ${}_2L_m(y,z)$

$$F(x,y,z;t)=\sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)_2L_m(y,z), \tag{3.1}$$

then the following relation holds true:

$$\exp(tz)_\eta L_n(x,y,z;t) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)_2L_m(y,z), \tag{3.2}$$

where ${}_\eta L_n(x,y,z;t) := n!\sum_{r=0}^{n}\frac{(-1)^r\eta_r(x,y,z;t)}{(r!)^2(n-r)!}$ *and* $\eta_r(x,y,z;t) := \sum_{s=0}^{r}\binom{r}{s}(x-tz)^{r-s}(2yt)^sC_{2s}(-2yt\frac{\partial}{\partial z})$.

Proof. For the polynomial ${}_2L_m(y,z)$ defined by (1.37), the multiplicative operator is given by (1.35).

In view of the relation (1.9) and (3.1), we get

$$\exp(t\hat{M})L_n(x-t\hat{M}) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)_2L_m(y,z), \tag{3.3}$$

which on substituting the multiplicative operator (1.35), yields

$$\exp(t(z+2D_y^{-1}\frac{\partial}{\partial z}))L_n(x-t(z+2D_y^{-1}\frac{\partial}{\partial z})) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{m!}L_n^{(m)}(x)_2L_m(y,z). \tag{3.4}$$

To explicitly evaluate the above expression we note the exponential operator

$$\exp(2tD_y^{-1}\frac{\partial}{\partial z})=C_0(-2yt\frac{\partial}{\partial z}). \tag{3.5}$$

Further, applying (3.5) in the l.h.s. of equation (3.4), we find

$$\exp(tz)n!\sum_{r=0}^{n}\frac{(-1)^r}{(r!)^2(n-r)!}\sum_{s=0}^{r}\frac{r!(-1)^s(x-tz)^{r-s}(2t)^s}{(r-s)!s!}(D_y^{-1}\frac{\partial}{\partial z})^sC_0(-2yt\frac{\partial}{\partial z}). \tag{3.6}$$

On account of the fact

$$(-1)^s\left(\frac{\partial}{\partial z}\right)^s C_n(x) \;=\; C_{n+s}(x), \tag{3.7}$$

equation (3.6) can be recast in the form

$$\exp(tz)n!\sum_{r=0}^{n}\frac{(-1)^r}{(r!)^2(n-r)!}\sum_{s=0}^{r}\frac{r!(x-tz)^{r-s}(2t)^s}{(r-s)!s!}(D_y^{-1})^s C_s(-2yt\frac{\partial}{\partial z}), \tag{3.8}$$

which on satisfying the property (2.14), gives the generating relation (3.2). □

Theorem 3.2. *If there exist the following bilateral generating relation involving ALP* $L_n^{(m)}(x)$ *and 2VGLeP* $R_m^{(\mu,\nu)}(y,z)$

$$F(x,y,z;t) = \sum_{m=0}^{\infty}\frac{t^m}{(m!)^2}L_n^{(m)}(x)R_m^{(\mu,\nu)}(y,z), \tag{3.9}$$

then the following relation holds true:

$${}_{\delta}L_n(x,y,z;t) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{(m!)^2}L_n^{(m)}(x)R_m^{(\mu,\nu)}(y,z), \tag{3.10}$$

where ${}_{\delta}L_n(x,y,z;t) := n!\sum_{r=0}^{n}\frac{(-1)^r\delta_r(x,y,z;t)}{(r!)^2(n-r)!}$,
$\delta_r(x,y,z;t) := \sum_{s=0}^{r}(-1)^s\binom{r}{s}x^{r-s}t^s\gamma_s(y,z;t)$
and $\gamma_s(y,z;t) := \sum_{p=0}^{s}(-1)^p\binom{s}{p}z^{\nu+s-p}y^{\mu+p}C_{2\nu+s-p}(-zt)C_{2\mu+p}(yt)$.

Proof. For the polynomial $R_m^{(\mu,\nu)}(y,z)$ defined by (1.40), the multiplicative operator is given by (1.41).

In view of the relation (1.9) and (3.9), we get

$$\exp(t\hat{M})L_n(x-t\hat{M}) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{(m!)^2}L_n^{(m)}(x)R_m^{(\mu,\nu)}(y,z), \tag{3.11}$$

which on substituting the multiplicative operator (1.41), yields

$$\exp(t(D_{z,\nu}^{-1}-D_{y,\mu}^{-1}))L_n(x-t(D_{z,\nu}^{-1}-D_{y,\mu}^{-1})) \;=\; \sum_{m=0}^{\infty}\frac{t^m}{(m!)^2}L_n^{(m)}(x)R_m^{(\mu,\nu)}(y,z). \tag{3.12}$$

Further, applying the exponential operator (2.20) in the l.h.s. of equation (3.12), we find

$$n!\sum_{r=0}^{n}\frac{(-1)^r}{(r!)^2(n-r)!}\sum_{s=0}^{r}\frac{r!(-1)^s x^{r-s}t^s}{(r-s)!s!}\sum_{p=0}^{s}\frac{s!(-1)^p(D_{z,\nu}^{-1})^{s-p}(D_{y,\mu}^{-1})^p}{(s-p)!p!}C_\nu(-zt)C_\mu(yt), \tag{3.13}$$

which on satisfying the property (2.22), gives the generating relation (3.10). □

Remark. *For* $\mu=\nu=0$, *equation (3.10) gives the bilateral generating relation* [8].

4. Concluding Remark

The operational techniques (including differential and integral operators) provide a systematic and analytic approach in the study of special functions. In the previous sections, we have shown how operational method may play a significant role in the derivation of bilateral generating relations. The method outlined in the previous sections is general and allow us to explore further extension considering other generating functions for families of polynomials and functions.

To illustrate this, we first recall the generating function [7]:

$$C_l(x-t) \;=\; \sum_{n=0}^{\infty} \frac{t^n}{n!} C_{n+l}(x), \tag{4.1}$$

which alongwith the operational rule (1.9), yields

$$C_l(x-t\hat{M}) \;=\; \sum_{n=0}^{\infty} \frac{t^n}{n!} C_{n+l}(x) p_m(x), \tag{4.2}$$

As an example, we consider the 2VLeP ${}_2L_m(y,z)$. The multiplicative operator for ${}_2L_m(y,z)$ is given by (1.35), which when substituted in equation (4.2), gives

$$C_l(x-t(z+2D_y^{-1}\frac{\partial}{\partial z})) \;=\; \sum_{n=0}^{\infty} \frac{t^n}{n!} C_{n+l}(x)_2L_m(y,z). \tag{4.3}$$

Further simplication can be derived within the context of the so far developed procedure as

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(l+r)!} \sum_{s=0}^{r} \frac{(-1)^s r!(x-tz)^{r-s}(2t)^s}{(r-s)!s!} (D_y^{-1}\frac{\partial}{\partial z})^s(1) \;=\; \sum_{n=0}^{\infty} \frac{t^n}{n!} C_{n+l}(x)_2L_m(y,z). \tag{4.4}$$

By using (1.13), it is evident that

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(l+r)!} \sum_{s=0}^{r} \frac{(-1)^s r!(x-tz)^{r-s}(2yt\frac{\partial}{\partial z})^s}{(r-s)!(s!)^2} \;=\; \sum_{n=0}^{\infty} \frac{t^n}{n!} C_{n+l}(x))_2L_m(y,z), \tag{4.5}$$

which gives the generating relation

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(l+r)!} L_r(2yt\frac{\partial}{\partial z}, x-zt) \;=\; \sum_{n=0}^{\infty} \frac{t^n}{(n!)} C_{n+l}(x)_2L_m(y,z), \tag{4.6}$$

or,equivalently

$${}_LC_n((2yt\frac{\partial}{\partial z}, x-zt) \;=\; \sum_{n=0}^{\infty} \frac{t^n}{(n!)} C_{n+l}(x)_2L_m(y,z). \tag{4.7}$$

We can also obtain the bilateral generating relations for the 2VALP $l_n^{(\nu)}(x,y)$, PsLP $L_n(x,y;r)$, GLKnP ${}_kL_n^{(\alpha,\beta)}(x,y)$ and 2VGLeP $R_n^{(\mu,\nu)}(x,y)$ by following the above procedure in connection with the generating relation (4.1).

It is evident that the method discussed is fairly powerful and very efficient in providing families of bilateral generating relations. Further it may be extended to obtain the bilateral generating relations of general families of polynomials and functions, as it will be shown in a forthcoming investigation.

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