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GENERALIZATION OF SOME INEQUALITIES FOR THE SPECIAL FUNCTIONS

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Abstract. It has been shown that if f is a differntiable, logarithmically convex function on nonnegative semi-axis, then the function $g(x) = \frac{f(a+bx)^e}{f(c+dx)^f}$, $a+bx \le c+dx$ and $eb \le fd$, is decreasing on its domain.

It has been shown that if $h(x) = \dfrac{\prod_{i=1}^{n} f(a_i+b_ix)^{c_i}}{f\Big(d+\Big(\sum_{i=1}^{n} b_i\Big)x\Big)^e}$ where

$f : R_+ \longrightarrow R_+$, differentiable log convex function and $c_i, e, i = 1, \ldots, n$ are positive real numbers and $a_i, b_i, d, i = 1, \ldots n$ are real numbers such that $e \ge c_i, i = 1, \ldots n, d + \sum_{i=1}^{n} b_i x \ge a_i + b_i x > 0, i = 1, \ldots n$. If $\alpha(x) = \dfrac{f'(d+(\sum_{i=1}^{n} b_i)x)}{f(d+(\sum_{i=1}^{n} b_i)x)}$ and $b_i > 0, i = 1, \ldots n$ then the function f is decreasing on its domain. In particular, if $x \in [0,1]$, then the following inequalities:

$$\frac{\prod_{i=1}^{n} f(a_i+b_i)^{c_i}}{f\Big(d+\Big(\sum_{i=1}^{n} b_i\Big)x\Big)^e} \le h(x) \le \frac{\prod_{i=1}^{n} f(a_i)^{c_i}}{f(d)^e}. \tag{1}$$

hold true.

Application of this paper is generalization the paper [9],[14],[15],[7].

1. Introduction and preliminaries

Logarithmically convex (log-convex) functions are of interest in many areas of mathematics and science. They have been found to play an important role in the theory of special functions and mathematical statistics (see e.g [3]).

In what follows the symbol R_+ and $R_>$ will stand for the nonnegative semi -axis and positive semi -axis, respectively.

Recall that a function $f : [c,d] \longrightarrow R_>$ is said to be log-convex if

$$f[ux+(1-u)y] \le [f(x)]^u[f(y)]^{1-u}(0 \le u \le 1)$$

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holds for all $x, y \in [c, d]$. It is well - know that a family of log-convex functions is closed under both addition and multiplication.

The *Euler gamma* function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The *digamma* (or *psi*) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)}, \tag{1.1}$$

where $\gamma = 0.57721\cdots$ denotes Euler's constant. Euler, gave another equivalent definition for the $\Gamma(x)$ (see [8]),

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})}, \quad x > 0, \tag{1.2}$$

where

$$\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x). \tag{1.3}$$

The p-analogue of the psi function is defined as the logarithmic derivative of the Γ_p function (see [9]), that is

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{1.4}$$

The function ψ_p defined in (1.4) satisfies the following properties (see [9]). It has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^{p} \frac{1}{x+k}. \tag{1.5}$$

It is increasing on $(0, \infty)$ and it is strictly completely monotonic on $(0, \infty)$. It's derivatives are given by

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p} \frac{(-1)^{n-1} \cdot n!}{(x+k)^{n+1}}. \tag{1.6}$$

Jackson (see [10, 11, 12, 13]) defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \; 0 < q < 1, \tag{1.7}$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty} (q-1)^{1-x} q^{\binom{x}{2}}, \; q > 1, \tag{1.8}$$

where $(a;q)_\infty = \prod_{j \geq 0} (1 - aq^j)$.

The q-analogue of the psi function is defined for $0 < q < 1$ as the logarithmic derivative of the q-gamma function, that is, $\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x)$. Many properties of the q-gamma function were derived by Askey [2]. It is well known that $\Gamma_q(x) \to \Gamma(x)$ and $\psi_q(x) \to \psi(x)$ as $q \to 1^-$. From (1.7), for $0 < q < 1$ and $x > 0$ we get

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n\geq 0} \frac{q^{n+x}}{1-q^{n+x}} = -\log(1-q) + \log q \sum_{n\geq 1} \frac{q^{nx}}{1-q^n} \tag{1.9}$$

and from (1.8) for $q > 1$ and $x > 0$ we obtain

$$\begin{aligned} \psi_q(x) &= -\log(q-1) + \log q \left(x - \tfrac{1}{2} - \sum_{n\geq 0} \tfrac{q^{-n-x}}{1-q^{-n-x}} \right) \\ &= -\log(q-1) + \log q \left(x - \tfrac{1}{2} - \sum_{n\geq 1} \tfrac{q^{-nx}}{1-q^{-n}} \right). \end{aligned} \tag{1.10}$$

A Stieltjes integral representation for $\psi_q(x)$ with $0 < q < 1$ is given in [4]. It is well-known that ψ' is strictly completely monotonic on $(0, \infty)$, that is,

$$(-1)^n (\psi'(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0,$$

see [1, Page 260]. From (1.9) and (1.10) we conclude that ψ'_q has the same property for any $q > 0$,

$$(-1)^n (\psi'_q(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0.$$

If $q \in (0,1)$, using the second representation of $\psi_q(x)$ given in (1.9) can be shown that

$$\psi_q^{(k)}(x) = \log^{k+1} q \sum_{n\geq 1} \frac{n^k \cdot q^{nx}}{1-q^n} \tag{1.11}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 1$, for all $k \geq 1$. If $q > 1$, from the second representation of $\psi_q(x)$ given in (1.10) we obtain

$$\psi'_q(x) = \log q \Big(1 + \sum_{n\geq 1} \frac{nq^{-nx}}{1-q^{-nx}} \Big) \tag{1.12}$$

and for $k \geq 2$,

$$\psi_q^{(k)}(x) = (-1)^{k-1} \log^{k+1} q \sum_{n\geq 1} \frac{n^k q^{-nx}}{1-q^{-nx}} \tag{1.13}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 0$, for all $q > 1$.

2. Main results and its applications

The following Theorem is the main result of these notes.

Theorem 2.1. *Let $f : R_+ \longrightarrow R_+$ differentiable log convex function and let $a + bx \leq c + dx$ and $eb \leq fd$,then the function*

$$g(x) = \frac{f(a+bx)^e}{f(c+dx)^f} \tag{2.1}$$

decreases on its domain. In particular, if $x \in [0,1]$, then the following inequalities:

$$\frac{f(a+b)^e}{f(c+d)^f} \leq \frac{f(a+bx)^e}{f(c+dx)^f} \leq \frac{f(a)^e}{f(c)^f} \tag{2.2}$$

hold true. If $a + bx \geq c + dx$ and $eb \geq fd$, then the function g is an increasing function on R_+ and the inequalities (2.2) are reserved.

Proof. We prove the theorem when $a + bx \leq c + dx$ and $eb \leq fd$. Logarithmic convexity of f implies that its logarithmic derivative

$$\alpha(x) = \frac{f^{'}(x)}{f(x)}$$

is an increasing function on R_+ ie that

$$\alpha(a + bx) \leq \alpha(c + dx) \tag{2.3}$$

Logarithmic differentiation of (2.1) gives

$$\begin{aligned}
\frac{g^{'}(x)}{g(x)} &= eb\frac{f^{'}(a+bx)}{f(a+bx)} - fd\frac{f^{'}(c+dx)}{f(c+dx)} \\
&\leq df\Big[\frac{f^{'}(a+bx)}{f(a+bx)} - \frac{f^{'}(c+dx)}{f(c+dx)}\Big] \\
&= df(\alpha(a+bx) - \alpha(c+dx)) \\
&= df\int_{c+dx}^{a+bx} (lnf(x))^{''}dx
\end{aligned} \tag{2.4}$$

This is conjunction with (2.3) yields $g^{'}(x) \leq 0$, because $g(x) > 0$ for all $x \in R_+$.

This proves the monotonicity property of the function g inequalities (2.2) now follow because for $0 \leq x \leq 1, g(1) \leq g(x) \leq g(0)$. The proof is completed. □

Theorem 2.2. *Let h be a function defined by*

$$h(x) = \frac{\prod\limits_{i=1}^{n} f(a_i + b_i x)^{c_i}}{f\Big(d + \Big(\sum\limits_{i=1}^{n} b_i\Big)x\Big)^e} \tag{2.5}$$

where $f : R_+ \longrightarrow R_+$, differentiable log convex function and $c_i, e, i = 1, \ldots, n$ are positive real numbers and $a_i, b_i, d, i = 1, \ldots n$ are real numbers such that $e \geq c_i, i = 1, \ldots n, d + \sum\limits_{i=1}^{n} b_i x \geq a_i + b_i x > 0, i = 1, \ldots n$.

If $\alpha(x) = \dfrac{f^{'}(d + (\sum\limits_{i=1}^{n} b_i)x)}{f(d + (\sum\limits_{i=1}^{n} b_i)x)}$ and $b_i > 0, i = 1, \ldots n$ then the function f is decreasing on its domain. In particular, if $x \in [0, 1]$, then the following inequalities:

$$\frac{\prod\limits_{i=1}^{n} f(a_i + b_i)^{c_i}}{f\Big(d + \Big(\sum\limits_{i=1}^{n} b_i\Big)x\Big)^e} \leq h(x) \leq \frac{\prod\limits_{i=1}^{n} f(a_i)^{c_i}}{f(d)^e}. \tag{2.6}$$

hold true.

If $\alpha(x) = \dfrac{f^{'}(d + (\sum\limits_{i=1}^{n} b_i)x)}{f(d + (\sum\limits_{i=1}^{n} b_i)x)}$ and $b_i < 0, i = 1, \ldots n$ then the function h is increasing on R_+ and the inequalities (2.6) are reserved.

Proof. Let g be a function defined by $g(x) = \ln h(x)$ for $x \in [0, \infty)$

$$g^{'}(x) = \sum_{i=1}^{n} c_i b_i \alpha(a_i + b_i x) - e(\sum_{i=1}^{n} b_i)\alpha(d + \sum_{i=1}^{n} b_i x)$$
$$= \sum_{i=1}^{n} b_i \Big(c_i \alpha(a_i + b_i x) - e\alpha(d + \sum_{i=1}^{n} b_i x) \Big).$$

It is easy to see that

$$\alpha(a_i + b_i x) \leq \alpha(d + \sum_{i=1}^{n} b_i x)$$

Multiplying both sides of inequality $e \geq c_i$ with $\alpha(d + \sum_{i=1}^{n} b_i x)$ we obtain

$$e\alpha(d + \sum_{i=1}^{n} b_i x) \geq c_i \alpha(d + \sum_{i=1}^{n} b_i x) \geq \alpha(a_i + b_i x).$$

Hence,

$$\sum_{i=1}^{n} b_i \Big(c_i \alpha(a_i + b_i x) - e\alpha(d + \sum_{i=1}^{n} b_i x) \Big) \leq 0.$$

We have $g^{'}(x) \leq 0$. It means that g is decreasing on $[0, \infty)$. This implies that h is decreasing on $[0, \infty)$.

Now for $x \in [0, 1]$ we have

$$h(1) \leq h(x) \leq h(0)$$

Thus

$$\frac{\prod_{i=1}^{n} f(a_i + b_i)^{c_i}}{f\Big(d + \Big(\sum_{i=1}^{n} b_i\Big)x\Big)^e} \leq \frac{\prod_{i=1}^{n} f(a_i + b_i x)^{c_i}}{f\Big(d + \Big(\sum_{i=1}^{n} b_i\Big)x\Big)^e} \leq \frac{\prod_{i=1}^{n} f(a_i)^{c_i}}{f(d)^e}.$$

□

3. Application of theorem 2.1 and 2.2

3.1 Inequalities involving the gamma function. Let$f(x) = \Gamma(x)$. It is well know that the function f is log -convex(see e.g [3]]. Making use theorem (2.1) we conclude that the function $\frac{\Gamma(a+bx)^c}{\Gamma(c+dx)^f}$ is decreases for all $x \geq 0$ and the inequalities

$$\frac{\Gamma(a+b)^c}{\Gamma(c+d)^f} \leq \frac{\Gamma(a+bx)^c}{\Gamma(c+dx)^f} \leq \frac{\Gamma(a)^c}{\Gamma(c)^f} \tag{3.1}$$

hold true for $x \in [0, 1]$. Inequalities (3.1) have been obtained in [14].

3.2 Inequalities involving the $q-$ gamma function.

Let$f(x) = \Gamma_q(x)$. It is well know that the function f is log -convex(see e.g [3]]. Making use theorem (2.1) we conclude that the function $\frac{\Gamma_q(a+bx)^c}{\Gamma_q(c+dx)^f}$ is decreases for all $x \geq 0$ and the inequalities

$$\frac{\Gamma_q(a+b)^c}{\Gamma_q(c+d)^f} \leq \frac{\Gamma_q(a+bx)^c}{\Gamma_q(c+dx)^f} \leq \frac{\Gamma_q(a)^c}{\Gamma_q(c)^f} \tag{3.2}$$

hold true for $x \in [0, 1]$.

Inequalities (3.2) have been obtained in [15].

3.3 Inequalities involving the $p-$ gamma function. Let$f(x) = \Gamma_p(x)$. It is well know that the function f is log -convex(see e.g [9]]. Making use theorem (2.1) we conclude that the function $\frac{\Gamma_p(a+bx)^c}{\Gamma_p(c+dx)^f}$ is decreases for all $x \geq 0$ and the inequalities

$$\frac{\Gamma_p(a+b)^c}{\Gamma_p(c+d)^f} \leq \frac{\Gamma_p(a+bx)^c}{\Gamma_p(c+dx)^f} \leq \frac{\Gamma_p(a)^c}{\Gamma_p(c)^f} \tag{3.3}$$

hold true for $x \in [0,1]$.

Inequalities (3.3) have been obtained in [9].

3.4 Inequalities for the Riemann zeta function. A formula which connects Euler's gamma function and Riemann's zeta function

$$\Gamma(x)\zeta(x) = \int\limits_0^\infty \frac{t^{x-1}}{e^t - 1} dt \tag{3.4}$$

is well know (see e.g [1]) It is well know function $f(x) = \Gamma(x)\zeta(x)$ is log -convex for $x \in R_+$(see e.g [3]]. Making use theorem (2.1) the inequalities

$$\frac{[\Gamma(a+b)\zeta(a+b)]^e}{[\Gamma(c+d)\zeta(c+d)]^f} \leq \frac{[\Gamma(a+bc)\zeta(a+bx)]^e}{[\Gamma(c+dx)\zeta(c+dx)]^f} \tag{3.5}$$

hold for $x \in [0,1]$

3.3 Inequalities involving the gamma function.

Let $h(x) = \frac{\prod\limits_{i=1}^n \Gamma(a_i+b_ix)^{c_i}}{\Gamma\left(d+\left(\sum\limits_{i=1}^n b_i\right)x\right)^e}$. It is well know that the function h is log -convex.

Making use theorem (2.2) we conclude that the function $\frac{\prod\limits_{i=1}^n \Gamma(a_i+b_ix)^{c_i}}{\Gamma\left(d+\left(\sum\limits_{i=1}^n b_i\right)x\right)^e}$ is decreases for all $x \geq 0$ and the inequalities

$$\frac{\prod\limits_{i=1}^n \Gamma(a_i+b_i)^{c_i}}{\Gamma\left(d+\left(\sum\limits_{i=1}^n b_i\right)x\right)^e} \leq \frac{\prod\limits_{i=1}^n \Gamma(a_i+b_ix)^{c_i}}{\Gamma\left(d+\left(\sum\limits_{i=1}^n b_i\right)x\right)^e} \leq \frac{\prod\limits_{i=1}^n \Gamma(a_i)^{c_i}}{\Gamma(d)^e}. \tag{3.6}$$

hold true for $x \in [0,1]$.

Inequalities (3.6) have been obtained in [7].

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