Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com
Volume 3, Issue 3 (2012), Pages 29-33

ON THE ZEROS OF POLYNOMIALS SATISFYING A THREE-TERM RECURRENCE RELATION WITH COMPLEX COEFFICIENTS

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Abstract. A general class of polynomials satisfying a three-term recurrence relation with complex coefficients is studied, with respect to their zeros. The method used is a functional-analytic one, which connects these zeros, with the eigenvalues of a specific linear operator in an abstract finite dimensional Hilbert space. In this way, inequalities for the real and imaginary part of the complex zeros of these polynomials can be obtained. As specific cases, the Laguerre and Ultraspherical polynomials with complex coefficients are treated. The obtained results are compared with some already known ones.

1. Introduction

The orthogonal polynomial sequences $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy a three term recurrence relation of the form [2]:

$$\begin{gathered}\tilde{a}_n Q_{n+1}(x) + \tilde{\beta}_n Q_n(x) + \tilde{c}_n Q_{n-1}(x) = xQ_n(x),\ n = 0, 1, 2, ... \\ Q_{-1}(x) = 0, \quad Q_0(x) = 1\end{gathered} \tag{1.1}$$

where the coefficients $\tilde{a}_n,\ \tilde{\beta}_n,\ \tilde{c}_n$ are real sequences with $\tilde{a}_n\tilde{c}_{n+1} > 0$. It is well known that after specific transformations, relation (1.1) can take the form:

$$\begin{gathered}a_n P_{n+1}(x) + a_{n-1}P_{n-1}(x) + \beta_n P_n(x) = xP_n(x),\ n = 1, 2, ... \\ P_0(x) = 0, \quad P_1(x) = 1.\end{gathered} \tag{1.2}$$

where the coefficients a_n, β_n, are real sequences with $a_n > 0$. It is also known, that the zeros of the polynomials $Q_n(x)$ or $P_n(x)$ are real and simple [2]. Moreover, the properties of their zeros have been extensively studied by many authors during the last 30 years.

However, the polynomial sequences which satisfy the three term recurrence relation (1.1), in the case where the coefficients are complex sequences have not been widely studied, although the location of complex zeros of orthogonal polynomials is a very important topic. For example, the location of complex zeros of Jacobi polynomials play an important role on numerical integration for the solution of singular integral equations, which arise in plane elasticity crack problems [12]. Also, the location of complex zeros of Laguerre polynomials is necessary in the numerical

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2000 *Mathematics Subject Classification.* 33C45.
Key words and phrases. complex zeros; Laguerre polynomials; Ultraspherical polynomials.

Submitted Jun 6, 2012. Published October 10, 2012.

technique used in [1], where the ODE/BA correspondence was used for the numerical solutions of the Bethe equations, defining eigenstates of Gaudin models when degeneracies are present.

As far as we know, there exist only few papers concerning the location or study of zeros of complex polynomials satisfying three–term recurrence relations. More precisely, Runckel [10, 11] and Gilewicz & Leopold [5], have considered the problem of the location of zeros of polynomials with complex coefficients. Also in [4], the asymptotic behavior of the zeros of classical orthogonal polynomials, when their parameters vary in the complex plane, has been investigated. Defazio in his thesis [3], among other things, studied also the behavior of the complex zeros of the Laguerre polynomials $L_n^{(a)}(x)$, for $a \leq -1$ and he called quasi-definite orthogonal polynomials, the orthogonal polynomials which satisfy relation (1.1) with complex coefficients. Finally, Petropoulou in [9], gave results concerning the complex zeros of orthogonal polynomials which satisfy (1.2), in the case where a_n is a purely imaginary sequence and β_n is real.

In this paper, we generalize the results of [9] assuming that the coefficients a_n and β_n of (1.2) are complex sequences. Using an analogues procedure as in [9], we will give bounds for $\mathrm{Re}[x_{nk}]$ and $\mathrm{Im}[x_{nk}]$, where x_{nk} are the k-th zeros, $k = 1, ..., n$ of the orthogonal polynomials $P_n(x)$. As specific cases, the Laguerre and Ultraspherical polynomials with complex coefficients are treated. The obtained results for the Laguerre polynomials are compared with some already known ones. These results are included in §3-4.

In §2, we present the method that we use. This is a functional analytic method, also used in [9], which have been introduced by Ifantis & Siafarikas in their study for the real zeros of orthogonal polynomials which satisfy (1.2) [7], [8]. The basic idea of this method is that the zeros of the polynomials satisfying (1.2) are the eigenvalues of a linear tridiagonal operator defined in an abstract finite dimensional Hilbert space and vice-versa.

2. The method

Let $\{e_k\}_{k=1}^n$ be an orthonormal base in a finite dimensional complex Hilbert space H_n with inner product denoted as usual by $(\cdot,\cdot)$ and let V be the truncated shift operator

$$Ve_k = e_{k+1}, \quad k = 1, ..., n-1, \quad Ve_n = 0.$$

The adjoint of V is the shift operator V^* defined by:

$$V^*e_k = e_{k-1}, \quad k = 2, ..., n, \quad V^*e_1 = 0.$$

We also define the diagonal operators A and B

$$Ae_k = a_k e_k, \quad Be_k = \beta_k e_k, \quad k = 1, ..., n.$$

It is known [7], that the zeros of the polynomials $P_{n+1}(x)$ defined by (1.2), in the case where a_n, β_n are real with $a_n > 0$, are the eigenvalues of the operator $T = AV^* + VA + B$, that is,

$$Tf_k = x_{nk} f_k, \quad \|f_k\| = 1 \tag{2.1}$$

and vice versa.

(i) If the sequences a_n and β_n are real, then the operator $T = AV^* + VA + B$ is self-adjoint and thus its eigenvalues are all real. Moreover, from (2.1) follows:

$$x_{nk} = (Tf_k, f_k), \quad k = 1, ..., n \tag{2.2}$$

(ii) If the sequences a_n and β_n are complex, say $a_n = a_{1n} + ia_{2n}$, $\beta_n = \beta_{1n} + i\beta_{1n}$, where a_{in}, β_{in} $i = 1, 2$ are real sequences, then the operator T can be written in the form $T = T_1 + iT_2$, where $T_i = A_iV^* + VA_i + B_i$, with $A_ie_k = a_{ik}e_k$ and $B_ie_k = \beta_{ik}e_k$, $i = 1, 2$, $k = 1, ..., n$.

Moreover, if x_{nk} is the k-th complex zero of $P_{n+1}(x)$ or equivalently the corresponding eigenvalue of the operator T, then relation (2.2) takes the form:

$$x_{nk} = ((A_1V^* + VA_1 + B_1)f_k, f_k) + i((A_2V^* + VA_2 + B_2)f_k, f_k) \tag{2.3}$$

Since the operators $T_1 = A_1V^* + VA_1 + B_1$ and $T_2 = A_2V^* + VA_2 + B_2$ are self-adjoint, the inner products (T_1f_k, f_k) and (T_2f_k, f_k) are real and as a consequence, $Re[x_{nk}] = (T_1f_k, f_k)$ and $Im[x_{nk}] = (T_2f_k, f_k)$ which immediately result into the following inequalities:

$$|\mathrm{Re}[x_{nk}]| \leq \|T_1\| \leq 2\|A_1\| + \|B_1\| = 2\sup_k |a_{1k}| + \sup_k |\beta_{1k}| \tag{2.4}$$

$$|\mathrm{Im}[x_{nk}]| \leq \|T_2\| \leq 2\|A_2\| + \|B_2\| = 2\sup_k |a_{2k}| + \sup_k |\beta_{2k}|, \tag{2.5}$$

where all the sup are taken over $k = 1, ..., n$.

Of course the above mentioned analysis, coincide with the analysis of [9] in the case where $a_{1n} \equiv 0$ and $\beta_{2n} \equiv 0$.

3. Complex Laguerre Polynomials

The Laguerre polynomials $L_n^{(a)}(x)$ satisfy the recurrence relation (1.1) for

$$\tilde{a}_k = -(k+1), \quad \tilde{\beta}_k = a + 2k + 1, \quad \tilde{c}_k = -(a+k), \quad k = 1, ..., n.$$

However, after specific transformations, the orthonormal Laguerre polynomials satisfy (1.2) with

$$a_k = \sqrt{k(k+a)}, \quad \beta_k = 2k + a - 1, \quad k = 1, ..., n.$$

By considering $a = a_1 + ia_2$, $a_1, a_2 \in \mathbb{R}$, we obtain what we'll call complex Laguerre polynomials. For these polynomials the following holds:

Theorem 3.1. *The zeros $x_{nk}(a) = Re[x_{nk}(a)] + iIm[x_{nk}(a)]$ of the complex Laguerre polynomials $L_n^{(a)}(x)$, $a = a_1 + ia_2$, satisfy the following inequalities:*

$$|\mathrm{Re}[x_{nk}(a)]| \leq 2\left[n^2(n+a_1)^2 + n^2a_2^2\right]^{1/4} + |2n + a_1 - 1| \tag{3.1}$$

$$|\mathrm{Im}[x_{nk}(a)]| \leq 2\left[n^2(n+a_1)^2 + n^2a_2^2\right]^{1/4} + |a_2| \tag{3.2}$$

Remark 3.2. *The results of this theorem are in accordance with the results of* [4] *and* [11]. *We mention though, that our method is simpler than those used in* [4] *and* [11].

Remark 3.3. *The Theorem 2.1 of* [9], *can be obtained by setting $a_{1n} = 0$ and $\beta_{2n} = 0$.*

Proof. For $a = a_1 + ia_2$, $a_1, a_2 \in \mathbb{R}$ the sequences a_k and β_k become:

$$a_k = [k(k+a_1) + ika_2]^{\frac{1}{2}}, \quad \beta_k = 2k + a_1 - 1 + ia_2, \quad k = 1, ..., n.$$

Especially the sequence a_k can be rewritten as

$$a_k = \left[k^2(k+a_1)^2 + k^2a_2^2\right]^{1/4} \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right),$$

where θ is the argument of the complex number $k(k+a_1) + ika_2$.

Therefore, from (2.4) and (2.5), we obtain the following inequalities for the real and the imaginary part of the zeros $x_{nk}(a)$:

$$|\mathrm{Re}[x_{nk}(a)]| \leq 2\sup_k \left[k^2(k+a_1)^2 + k^2a_2^2\right]^{1/4} + \sup_k |2k + a_1 - 1|$$

$$= 2\left[n^2(n+a_1)^2 + n^2a_2^2\right]^{1/4} + |2n + a_1 - 1|$$

and

$$|\mathrm{Im}[x_{nk}(a)]| \leq 2\sup_k \left[k^2(k+a_1)^2 + k^2a_2^2\right]^{1/4} + |a_2|$$

$$= 2\left[n^2(n+a_1)^2 + n^2a_2^2\right]^{1/4} + |a_2|$$

□

4. Complex Ultraspherical Polynomials

The Ultraspherical polynomials $P_n^{(\lambda)}(x)$ satisfy the recurrence relation (1.1) for

$$\tilde{a}_n = \frac{n+1}{2(n+\lambda)}, \quad \tilde{\beta}_n = 0, \quad \tilde{c}_n = \frac{n+2\lambda-1}{2(n+\lambda)}.$$

However, after specific transformations, the orthonormal Ultraspherical polynomials satisfy (1.2) with

$$a_n = \frac{1}{2}\sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1)}}, \quad \beta_n = 0.$$

By considering $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$, we obtain what we'll call complex Ultraspherical polynomials. For these polynomials the following holds:

Theorem 4.1. *The zeros $x_{nk}(\lambda)$ of the complex Ultraspherical polynomials $P_n^{(\lambda)}(x)$ for $\lambda = \lambda_1 + i\lambda_2$ satisfy the following inequalities:*

$$|\mathrm{Re}[x_{nk}(\lambda)]|, |\mathrm{Im}[x_{nk}(\lambda)]| \leq (a_{n_1}^2 + a_{n_2}^2)^{\frac{1}{4}} \tag{4.1}$$

where

$$a_{n_1} = \frac{n[(n+2\lambda_1-1)((n+\lambda_1)(n+\lambda_1-1)-\lambda_2^2) + 2\lambda_2^2(2n+2\lambda_1-1)]}{[((n+\lambda_1)(n+\lambda_1-1)-\lambda_2^2)^2 + \lambda_2^2(2n+2\lambda_1-1)^2]^{\frac{1}{2}}} \tag{4.2}$$

and

$$a_{n_2} = \frac{n\lambda_2[2((n+\lambda_1)(n+\lambda_1-1)-\lambda_2^2) - (2n+2\lambda_1-1)(n+2\lambda_1-1)]}{[((n+\lambda_1)(n+\lambda_1-1)-\lambda_2^2)^2 + \lambda_2^2(2n+2\lambda_1-1)^2]^{\frac{1}{2}}} \tag{4.3}$$

Proof. For $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ the sequence a_k eventually becomes:

$$a_k = \frac{1}{2}(a_{k_1}^2 + a_{k_2}^2)^{\frac{1}{4}} \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right), \quad k = 1, 2, ..., n$$

where a_{ki}, $i = 1, 2$ are given by (4.2) and (4.3), respectively.

Therefore, from (2.4) and (2.5) we obtain the following inequalities for the real and the imaginary part of the zeros $x_{nk}(\lambda)$:

$$|\mathrm{Re}[x_{nk}(\lambda)]|, |\mathrm{Im}[x_{nk}(\lambda)]| \le 2\frac{1}{2}\sup_k (a_{k_1}^2 + a_{k_2}^2)^{\frac{1}{4}} = (a_{n_1}^2 + a_{n_2}^2)^{\frac{1}{4}}$$

with a_{n_1} and a_{n_2} given by (4.2) and (4.3), respectively. □

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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