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ON HYPERSHERICAL LEGENDRE POLYNOMIALS AND HIGHER DIMENSIONAL MULTIPOLE EXPANSIONS

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Abstract. The Green function for the Laplace operator in N dimensions is obtained. It is used as the generating function for hyperspherical Legendre polynomials, that extend to N dimensions the original Legendre polynomials in three-dimensional space. The main properties, such as particular values, explicit coefficients of expansion in powers, ordinary differential equation and recurrence and differentiation formulas are extended from the original Legendre polynomials in three-dimensional space to the hyperspherical Legendre polynomials in any dimension. The hyperspherical Legendre polynomials are used together with hyperspherical and hypercylindrical coordinates to specify N-dimensional monopoles, dipoles, quadropoles, octupoles and multipoles of any order. As a further application, the circle and sphere theorems involving the reciprocal point are extended to an hypersphere theorem: the latter specifies the effect of inserting in an uniform field an hypersphere with either tangential or orthogonal boundary condition at the surface.

1. Introduction

The theory of the potential [1,6,10,12] specifying the potential field due to an arbitrary source distribution is generalized to N dimensions, for irrotational fields [5,7,18,19] starting with the Green function for the Laplace operator (section 2). The Laplace operator with radial symmetry in N dimensions (subsection 2.1), forced by a centrally-symmetric N-dimencional Dirac distribution [1,8,20] specifies the Green function (subsection 2.2). The latter includes the logarithmic potential in the plane, and inverse distance potential in space, and extends them to any dimension (subsection 2.3). Likewise the Poisson integral is extended to unbounded space (subsection 2.3) or a compact region (subsections 2.4-2.5) in N-dimensional space.

The multipolar expansion in N-dimensional space (subsection 3.1) specifies the hyperspherical Legendre polynomials (subsection 3.2) that extend (section 3) the original Legendre polynomials [9,13] beyond three dimensions. The Green function for the N-dimensional Laplace operator is thus the generating function for hyperspherical Legendre polynomials (section 3) from which can be obtained: (subsection 3.2) convergence properties; (subsection 3.4) values at particular points; (subsection 3.5) explicit coefficients for the hyperspherical Legendre polynomials; (subsection

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3.6) explicit expressions in terms of cosines of angles and multiple angles; (subsection 3.7) one recurrence formula; (subsection 3.8) four differentiation formulas; (subsection 3.9) the hyperspherical Legendre differential equation; (subsection 3.10) integral and differential representations [2,3,4,16]. These formulas for hyperspherical Legendre polynomials reduce in the three-dimensional case to the well-known results for the original Legendre polynomials.

As an application the multipolar expansion (subsection 4.1) is considered in an N dimensional space (section 4), using hyperspherical (subsection 4.2) or hypercylindrical (subsection 4.3) coordinates. The lowest order terms, viz. monopoles (subsection 4.4), dipoles (subsection 4.5), quadrupoles (subsection 4.6) and octupoles (subsection 4.7) as well as multipoles of any order, are considered in N dimensions, in comparison with the plane and space cases. A further example of extension to N dimensions is the generalization (section 5) of the circle and sphere theorems involving the reciprocal point [1,7,8,10,11,12,14] into the hypersphere theorem (subsection 5.2); this involves the reciprocal potential that is an harmonic function (subsection 5.3) and can be used to determine the effect on a uniform field of the introduction of an hypersphere, to which the field is either tangent (subsection 5.1) or orthogonal (subsection 5.4). As the briefest of concluding discussions a Table indicates the asymptotic decay of multipoles of all orders in any dimension (subsection 5.5).

2. Green function and source distributions

The Green function (subsection 2.2) is the fundamental solution for the Laplace operator with hyperspherical symmetry (subsection 2.1). It can used to specify the potential due to an arbitrary source distribution in free space (subsection 2.3). It is extended to a bounded region by the (subsection 2.4) Green second identity, leading to source distributions on the boundary (subsection 2.5).

2.1. **Laplace operator with hyperspherical symmetry.**

Consider an N-dimensional space (1a) with coordinates (1b) for which the radial distance is given by (1c):

$$i = 1, \dots, N; \ \vec{x} \equiv (x_1, x_2, \dots, x_N) \in |R^N : \qquad R = \textstyle\sum_{n=1}^{N} (x_i)^2 \equiv x_i\, x_i\,, \tag{1a-c}$$

where a repeated index (1c) means summation over (1a). The derivatives of first (second) order of the radial distance are (2a) [(2b)], the latter involving the identity matrix δ_{ij}:

$$\partial R/\partial x_i = x_i/R, \qquad \partial^2 R/\partial x_i\, \partial x_j = -\, x_i\, x_j\, /R^3 + \delta_{ij}\, /R\,. \tag{2a,b}$$

They specify the first (3b) [second (3c)], order derivatives of a scalar function with hyperspherical symmetry (3a):

$$\Psi\,(x_1, \dots, x_N) = \Phi\,(R) : \partial\Psi/\partial x_i = (d\Phi/dR)\,(\partial R/\partial x_i) = (d\Phi/dR)\, x_i/R\,, \tag{3a,b}$$

$$\partial^2\Psi/\partial x_i\, \partial x_j = \left(d^2\Phi/dR^2\right)\, x_i x_j/R^2 + \,(d\,\Phi/dR)\, \left[R^{-1}\,\delta_{ij} - R^{-3}\, x_i x_j\right]\ ; \tag{3c}$$

summing over $i,\ j = 1, ..., N$ in (3c), and noting (4a) leads to (4b):

$$\delta_{ii} = N : \qquad \partial^2\Psi/\partial x_i\,\partial x_i = \ d^2\Phi/dR^2 + \ [(N-1)/R]\ d\,\Phi/dR. \qquad (4\text{a,b})$$

This is the Laplacian with hyperspherical symmetry in N dimensions:

$$\nabla^2\ \Psi\ \equiv\ \sum_{i=1}^{N} \frac{\partial^2\Psi}{\partial x_i^2} = \ \frac{d^2\Phi}{dR^2} + \ \frac{N-1}{R}\ \frac{d\Phi}{d\,R} = \ \frac{1}{R^{N-1}}\ \frac{d}{dR}\ \left(R^{N-1}\ \frac{d\Phi}{dR}\right)\ \equiv\ \nabla^2\ \Phi, \ (5)$$

that includes as particular cases in one / two / three dimensions the line (6a) / cylindrical (6b) / spherical (6c) Laplacian:

$$N = 1 : \nabla^2 = d^2/dx^2; N = 2 : r^{-1}\ (d/dr)\ [r\ d\Phi/dr] = d^2\Phi/dr + r^{-1} d\,\Phi/dr, \ (6\text{a,b})$$

$$N = 3 : \quad \nabla^2 = R^{-2}\ (d/dR)\ \left[R^2 d\,\Phi/dR\right] = d^2\Phi/dR^2 + \ (2/R)\ d\Phi/dR. \quad (6\text{c})$$

The Laplacian with line (6a) / cylindrical (6b) / spherical (6c) symmetry is the first term of the Laplacian respectively in Cartesian/cylindrical/spherical coordinates.

2.2. **Green function for the Laplace operator in any dimension.**

The Green function for the N-dimensional Laplacian is the fundamental solution of the Poisson equation forced by N-dimensional Dirac impulse distribution:

$$\sum_{i=1}^{N} \partial^2 G\,(x_n; y_n)\,/\partial x_i^2 \equiv \partial^2 G\,\left(\vec{x}; \vec{y}\right)/\partial\,\vec{x}^2 = \delta\left(\vec{x} - \vec{y}\right) \equiv \prod_{i=1}^{N} \delta\,(x_i - y_i)\,. \ (7)$$

The latter has hypersherically symmetry and involves the area σ_N of the unit hypersphere in N-dimensions, and thus forces the Laplacian with hyperspherical symmetry (5) whose solution is the Green function:

$$R^{1-N}\ \ (d/dR)\ \ \left[R^{N-1}\ dG/dR\right] = \ \ \sigma_N^{-1}\ \ R^{1-N}\ \ \delta\,(R)\ . \qquad (8)$$

A first integration (28a) involves the Heaviside unit jump distribution (28b):

$$R > 0 : \quad R^{N-1}\,dG/dR = \sigma_N^{-1}\ \ H\,(R) = 1/\sigma_N, \quad G\,(R) = \ \sigma_N^{-1}\ \int R^{1-N}\,dR\,, \quad (9\text{a-c})$$

leading to (9c) by a second integration; an arbitrary constant was omitted since the potential can be defined to within a constant. The integral (9c) is specified by (9a-c) [(10d,e)] for the plane $N = 2$ (other dimensions $N \neq 2$), viz.:

$$R \equiv \left|\vec{x} - \vec{y}\right|,\ N = 2,\ \sigma_2 = 2\pi :$$

$$G_2\left(\vec{x}; \vec{y}\right) = (1/\sigma_2)\log R = (2\pi)^{-1}\log\left|\vec{x} - \vec{y}\right|, \ (10\text{a-d})$$

$$2 \neq N = 1\,,\,3\,,\,4\,,.... : \quad G_N\ \left(\vec{x}\ ;\ \vec{y}\right)\ \ =\ \ [(2-N)\ \ \sigma_N]^{-1}\ \left|\vec{x} - \vec{y}\right|^{2-N}\,. \ (10\text{e,f})$$

Thus has been obtained the Green function for the Laplace operator (7)≡(8) in N-dimensions (9c), that includes: (ii) the logarithmic potential (10d) in the plane (10b) involving the perimeter of the unit circle (10c); (i/iii) the one (three) dimensional Green function, viz. a ramp function (11a,b) [the inverse distance (12c)]:

$$N=1: \qquad G_1\ (x;y)=\begin{cases} 0 & \text{if} \quad x \ \leq \ y\ , \\ x\ -\ y & \text{if} \quad x\ \geq\ y\ , \end{cases} \tag{11a,b}$$

$$N=3\,,\, \sigma_3=4\pi : G_3\left(\vec{x}\,;\ \vec{y}\right)=-\left\{\sigma_3\left|\vec{x}-\vec{y}\right|\right\}^{-1}= \\ =-\left\{4\pi\ \left|\ \vec{x}-\vec{y}\ \right|\right\}^{-1}, \tag{12a-c}$$

with the latter (12a) involving the area of the unit sphere (12b); (iv) both are particular cases of the area of the unit hypersphere, that appears in the Green function (10f) in higher $N \geq 4$ dimensions (10e).

2.3. **Arbitrary source distribition in free space.**

The solution of the Poisson equation (32a) forced by an arbitrary source distribution is (13b) the convolution of the latter with the Green function:

$$\nabla^2\Phi=q\left(\vec{x}\right), \quad \Phi\left(\vec{x}\right)=q*G\left(\vec{x}\right)\equiv\int_{-\infty}^{+\infty}q\left(\vec{y}\right)\ G\left(\vec{x};\vec{y}\right)d\,\vec{y}\,; \tag{13a,b}$$

substituting (10-12) it follows that the potential as a solution of the Poisson equation (13a) forced by an arbitrary source distribution in free (i.e. unbounded) space, is specified in one / two / three / more dimensions by (14a/b/c/d) by the Poisson integral:

$$N=1,\ \sigma_1=1: \qquad \Phi\ (x)\ =\ \int_x^{\infty}(x-y)\ \ q\ (y)\ \ dy, \tag{14a}$$

$$N=2,\ \sigma_2=2\pi: \qquad 2\pi\ \Phi\left(\vec{x}\right)\ =\ \int q\left(\vec{y}\right)\ \log\left|\vec{x}-\vec{y}\right|\ d^2\ \vec{y}\,, \tag{14b}$$

$$N=3,\ \sigma_3=4\pi: \qquad -\ 4\pi\ \Phi\left(\vec{x}\right)\ =\ \int q\left(\vec{y}\right)\ \left|\vec{x}-\vec{y}\right|^{-1}\ d^2\ \vec{y}\,, \tag{14c}$$

$$2\neq N=1\,,\ 3\,,\ 4:\ \sigma_N\ (2-N)\ \ \Phi\left(\vec{x}\right)=\int q\left(\vec{y}\right)\ \left|\vec{x}-\vec{y}\right|^{2-N}\ d^N\ \vec{y}\,, \tag{14d}$$

as the convolution of the source with the Green's function (11a,b)/(10d)/(12c)/(10f) involving the perimeter of the unit circle / area of the unit sphere / hypersphere respectively in (14b/c/d). The case of a source distribution in a bounded region leads to additional sources on the boundary as shown next.

2.4. **Poisson equation in a bounded versus unbounded region.**

The Green first identity can be used to prove unicity theorems for the inner problems of Cauchy-Dirichlet, Neumann or mixed type; also the second Green identity can be applied to the Poisson equation generalizing the Poisson integral from free space (subsection 2.3) to bounded (subsection 2.4) regions. Choosing: (i) the function Ψ to be the Green function (15a) for the Poisson equation (15b)≡(7)

in any number of dimensions, i.e. the potential due to a point monopole of unit magnitude; (ii) the potential Φ due to an arbitrary source distribution (13a)≡(15c):

$$\Psi \equiv G: \qquad \nabla^2\Psi = \delta\left(\vec{x} - \vec{y}\right), \qquad \nabla^2\Phi = q\left(\vec{x}\right); \tag{15a-c}$$

the Green second identity leads to (15d) where is used the substitution property the N-dimensional Dirac unit impulse:

$$\begin{aligned}\int_{\partial D}(G\,\partial\Phi/\partial n - \Phi\,\partial G/\partial n)\;dS &= \int_D (G\,\nabla\Phi - \Phi\,\nabla G)\,dV \\ = \int_D G\,q\,dV - \int_D \Phi\left(\vec{y}\right)\,\delta\left(\vec{x} - \vec{y}\right)\,d^N\vec{y} &= \int_D G\,q\,dV - \Phi\left(\vec{x}\right). \end{aligned} \tag{15d}$$

The preceding result may be re-stated: the solution of the Poisson equation (15c) ≡ (16b) in a domain D satisfies the integral identity (15d) ≡ (16b):

$$\begin{aligned} D: \quad \nabla^2\Phi = q; \quad \Phi\left(\vec{x}\right) &= \int_D G\left(\vec{x};\vec{y}\right)\,q\left(\vec{y}\right)\,d^N\vec{y} \\ &+ \int_{\partial D}\left\{\Phi\left(\vec{y}\right)\left[\partial G\left(\vec{x};\vec{y}\right)/\partial\vec{y}\right]d\vec{S} - G\left(\vec{x};\vec{y}\right)\left[\partial\Phi\left(\vec{y}\right)/\partial\vec{y}\right].d\vec{S}\right\}, \end{aligned} \tag{16a,b}$$

where the: (i) volume integral represents the real source distribution in the domain D; (ii) the surface integral represents an equivalent source distribution on the boundary consisting of monopole (dipole) terms involving the Green function (its normal derivative). In particular in one (17a) / two (17b)/ three (17c) / higher (17d) dimensions the volume and surface sources (i)+ (ii) lead to the total potential:

$$N = 1\,,\ \sigma_1 = 1: \qquad \Phi\,(x) = \int_x^\infty (x-y)\;q\,(y)\;\;dy + \Phi\,(y)\,, \tag{17a}$$

$$\begin{aligned} N = 2\,,\ \sigma_2 = 2\pi: \qquad 2\,\pi\,\Phi\left(\vec{x}\right) &= \int_D q\left(\vec{y}\right)\;\log\left|\vec{x} - \vec{y}\right|\;d^2\vec{y} \\ + \int_{\partial D}\left\{\Phi\left(\vec{y}\right)\left[\left(\vec{x} - \vec{y}\right).\,\vec{n}\right]\left|\vec{x} - \vec{y}\right|^{-2} \right. &- \left.\left[\vec{n}\,.\left(\partial\Phi/\partial\,\vec{y}\right)\right]\log\left|\vec{x} - \vec{y}\right|\right\}\,dS, \end{aligned} \tag{17b}$$

$$\begin{aligned} N = 3,\ \sigma_3 = 4\pi: \qquad 4\,\pi\,\Phi\left(\vec{x}\right) &= -\int_D q\left(\vec{y}\right)\;\left|\vec{x} - \vec{y}\right|^{-1}\;d^3\vec{y} \\ + \int_{\partial D}\left\{\left[\vec{n}\,.\left(\partial\Phi/\partial\,\vec{y}\right)\right] - \Phi\left(\vec{y}\right)\right.&\left.\left[\left(\vec{x} - \vec{y}\right).\,\vec{n}\right]\left|\vec{x} - \vec{y}\right|^{-3}\right\}\left|\vec{x} - \vec{y}\right|^{-1}\,dS, \end{aligned} \tag{17c}$$

$$\begin{aligned} 2 \neq N = 3\,,\,4,..: \quad [\sigma_N\;(2-N)]\;\;\Phi\left(\vec{x}\right) &= \int_D q\left(\vec{y}\right)\;\left|\vec{x} - \vec{y}\right|^{2-N}\;d^N\vec{y} \\ &+ \int_{\partial D}\left\{\Phi\left(\vec{y}\right)(2-N)\left[\left(\vec{x} - \vec{y}\right).\,\vec{n}\right]\left|\vec{x} - \vec{y}\right|^{-N}\right. \\ &- \left.\left[\vec{n}\,.\left(\partial\Phi/\partial\,\vec{y}\right)\right]\right\}\left|\vec{x} - \vec{y}\right|^{2-N}\,dS_n, \end{aligned} \tag{17d}$$

where $\vec{n}$ is the unit outer normal to the boundary and respectively the Green functions (11a,b), (10d), (12c), (10f) were used.

2.5. **Volume and surface source distributions.**

The preceding result (16b) $\equiv$ (17a-d) is not the solution of Poisson's equation (16a) in the region D, because it is not possible specify independently the monopole strength Φ and dipole moment $-\partial\Phi/\partial n$ on the boundary. According to the unicity theorem for the Poisson or Laplace equation, only one of Φ and $\partial\Phi/\partial n$ can be specified on the boundary ∂D, and the other is then determined by the solution of the problem. Thus if both Φ and $\partial\Phi/\partial n$ are specified the two conditions will be incompatible (redundant) if the solutions of the Cauchy-Dirichlet and Neumann problems with given Φ and given $(\partial\Phi/\partial n)$ are distinct (coincide). It follows that the integral identity (16b) $\equiv$ (17a-d) is equivalent to Poisson's equation (16a) in region D, and not a solution of it; the region D can be compact or non-compact if an additional asymptotic condition is met (subsection 5.5). The integral equation (16b) for the potential becomes a solution of the Poisson equation (16a) if the surface integrals vanish, e.g. if $\partial D \equiv S_\infty$ is the surface at infinity, and the asymptotic condition is met by the potential. Omitting the boundary integral i.e. the last term on the r.h.s. of (16b), it follows that the solution of the Poisson equation (16a)$\equiv$(13a) in an unbounded medium is the first term of (16b), that coincides with (13b)$\equiv$(14a-d); the latter was obtained using the convolution property of distributions, instead of the Green second theorem for the Poisson equation, both being linear processes. Thus the potential as a solution of the Poisson equation (16a) with sources is specified by: (i) in one/two/three/more dimensions by (17a/b/c/d) in a compact domain; (ii) this applies as well in a non-compact domain if the potential satisfies suitable asymptotic conditions (subsection 5.5); (iii) it simplifies to (14a/b/c/d) in free space. The integral identity (17a/b/c/d) for the potential is equivalent to the Poisson equation. It is not a solution of the Poisson equation, unless the monopole Φ and dipole $\partial\Phi/\partial n$ potentials can be specified consistently on the boundary.

3. Hyperspherical or generalized Legendre polynomials

The irrotational fields are specified by a scalar potential in any dimension, leading to a straightforward generalization to dimensions higher than the third, that requires a modest extension of ordinary concepts; a starting point can be the use the Poisson integral (subsection 3.1) obtained from the Green function (subsection 2.2) for the Laplace operator with hyperspherical symmetry: it specifies the multipolar expansion in any dimension (subsection 3.2), that involves the hyperspherical Legendre polynomials. The properties of hyperspherical harmonics of arbitrary order are related to those of hyperspherical Legendre polynomials: (i) the free-space Green function is the generating function (subsection 3.3); (iii) it specifies the values of hyperspherical Legendre polynomials at particular points (subsection 3.4); (ii) the values at all points follow from explicit formulas for all the coefficients of the polynomial (subsection 3.5); (iv) these formulas involve powers, and if cos θ is used as argument can be written as linear combinations of cosines of multiple angles (subsection 3.6); (v) the hyperspherical Legendre polynomials of higher degree can also be obtained from those of lower degree using a recurrence formula (subsection 3.7); (vi) the recurrence formula can be combined with differentiation formulas, e.g.

in the rule of generation of higher order multidimensional multipoles (subsection 3.8); (vii) the original Legendre differential equation is generalized to the hyperspherical Legendre differential equation (subsection 3.9); (viii) the hyperspherical Legendre functions are a generalization allowing all parameters, viz. degree n and dimension α, to be complex (subsection 3.10), and relate to differential (integral) representations like the Rodrigues (Schlaffi) formulas.

3.1. **Multi-dimensional potential, field and force.**

The Green function (10f) for Laplace operator (7) $\equiv$ (8) applies in all dimensions except for the logarithmic potential in two dimensions (10c). It leads to the irrotational field:

$$N = 1, 2, , ... : \qquad \nabla\Phi = (1/\sigma_N) \int_D q\left(\vec{y}\right) \left|\vec{x} - \vec{y}\right|^{1-N} d^N \vec{y}\,, \quad (18)$$

for an arbitrary source (13b), that applies in any dimension, because (18) follows from (14a,c,d) [(14b)] for $N \neq 2$ $(N = 2)$. The force between two source distributions is:

$$\begin{aligned}\vec{F} &= (1/\sigma_N) \int_{D'} q'\left(\vec{x}\right) \nabla\Phi\left(\vec{x}\right) d^N \vec{x} \\ &= (1/\sigma_N)^2 \int_{D'} d^N \vec{x} \int_D d^N \vec{y} \left|\vec{x} - \vec{y}\right|^{1-N} q'\left(\vec{x}\right) q\left(\vec{y}\right). \qquad (19)\end{aligned}$$

The case of point sources (20a,b):

$$q\left(\vec{y}\right) = q\,\delta\left(\vec{x} - \vec{y}\right), \quad q'\left(\vec{x}\right) = q'\,\delta\left(\vec{x} - \vec{a}\right): \qquad (20\text{a,b})$$

$$N \neq 2: \qquad \Phi\left(\vec{x}\right) = \{ q/ [\sigma_N (2 - N)] \} \left|\vec{x} - \vec{y}\right|^{2-N}; \qquad (20\text{c-d})$$

$$N = 2: \qquad \Phi\left(\vec{x}\right) = (q/2\pi) \log \left|\vec{x} - \vec{y}\right|; \qquad (20\text{e-f})$$

$$N = 1\,, 2\,, .. : \nabla\Phi\left(\vec{x}\right) = (q/\sigma_N)\left|\vec{x} - \vec{y}\right|^{1-N}, \vec{F} = \left(q' q/\sigma_N^2\right)\left|\vec{x} - \vec{y}\right|^{1-N}, \quad (20\text{g,h})$$

leads to: (i) the potential (20e,f) [(20c,d)] in two (other) dimensions; (ii/iii) the field (20g) and the force (20h) in both cases.

3.2. **Definition of generalized or hyperspherical Legendre polynomials.**

The Green function corresponds to the potential of a unit point source $q = 1$ in two (20e,f)≡(21c,d) [other (20c,d)≡(21f,g)] dimensions and depends only on the modulus (21a) and angle (21e) of the position vectors of source and observer:

$$\left|\vec{x}\right| = R, \left|\vec{y}\right| = y, N = 2: \; G\left(\vec{x}; \vec{y}\right) = (2\pi)^{-1} \log\left|R^2 + y^2 - 2Ry\cos\theta\right|, \quad (21\text{a-d})$$

$$\begin{aligned}\vec{x}\,.\,\vec{y} = Ry\cos\theta, N \neq 2: \; G\left(\vec{x}; \vec{y}\right) &= [\sigma_N (N-2)]^{-1} \left|R^2 + y^2 - 2Ry\cos\theta\right|^{1-N/2} \\ &= \left[R^{1-N/2}/\sigma_N (N-2)\right] \left[1 - 2 (y/R) \cos\theta + (y/R)^2\right]^{1-N/2}. \qquad (21\text{e-g})\end{aligned}$$

The non-plane case (21g) leads to the hyperspherical Legendre polynomials that are defined by (22c):

$$\alpha = (N-3)/2\,;\ a<1: \quad \left|1-2\,a\ \cos\theta + a^2\right|^{-1/2-\alpha} = \textstyle\sum_{m=0}^{\infty} a^m\, P_{m,\alpha}\ (\cos\theta)\ , \tag{22a-c}$$

so that they coincide $1-\ N/2 = -\,1/2-\alpha$ with the original Legendre polynomials (24a) for $\alpha = 0$, i.e. in three dimensions $N = 3$ in (22a). Substitution of (22c) in (21g) with the parameters (22a,23a):

$$a \equiv y/R = \left|\vec{y}\right| / \left|\vec{x}\right| < 1\ : \\ G\left(\vec{x};\vec{y}\right) \equiv\ [\sigma_N\ (N-2)]^{-1}\ \textstyle\sum_{m=0}^{\infty} \left|\vec{x}\right|^{1-N/2-m}\ \left|\vec{y}\right|^{m}\ P_{m,\,(3-N)/2}\ (\cos\ ,\theta)\ , \tag{23a,b}$$

leads to the near field (23a) multipolar expansion (23b). Expanding (22c) by the binomial theorem to $O\left(a^4\right)$ leads to (24b):

$$P_{m,0}\ (\cos\ \theta) = P_m\ (\cos\ \theta):\qquad \left|1-2\,a\ \cos\ \theta + a^2\right|^{-1/2-\alpha} \\ = 1+\ (1/2+\alpha)\ (2\ \cos\ \theta -\ a)\ a\ \ +\ (1/2+\alpha)\ (3/2+\alpha)\ \left(a^2/2\right)\ (2\ \cos\ \theta - a)^2 \\ +(1/2+\alpha)\ (3/2+\alpha)\ (5/2+\alpha)\ \left(a^3/6\right)\ (2\ \cos\ \theta - a)^3 + O\ \left(a^4\right)\ . \tag{24a, b}$$

This identifies the hyperspherical Legendre polynomials of four lowest degrees:

$$P_{0,\alpha}\ (\cos\ \theta) = 1, \qquad P_{1,\alpha}\ (\cos\ \theta) =\ (1+2\alpha)\ \cos\ \theta, \tag{25a,b}$$

$$P_{2,\alpha}\ (\cos\ \theta) =\ (1/2+\alpha)\ \left[(3+2\alpha)\ \cos^2\theta - 1\right]\ , \tag{25c}$$

$$P_{3,\alpha}\ (\cos\ \theta) =\ (1/2+\alpha)\ (1+2\alpha/3)\ \cos\ \theta\ \left[(5+2\alpha)\cos^2\ \theta - 3\right]\ ; \tag{25d}$$

the first four hyperspherical (25a-d) reduce to the original Legendre polynomials for $\alpha = 0$ in (24a) or three dimensions $N = 3$ in (22a):

$$\alpha = 0:\ P_{0,0}\,(\cos\theta) = 1,\ P_{1,0}\,(\cos\theta) = \cos\theta, \\ P_{2,0}\,(\cos\theta) = \left(3\cos^2\theta - 1\right)/2,\ P_{3,0}\,(\cos\theta) = \cos\theta\left(5\cos^2\theta - 3\right)/2, \tag{26a,e}$$

The generating function (22c) for the hyperspherical Legendre polynomials has been used in (23b) in the far-field $a < 1$ or $y < R$ in (23a); it can also be applied in the near field (27a) leading to (27b):

$$a > 1: a^{-1-2\alpha}\ \left|1-\ (2/a)\ \cos\ \theta + 1/a^2\right|^{-1/2-\alpha} = \textstyle\sum_{m=0}^{\infty} a^{-1-2\alpha-m}\, P_{m,\alpha}\ (\cos\ \theta)\ . \tag{27a,b}$$

The corresponding near-field (28a) multipolar expansion for the N-dimensional Green function (21g) is (28b):

$$1/a \equiv R/y = \left|\vec{x}\right| / \left|\vec{y}\right| < 1 : \quad G\left(\vec{x};\vec{y}\right) =$$
$$\left[\sigma_N\ (N-2)\right]^{-1}\ \textstyle\sum_{m=0}^{\infty} \left|\vec{x}\right|^{-1+N/2+m} \left|\vec{y}\right|^{2-N-m} P_{m,\alpha}\left(\cos\ \theta\right)\ . \quad (28a,b)$$

Thus the Green function (21e-g) for the unit monopole in $N \neq 2$ dimensions (10e,f) has multipolar expansion (22b) [(28b)] for observer in the far-field (22a) [near field (28a)]. Both multipolar expansions involve the hyperspherical or generalized Legendre polynomials of which the four of lowest degree are (25a-d). The hyperspherical Legendre polynomials of all degrees appear both in the far (near) field multipolar expansion (23a,b) [(28a,b)], which is valid for a $(1/a)$ satisfying (30a-d). The multipolar expansion in the remaining two-dimensional case $N = 2$ corresponds to the Laurent series for functions of a complex variable [6]. The general N-dimensional case is considered with regard to several properties of the hyperspherical Legendre polynomials of arbitrary degree which relate to hyperspherical harmonics of any order and are obtained next (subsections 3.3-3.10).

3.3. **Hyperspherical Legendre generating function.**

The generating function for hyperpherical Legendre polynomials (22b) [(27b)] is specified by (29b) [(29d)] in the far (29a) [near (29c)] field:

$$a < 1 : \quad L\left(a,z;\alpha\right) \equiv \left|1 - 2\,az + a^2\right|^{-1/2-\alpha} = \textstyle\sum_{m=0}^{\infty} a^m P_{m,\alpha}\left(z\right); \qquad (29a,b)$$

$$a > 1 : L\left(a,z;\alpha\right) = a^{-1-2\alpha}\left[1 - 2z/a + 1/a^2\right]^{-1/2-\alpha} = \textstyle\sum_{m=0}^{\infty} a^{-1-2\alpha-m} P_{m,\alpha}\left(z\right)\ . \qquad (29c,d)$$

The series expansion (29b) [(29d)] applies in a circle with centre at the origin $a = 0$ and radius such as to exclude any singularities of the function. For absolute convergence this requires the expression in modulus to be non-zero (30a,b):

$$z \equiv \cos\theta : \ 1 - 2\,a\,z + a^2 = \left(1-a\right)^2 + 2a\ \left(1 - \cos\theta\right) > 0, \quad a \neq 1 \neq \cos\theta\,, \quad (30a\text{-}d)$$

which holds for (30c,d); for uniform convergence of (29b) [(29d)] the condition (30e) must hold in the closed interior (30f) [exterior (30g)] of the unit circle:

$$\left|1 - z\right| = \left|1 - \cos\ \theta\right| \ \geq\ \varepsilon\ >\ 0 : \quad \left|a\right|\ \leq\ 1 -\ \varepsilon\,;\ \left|a\right|\ \geq\ 1\ +\ \varepsilon\,, \qquad (30e\text{-}g)$$

and (30d) is strengthened to (30e). Thus the far (29b) [near (29d)] field expansion hyperspherical Legendre polynomials is: (i) absolutely convergent for (30c,d); (ii) is totally (i.e. absolutely and uniformly) convergent for (30e,f) [(30e,g)]. In the latter case (ii) it can be differentiated term-by-term, integrated or deranged, and limits may be taken under integral sign.

3.4. **Particular values of the hyperspherical Legendre polynomials.**

The generating function (29b) specifies:

$$z=1: \quad L\,(a,1;\,\alpha) = \,|1-a|^{-1-2\alpha} = \,\textstyle\sum_{m=0}^{\infty} (-)^m \begin{pmatrix} -1-2\alpha \\ \mathrm{m} \end{pmatrix} a^m, \tag{31}$$

$$z=-1: \quad L\,(a,-1;\,\alpha) = \,|1\,+\,a|^{-1-2\alpha} = \,\textstyle\sum_{m=0}^{\infty} \begin{pmatrix} -1-2\alpha \\ \mathrm{m} \end{pmatrix} a^m, \tag{32}$$

$$z=0: \quad L\,(a,0;\,\alpha) = \,\left|1+a^2\right|^{-1/2-\alpha} = \,\textstyle\sum_{m=0}^{\infty} \begin{pmatrix} -1/2-\alpha \\ \mathrm{m} \end{pmatrix} a^{2m}, \tag{33}$$

that is respectively the values of the hyperspherical Legendre polynomials at the points $z = \pm\,1,\ 0$:

$$z=1, \theta=0: \quad P_{m,\alpha}\,(1) = (-)^m \begin{pmatrix} -1-2\alpha \\ \mathrm{m} \end{pmatrix}$$
$$= (2\alpha+1)\,(2\alpha+2)\,....\,(2\alpha+m)\,/m!\,, \tag{34}$$

$$z=-1,\ \theta=\pi: \quad P_{m,\alpha}\ (-1)\ = \begin{pmatrix} -1-2\alpha \\ \mathrm{m} \end{pmatrix}$$
$$=\ (-)^m\ (2\alpha+1)\ (2\alpha+2)\ ....\ (2\alpha+m)\,/\ m\,!\,, \quad = (-)^m\ P_{m,\alpha}(1), \tag{35}$$

$$z=0\,,\ \theta=\pm\,\pi/2: \quad P_{2m+1,\alpha}\ (0) = 0\ ,$$
$$P_{2m,\alpha}\,(0) = \begin{pmatrix} -1/2-\alpha \\ \mathrm{m} \end{pmatrix} = (-)^m\,2^{-m}\,(2\alpha+1)\,(2\alpha+3)\,....\,(2\alpha+2m-1)\,/m!\,. \tag{36a,b}$$

These values (34,35,36a,b) can be checked for the first four hypersherical Legendre polynomials (25a-d). They simplify (24a) to:

$$\alpha=0: \qquad P_m\ (1) = 1, \qquad P_m\ (-1) = \ (-)^m\,,$$
$$P_{2m+1}\ (0) = 0,\ P_{2m}\ (0) = (-)^m\ (2m-1)\,!!/\ (2m)\,!!, \tag{37a-d}$$

for the original Legendre polynomials. In (37d) can be used:

$$(-)^m\ P_{2m}\,(0) = \ (2^{-m}/m\,!)\ (2m-1)\,!! =$$
$$= 1.3.5\ldots(2m-1)\,/\,(1.2.4\ldots2m) \equiv \ (2m-1)\,!!/\ (2\,m)\,!\,!, \tag{37e}$$

either double or single factorials.

3.5. **Explicit coefficients for the hyperspherical Legendre polynomials.**

Applying twice the binomial series to the generating function (29b) leads to the double sum (38b):

$$m=j+k: \quad \left|1-2\,a\,z+a^2\right|^{-1/2-\alpha} = \,\textstyle\sum_{j=0}^{\infty} \begin{pmatrix} -1/2-\alpha \\ \mathrm{j} \end{pmatrix} a^j\ (a-2\,z)^j$$
$$= \textstyle\sum_{j=0}^{\infty} \sum_{k=0}^{j} \begin{pmatrix} -1/2-\alpha \\ \mathrm{j} \end{pmatrix} \begin{pmatrix} j \\ \mathrm{k} \end{pmatrix} a^{j+k}\ (-2\,z)^{j-k}$$
$$= \textstyle\sum_{j=0}^{\infty} \sum_{k=0}^{j} \Big\{ a^{j+k}\ (-)^{2j-k}\ 2^{j-k}\ z^{j-k}$$
$$(\alpha+1/2)\ (\alpha+3/2)\,...\,(\alpha+j-1/2)\,/\,[k!\ (j-k)!] \Big\}$$

$$= \sum_{m=0}^{\infty} a^m \ \sum_{k=0}^{\leq m/2} (-)^k \ 2^{m-2k} \ z^{m-2k} \ (\alpha+1/2) \ (\alpha+3/2) \\ ... \ (\alpha+m-k-1/2) \ /[k\,! \ (m-2k)\,!] \qquad (38a,b)$$

where a change of index of summation was made (38a) and the terms with negative factorial $(m-2\,k)\,! \ = \infty$ for $2\,k \ > \ m$ are ommitted. Comparing (38b) and (29b) gives the explicit formula for the coefficients of the hyperspherical Legendre polynomials:

$$P_{m,\alpha} \ (z) \\ = \sum_{k=0}^{\leq m/2} (-)^k \, 2^{-k} z^{m-2k} \left[(2\alpha+1) \ (2\alpha+3) ... \ (2\alpha+2m-2k-1) \,/ k! \, (m-2k)\,!\right]. \qquad (39)$$

The case $\alpha = 0$ of (39) leads (24a) to the original Legendre polynomials:

$$P_m \ (z) = \ \sum_{k=0}^{\leq m/2} (-)^k \ 2^{-k} \ z^{m-2k} \ (2m-2k-1)\,!\,!/ \ [k! \ (m-2k) \ !] \ , \qquad (40a)$$

$$= \ \sum_{k=0}^{\leq m/2} (-)^k \ 2^{-m} \ z^{m-2k} \ (2m-2k)\,! \ / \left[(m-k)\,! \ k\,! \ \ (m-2k)\,!\right] , \ (40b)$$

where was made the simplification:

$$(2m-2k-1)\,!! \ \equiv \ (2m-2k-1) \ (2m-2k-3) \ .... \ 3.1 = \\ [(2m-2k)\,! \,/ \ (m-k)\,!] \ 2^{k-m}. \qquad (40c,d)$$

The general formula (39) can be checked against the hyperspherical Legendre polynomials of degrees up to three (25a-d), and apply as well to higher degrees.

3.6. **Explicit expressions in terms of cosines of angles and multiple angles.**

The explicit expression (39) for spherical harmonics is a polynomial of degree n of cos θ, viz.:

$$P_{m,\alpha} \ (\cos \ \theta) = \ \sum_{k=0}^{\leq m/2} (-)^k \ 2^{-k} \ \cos^{m-2k} \ \theta \ (2\alpha+1) \ (2\alpha+3) \\ ... \ (2\alpha+2m-2k-1) \,/ \ [k\,! \ (m-2k)\,!] \,. \qquad (41)$$

The powers of cosines can be expressed as linear combinations of cosines of multiple angles. Thus the hyperspherical Legendre polynomials can be expressed, as an alternative to (41), by a linear combination of cosines of angles multiple of θ up to mθ. This can be obtained most simply re-writing the generating function (29b,30a) in the form:

$$L \ (a, \ \cos \ \theta\,; \ \alpha) = \ \left|1 - a \ \left(e^{i\theta} + e^{-i\theta}\right) \ + a^2\right|^{-1/2-\alpha} \\ = \left|1 - a \ e^{i\theta}\right|^{-1/2-\alpha} \ \left|1 - a\, e^{-i\theta}\right|^{-1/2-\alpha} ; \qquad (42)$$

using again the binomial series, leads (38a) to the double sum:

$$\sum_{m=0}^{\infty} a^m\ P_{m,\alpha}\ (\cos\ \theta) =$$
$$\sum_{j,k=0}^{\infty}\ (-)^{j+k}\ \ a^{j+k} \begin{pmatrix} -1/2-\alpha \\ \mathrm{j} \end{pmatrix} \begin{pmatrix} -1/2-\alpha \\ \mathrm{k} \end{pmatrix} \exp\ \ [i\ \ (j-k)\ \ \theta]$$
$$= \sum_{m=0}^{\infty} a^m \sum_{k=0}^{m} \{\, (1/2+\alpha)\ \ (3/2+\alpha)\ ...\ (\alpha+k-1/2)$$
$$(1/2+\alpha)\,(3/2+\alpha)\,...\,(\alpha+m-k-1/2)\ /\ [k!\ \ (m-k)\ !]\,\}\ \exp\ \ [i\ \ (m-2k)\ \ \theta]\,. \tag{43a,b}$$

This simplifies to (44a) [(44b)] for even (odd) m:

$$P_{2m,\alpha}\ (\cos\ \theta) = \left\{2^{-2m}/\ (m\,!)^2\right\}\ [(1+2\alpha)\ \ (3+2\alpha)\ ...\ (2m+2\alpha-1)]^2$$
$$+\,2^{1-2m}\ \sum_{k=0}^{m-1}\cos\ [2\ (m-k)\ \theta]\ \{(1+2\alpha)\ \ (3+2\alpha)\ ...\ (2\alpha+2k-1)$$
$$(1+2\alpha)\ (3+2\alpha)\ ...\ \ \ (2\alpha+4m-2k-1)\}\ /\,[k!\ \ (2m-k)\,!\,]\}\ \ , \tag{44a}$$

$$P_{2m+1,\alpha}\ (\cos\ \theta) =\ \ 2^{-2m} \sum_{k=0}^{m}\cos\ \ [(2m-2k+1)\ \theta]\ \ \{(1+2\alpha)\ \ (3+2\alpha)$$
$$....\ \ (2k+2\alpha-1)\ \ (1+2\alpha)\ \ \ (3+2\alpha)\ ...\,(2\alpha+4m-2k+1)\,/\ \ [k\,!\,(2m-k+1)!]\}\ . \tag{44b}$$

These are the explicit formulas (44a,b) for the hyperspherical Legendre polynomials in terms of cosines of multiples of the angle θ, instead of powers of cos θ in (41). From (44a) [(44b)] follow alternative expressions for the hyperspherical Legendre polynomials of degree two (25c)≡(45a) [three (25d)≡(45b)]:

$$P_{2,\alpha}\ (\cos\ \theta) =\ (1/2+\alpha)\ \ [(3/2+\alpha)\ \cos\ (2\,\theta)+1/2+\alpha]\ , \tag{45a}$$

$$P_{3,\alpha}\ (\cos\ \theta) =\ (1/2+\alpha)\ \ (3/2+\alpha)\ \ \{\,[\,(5/2+\alpha)\,/3]\ \ \cos\ (3\,\theta)+(\alpha+1/2)\cos\ \theta\}\ . \tag{45b}$$

The particular case (46a) leads from (41) [(44a,b)] to (46b) [(47a,b)]:

$$\alpha = 0: \quad P_{m,0}\ (\cos\ \theta)\ \equiv\ P_m\ (\cos\ \theta) =$$
$$\sum_{k=0}^{\le m/2} (-)^k\ 2^{-m}\ \{(2m-2k)\,!/\ \ \ [k!\ \ (m-k)\,!\ \ (m-2k\,!)\,]\}\cos^{2m-k}\ \theta, \tag{46a,b}$$

$$P_{2m,0}\ (\cos\ \theta) \equiv P_{2m}\ (\cos\ \theta) = 2^{1-4m}\ \left[(2m)\,!/\,(m\,!)^2\right]^2 + 2^{1-4m}\ \sum_{k=0}^{m-1}$$
$$\left\{\,(2k)\,!\ (4m-2k)\,!\,/\ [k!\ \ (2m-k)\,!]^2\right\}\ \cos\ \ [2\ \ (m-k)\ \ \theta]\ , \tag{47a}$$

$$P_{2m+1}\ (\cos\ \theta)\ \ \equiv P_{2m+1}\ (\cos\ \theta) = 2^{-1-4m}\ \ \ \sum_{k=0}^{m}\cos\ [\,(2m-2k+1)\ \theta]$$
$$\left\{\,(2k)\,!\ (4m-2k+2)\,!\,/\ [k!\ \ (2m-k+1)\,!]^2\right\}\ , \tag{47b}$$

that apply to the original Legendre polynomials.

3.7. **Recurrence formula for the hyperspherical harmonics.**

The hyperspherical Legendre polynomials of higher degree may be obtained from those of lower degree via a recurrence formula. The latter can be obtained differentiating with regard to a the uniformly convergent (30e,f) series expansion for the generating function (29b) leading to:

$$\sum_{m=1}^{\infty} m\, a^{m-1} P_{m,\alpha}\,(z) = \,(d/da)\,\left\{\,\left|1 - 2\,a\,z + a^2\right|^{-1/2-\alpha}\right\}$$
$$= \,(1+2\alpha)\,\,(z-a)\,\,\left|1 - 2\,a\,z + a^2\right|^{-3/2-\alpha}; \qquad (48a)$$

this is equivalent to:

$$\left(1 - 2\,a\,z + a^2\right)\,\sum_{m=0}^{\infty}(m+1)\,\,a^m\,\,P_{m+1,\alpha}\,(z) =$$
$$(1+2\,\alpha)\,\,(z-a)\,\sum_{m=0}^{\infty} a^m\,\,P_{m,\alpha}\,(z)\,\,. \qquad (48b)$$

Equating the coefficients of powers of a in (48b) leads to:

$$(m+1)\,\,P_{m+1,\alpha}\,(z) - 2\,m\,z\,P_{m,\alpha}\,(z) + \,(m-1)\,\,P_{m-1,\alpha}\,(z) =$$
$$(1+2\alpha)\,\,\left[z\,P_{m,\alpha}\,(z) - P_{m-1,\alpha}\,(z)\right]\,. \qquad (48c)$$

This simplifies to the recurrence formula (49) for hyperspherical Legendre polynomials:

$$(m+1)\,\,P_{m+1,\alpha}\,(z) = (1+2\,m+2\,\alpha)\,zP_{m,\alpha}\,(z) - (m+2\,\alpha)\,P_{m-1,\alpha}\,(z)\,\,. \qquad (49)$$

The particular case (50a):

$$\alpha = 0:\,\,(m+1)\,\,P_{m+1}\,(z)\,\,=\,\,(1+2\,m)\,\,z\,P_m\,(z)\,\,-\,\,m\,P_{m-1}\,(z)\,\,, \qquad (50a,b)$$

is the recurrence formula (50b) for the original Legendre polynomials. The more general formula (49) can be used to obtain the hyperpherical Legendre polynomials of higher degree from those of lower degree, e.g. (25b,c,d) from (25a).

3.8. **Four differentiation formulas for hyperspherical Legendre polynomials.**

The series for the generating function (29b) is uniformly convergent with regard to a (z) in (30f) [(30e)], and thus can be differentiated term-by-term (48a) [(51a)] with regard to a (z):

$$P^{'}_{m,\alpha}\,(z) \equiv d\,\,\left[P_{m,\alpha}\,(z)\right]/dz: \quad \sum_{m=0}^{\infty} a^m\,\,P^{'}_{m,\alpha}\,(z)\,\,=\,\,(d/dz)$$
$$\left\{\,\left|\,1 - 2az + a^2\right|^{-1/2-\alpha}\right\} = \,(1+2\alpha)\,\,a\,\,\left|\,1 - 2az + a^2\right|^{-3/2-\alpha}. \qquad (51a,b)$$

Using (29b) in (51b) leads (48b) to the equivalent form (51c):

$$(z-a)\ \sum_{m=0}^{\infty} a^m\ P'_{m,\alpha}\ (z) = \ (1+2\alpha)\ \left[(z-a)\,/\,\left(1-2\,a\,z+a^2\right)\right]\ a \sum_{m=0}^{\infty} a^m\ P_{m,\alpha}\ (z) = \sum_{m=0}^{\infty} (m+1)\ a^{m+1}\,P_{m+1,\alpha}\ (z)\ . \quad (51c)$$

Equating the coefficients of powers of a leads to:

$$mP_{m,\alpha}\,(z) = zP'_{m,\alpha}\,(z) - P'_{m-1,\alpha}\,(z)\,;\quad \alpha=0 : mP_m\,(z) = zP'_m\,(z) - P'_{m-1}\,(z)\,, \quad (52a\text{-}c)$$

that is the first differentiation formula for hyperspherical (52a) [original (52b)] Legendre polynomials; since it is independent of α it coincides with the differentiation formula (52c) for the original Legendre polynomials.

Differentiating the recurrence formula (49) with regard to z leads to:

$$(m+1)\ P'_{m+1,\alpha}\ (z)\ = \ (1+2\,m+2\,\alpha)\ z\,P'_{m,\alpha}\ (z) + \ (1+2m+2\alpha)\ P_{m,\alpha}\ (z)\ - \ (m+2\alpha)\ P'_{m-1,\alpha}\ (z)\ . \quad (53)$$

Substituting $P'_{m,\alpha}\ (z)$ from (52a) leads to the second differentiation formula for generalized (54a) [original (54b)] Legendre polynomials:

$$P'_{m+1,\alpha}\ (z) - P'_{m-1,\alpha}\ (z) = \ (1+2m+2\alpha)\ P_{m,\alpha}\ (z)\ . \quad (54a)$$

$$\alpha = 0 : \quad P'_{m+1}\ (z)\ -\ P'_{m-1}\ (z)\ =\ (1+2\,m)\ P_m\ (z)\ . \quad (54b,c)$$

Adding (54a) to (52a) yields the third differentiation formula for hyperspherical (54a) [original (54b)] Legendre polynomials:

$$P'_{m+1,\alpha}\,(z)\ -\ z\ P'_{m,\alpha}\ (z)\ =\ (1+m+2\alpha)\ P_{m,\alpha}\ (z)\ , \quad (55a)$$

$$\alpha = 0 : \quad P'_{m+1}\ (z)\ -\ z\,P'_m\ (z)\ =\ (1+m)\ \ P_m\ (z)\ . \quad (55b,c)$$

There is one recurrence (49) [(50a,b)] and there are four differentiation (52a;54a; 55a;58) [(52b,c; 54b,c; 55b,c; 57b,c)] formulas for hyperspherical (original) Legendre polynomials. The last (58) [(57b,c)] is obtained next.

The rule for generation of multipoles in three-dimensions corresponds to the differentiation formula for Legendre polynomials (56b):

$$z \equiv \cos\ \theta : \quad (m+1)\ \left[P_{m+1}\ (z) - z\ P_m\ (z)\right]\ =\ \left(z^2-1\right)\ P'_m\ (z)\ , \quad (56a,b)$$

using the variable (56a). The extension to hyperspherical Legendre polynomials is obtained multiplying (52a) by z and adding (55a) with m replaced by $m-1$:

$$\left(z^2-1\right)\ P'_{m,\alpha}\ (z) = \ m\,z\,P_{m,\alpha}\ (z)\ -\ (m+2\alpha)\ P_{m-1,\alpha}\ (z)\ . \quad (57)$$

Substituting $P_{m-1,\alpha}$ from (49) in (57) yields the fourth differentiation formula for generalized (58) [original (56b)] Legendre polynomials:

$$\left(z^2-1\right) P'_{m,\alpha}(z) = (m\ +1)\, P_{m+1,\alpha}\ (z) - (1+m+2\alpha)\, z\, P_{m+1,\alpha}(z)\,. \quad (58)$$

The recurrence formula (58) for hyperspherical Legendre polynomials, corresponds to the rule of generation of multidimensional multipoles, and it reduces to (56b) for $\alpha = 0$.

3.9. **Hyperspherical Legendre differential equation.**

The hyperspherical Legendre polynomials satisfy a linear second-order ordinary differential equation with variable coefficient that generalizes the original Legendre differential equation. It can be obtained differentiating (57):

$$\left(z^2-1\right) P''_{m,\alpha}(z) + 2z\, P'_{m,\alpha}(z) = mz\, P'_{m,\alpha}(z) + m\, P_{m,\alpha}(z) - (m+2\alpha)\, P'_{m-1,\alpha}(z). \quad (59)$$

Substituting $P'_{m-1,\alpha}(z)$ from (52a) leads to:

$$\left(z^2-1\right)\ P''_{m,\alpha}(z) + 2z\, P'_{m,\alpha}(z) = mz\, P'_{m,\alpha}(z) + m\, P_{m,\alpha}(z) + (m+2\alpha)\left[m\, P_{m,\alpha}(z) - z\, P'_{m,\alpha}(z)\right], \quad (60)$$

that simplifies to:

$$\left(1-z^2\right)\ P''_{m,\alpha}(z) - 2\,(1+\alpha)\, z\, P'_{m,\alpha}(z) + m\,(m+1+2\alpha)\ P_{m,\alpha} = 0. \quad (61)$$

The differential equation (61) satisfied by the hyperspherical Legendre polynomials simplifies to (62b) for (62a).

$$\alpha = 0: \qquad \left(1-z^2\right)\ P''_{m}(z) - 2zP'_{m}\,(z) + m\,(m+1)\ P_m(z) = 0, \quad (62a,b)$$

that is the original Legendre differential equation.

3.10. **Rodrigues and Schlaffi integrals.**

Since (29b) is the Stirling-Maclaurin series series expansion of the generating function in powers of a, the coefficient is given by:

$$P_{m,\alpha}\ (z) = \lim_{a\to 0}\ (m\,!)^{-1}\ (\partial^m/\partial a^m)\ \left\{\left[1-2a\,z+a^2\right]^{-1/2-\alpha}\right\}. \quad (63a)$$

The Cauchy third theorem may be used in (63a) leading to a loop integral going clockwise the around the point $\zeta = a$:

$$(m\,!)^{-1}\ (\partial^m/\partial a^m)\ \left[\left(1-2az+a^2\right)^{-1/2-\alpha}\right]$$
$$= (2\,\pi\,i)^{-1}\ \int^{(a+)}\left(1-2\,\zeta\,z+\zeta^2\right)^{-1/2-\alpha}\ (\zeta-a)^{-m-1}\ d\zeta\,; \qquad (63b)$$

substitution of (63b) in (63a) specifies the hyperspherical Legendre polynomial as complex loop integral going clockwise around the origin:

$$P_{m,\alpha}\ (z)\ =\ (2\,\pi\,i)^{-1}\ \int^{(0+)}\left(1-2\,\zeta\,z+\zeta^2\right)^{-1/2-\alpha}\ \zeta^{-m}\,d\zeta\,. \qquad (63c)$$

The expression (63c) holds for complex α, not necessarily an integer, and may be used to define the hyperspherical Legendre functions; likewise m may also be complex in (63c), leading to fractional derivatives [2,3,4,16]. The case of non-integral m or α leads to branch-points and branch-cuts in the integrand (63c) and affects the path of integration. Differentiating term-by-term after using the binomial theorem leads to:

$$(d^m/dz^m)\ \left(z^2-1\right)^m = (d^m/dz^m)\ \textstyle\sum_{k=0}^{m}(-)^k\ z^{2m-2k}\,m\,!\,/\,[k\,!\ (m-k)\,!]$$
$$= \textstyle\sum_{k=0}^{m}(-)^k\ z^{m-2k}\ (2m-2k)\,!\,m\,!/\ [(m-2k)\,!\ k\,!\ (m-k)\,!]\,. \qquad (64a)$$

Comparison with (46b) specifies the Rodrigues [15] formula (64c) for Legendre polynomials (64b):

$$P_m\ (z) =\ (2^{-m}/m\,!)\ d^m\ \left[\left(z^2-1\right)^m\right]/dz^m$$
$$= (2\pi\,i)^{-1}\ 2^{-m}\ \int^{(z+)}\left(\zeta^2-1\right)^m\ (\zeta-z)^{-m}\ d\zeta\,, \qquad (64b,c)$$

corresponding to (64c) the Schlaffi [17] integral, on account of the Cauchy third theorem. The extension of (64b) [(64c)] to non-integral m involves [2-4] differintegrations [deformation of the path of integration around the branch-points $\zeta = \pm\,1$ and branch-cuts associated with them].

4. Multipoles in hyperspherical and hypercylindrical coordinates

The multidimensional multipolar expansion is specified: (i) exactly to all orders in terms of hyperspherical Legendre polynomials (section 3); (ii) using the Taylor series for the Green function (subsection 4.1), for the four lowest orders, viz. multidimensional monopoles (subsection 4.4), dipoles (subsection 4.5), quadrupoles (subsection 4.6) and octupoles (subsection 4.7). Their potentials and fields in any dimension can be expressed in terms of hyperspherical (subsection 4.2) or hypercylindrical (subsection 4.3) coordinates.

4.1. **Multidimensional multipolar expansion.**

The multipolar expansion may also be obtained in integral form, using derivatives of the Green function (10f) of the Laplace operator:

$$(\partial/\partial x_i)\ \left[\sigma_N\ G\left(\vec{x},\vec{y}\right)\right] = (2-N)^{-1}\,\partial/\partial x^i\left\{\left|\vec{x}-\vec{y}\right|^{2-N}\right\}$$
$$= \left|\vec{x}-\vec{y}\right|^{-N}(x_i-y_i)\,, \qquad (65a)$$

$$\left(\partial^2/\partial x_i\partial x_j\right)\left[\sigma_N G\left(\vec{x};\vec{y}\right)\right] =$$
$$-N\left|\vec{x}-\vec{y}\right|^{-N-2}(x_i-y_i)\ (x_j-y_j)+\left|\vec{x}-\vec{y}\right|^{-N}\delta_{ij}, \quad (65b)$$

where (65b) involves the identity matrix. This specifies the first three terms of the Taylor series:

$$e_{ri} = x_i/\left|\vec{x}\right|: \qquad \sigma_N\,G\left(\vec{x};\vec{y}\right) = \left|\vec{x}\right|^{2-N}/\,(2-N) + x_i\,y_i\,\left|\vec{x}\right|^{-N}$$
$$+\ (1/2)\ x_i\,x_j\ \left(\delta_{ij}\,y^2 - N\ y_j\ y_i\right)\ \left|\vec{x}\right|^{-N-2} + O\ (x_i\,x_j\,x_k)\ . \quad (66a,b)$$

Substituting in the generalized Poisson integral (13b) specifies:

$$\Phi\left(\vec{x}\right) = \left|\vec{x}\right|^{2-N}\left\{P_0/\,(2-N)+\left|\vec{x}\right|^{-2}\vec{P}_1\,.\,\vec{x}+\left|\vec{x}\right|^{-4}P_{ij}x_ix_j+O\left(\left|\vec{x}\right|^{-3}\right)\right\} \quad (67)$$

the N-dimensional multipole expansion for the potential of an arbitrary source distribution:

$$\sigma_N\,P_0 \equiv \int_D\ q\left(\vec{y}\right)\ d^N\vec{y}\,, \quad \sigma_N\ \vec{P}_1 \equiv \int_D\ \vec{y}\ \ q\left(\vec{y}\right)\ d^N\vec{y}\,, \quad (68a,b)$$
$$2\,\sigma_N\,P_{ij} \equiv \int_D\ \left(\delta_{ij}\,y^2 - N\ y_i\ y_j\right)\ q\left(\vec{y}\right)\ d^N\vec{y}\,, \quad (68c)$$

with moments for the monopole (68a), dipole (68b) and quadrupole (68c) terms. The potentials and fields of multipoles in the plane (space) are conveniently represented using polar [spherical or cylindrical] coordinates. Their extension to hypersherical (subsection 4.2) and hypercylindrical (subsection 4.3) coordinates likewise specifies the potential and fields of multi-dimensional monopoles (subsection 4.4), dipoles (subsection 4.5) quadrupoles (subsection 4.6), octupoles and higher-order multipoles (subsection 4.7).

4.2. **Radial, longitude and multi-colatitude coordinates.**

The polar (spherical) coordinates are specified by (69a,b) [(70a-c)]:

$$0 \le r < \infty\ ,\ 0 \le \varphi < 2\pi: \quad x_1 = r\ \cos\varphi, \quad x_2 = r\ \sin\ \varphi\,, \qquad (69a,b)$$

$$0 \leq R < \infty, \quad 0 \leq \theta \leq \pi, \ 0 \leq \varphi \leq 2\pi : \quad x_1 = R \ \cos\theta, \quad x_2 = R \ \sin\theta\cos\varphi, \quad x_3 = R\sin\theta\sin\varphi, \tag{70a-c}$$

where R (r) is the distance from the origin (axis) and φ (θ) the longitude (colatitude). The four-dimensional spherical coordinates have two colatitudes:

$$0 \leq R < \infty, \quad 0 \leq \theta_1, \theta_2 < \pi, \quad 0 \leq \varphi < 2\pi : \qquad x_1 = R \ \cos \ \theta_1 \ , \\ x_2 = R \ \sin \ \theta_1 \cos\theta_2, \ x_3 = R\sin\theta_1\sin\theta_2\cos\varphi, \ x_4 = R\sin\theta_1\sin\theta_2\sin\varphi. \tag{71a-d}$$

The hyperspherical coordinates:

$$0 \leq R < \infty, \quad 0 \leq \theta_1, \, \ldots, \theta_{N-2} \leq \pi, \ 0 \leq \ \varphi < 2\pi : \quad x_1 = R \ \cos \ \theta_1, \tag{72a}$$

$$n = 2, \, \ldots, N-2 : \qquad x_n = R \ \sin \ \theta_1 \ \sin \ \theta_2 \ldots \ \sin \ \theta_{n-1} \ \cos \ \theta_n, \tag{72b}$$

$$\{x_{N-1} \, , \ x_N\} = R \ \sin \ \theta_1 \ \sin \ \theta_2 \ldots. \ \sin \ \theta_{N-3} \ \sin \ \theta_{N-2} \ \{\cos \varphi \ , \ \sin\varphi\} \ , \tag{72c-d}$$

have one longitude, (N-2) colatitudes and one radial distance:

$$R \equiv \left\{ \ \left| (x_1)^2 + \ldots \ + (x_N)^2 \right|^{1/2} : \quad \left| (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 \right|^{1/2}, \\ \left| (x_1)^2 + (x_2)^2 + (x_3)^2 \right|^{1/2}, \qquad \left| (x_1) + \ (x_2)^2 \right|^{1/2} \right\}, \tag{73a-d}$$

that applies in two (69a,b;73d), three (70a-c;73c), four (71a-d;73b) and N dimensions (72a-d;73a). The last $N - M - 1$ dimensions of hyperspherical coordinates (74a):

$$0 \leq \varphi < 2\pi, \quad 0 \leq \ \theta_M, \ \theta_{M+1}, \ldots., \ \theta_{N-2} \ \leq \ \pi : \\ R_M = R \cos \ \theta_1 \ \cos \ \theta_2 \ldots \ \cos \ \theta_{M-1}, \tag{74a,b}$$

lie on an hypersphere of dimension $N - M - 1$ with radius (74b).

4.3. **Hyperspherical and hypercylindrical coordinates.**

The cylindrical coordinates in space add a Cartesian coordinate z orthogonal to the plane (69a,b) of polar coordinates (69a-c):

$$-\infty < z < +\infty, 0 < r < \infty, 0 \leq \varphi < 2\pi : \ x_1 = z, \ x_2 = R \ \cos \ \varphi, \ x_3 = R\sin\varphi. \tag{75a-c}$$

The four-dimensional cylindrical coordinates have one colatitude (69a-d):

$$-\infty < z < + \infty, \ -0 \ \leq r < \infty, \quad 0 \leq \varphi \ \leq 2\pi, \quad 0 \leq \theta \ \leq \pi : \\ x_1 = z, \quad x_2 = r \ \cos \ \theta, \quad x_3 = r \ \sin \ \theta \ \cos \ \varphi \, , \quad x_4 = r \ \sin \ \theta \ \sin \ \varphi \, . \tag{76a-d}$$

The hypercylindrical coordinates:

$$-\infty < z < \infty,\ 0 \le r < \infty,\ 0 \le \theta_1\theta_2, ..., \theta_{N-3} \le \pi,\ 0 \le \varphi \le 2\pi :$$

$$x_1 = z \quad , x_2 = r\ \cos\ \theta_1, \tag{77a,b}$$

$$n = 2, \dots, N-3 : \qquad x_{n+1} = r\ \sin\ \theta_1\ \sin\ \theta_2 \dots \sin\ \theta_{n-1}\ \cos \theta_n\,, \tag{77c}$$

$$\{x_{N-1},\ x_N\} = r\ \sin\ \theta_1\ \sin\ \theta_2,\ \dots\ \sin\ \theta_{N-3}\ \{\cos\varphi\,,\ \sin\varphi\}\ , \tag{77d,e}$$

have one Cartesian coordinate, one longitude, $N-3$ colatitudes and one radial distance from the axis:

$$r \equiv \left\{ \left|(x_2)^2 + \dots + (x_N)^2\right|^{1/2} : \left|(x_2)^2 + (x_3)^2 + (x_4)\right|^{1/2}, \left|(x_2)^2 + (x_3)^2\right|^{1/2} \right\}, \tag{78a-c}$$

that applies in three (75a-c;78c), four (76a-d;78b) and N dimensions (77a-e;78a). The hyperspherical (72a-d;73a) [hypercylindrical (77a-e;78a)] coordinates have: (i) the same longitude φ and $(N-3)$ colatitudes $\theta_1, \dots,\ \theta_{N-3}$; (ii) they differ only on the distance from the origin (73a) [from the axis (78a)]; (ii) the last colatitude θ_{N-2} in hyperspherical coordinates is replaced by the Cartesian coordinate z for hypercylindrical coordinates. These are related by the first colatitude as for spherical and cylindrical coordinates in space:

$$R^2 = r^2 + z^2, \qquad z = R\ \cos\ \theta_1, \qquad r = R\ \sin\ \theta_1\,. \tag{79a-c}$$

$$\vec{e}_R = \vec{e}_z \cos\theta_1 + \vec{e}_r \sin\theta_1,\ \ \vec{e}_{\theta_1} = -\ \vec{e}_z \sin\theta_1 + \vec{e}_r \cos\theta_1, \tag{79d-e}$$

The last $N-M-2$ hypercylindrical coordinates (80a):

$$0\ \le \varphi < 2\pi, 0 \le \theta_M, \theta_{M+1}, ..., \theta_{N-3} \le \pi : r_M = r \sin\theta_1 \sin\theta_2 ... \sin\theta_{M-1}, \tag{80a,b}$$

lie on an hypersphere of dimension $N-M-2$ with radius (80b).

4.4. **Potential and field of a multidimensional monopole.**

The multidimensional monopole has a potential corresponding to the first term (68a) of the multipolar expansion (67) involving (25a) the hyperspherical Legendre polynomial of degree zero:

$$\begin{aligned}(2-N)\ \Phi_0\left(\vec{x}\right) &= (P_0/\sigma_N)\ \left|\vec{x}\right|^{2-N} = (P_0/\sigma_N)\ R^{2-N} \\ &= (P_0/\sigma_N)\ R^{2-N}\ \sigma_N^{-1}\ P_{0,\,(N-3)/2}\,(\cos\,\theta_1)\ ;\end{aligned} \tag{81}$$

the corresponding to the field is (82a) [(82b)] in hyperspherical (hypercylindrical) coordinates:

$$\vec{U}_0 = \nabla\,\Phi_0 = (P_0/\sigma_N)\ x_i\ \left|\vec{x}\right|^{-N} = (P_0/\sigma_N)\ R^{1-N}\ \vec{e}_R, \tag{82a}$$

$$\vec{U}_0 = (P_0/\sigma_N) \ R^{1-N} \left(\vec{e}_z \ \cos\theta_1 + \vec{e}_r \ \sin\theta_1\right) = (P_0/\sigma_N) \ \left|r^2+z^2\right|^{1/2-N/2} \left(\vec{e}_z \ z + \vec{e}_r \ r\right) ; \quad (82b)$$

the multidimensional monopole:

$$S_N = \sigma_N R^{N-1}; \quad \int_{\left|\vec{x}\right|=R} \left[\left(\vec{U}_0 \cdot \vec{x}\right)/R\right] dS = \sigma_N^{-1} P_0 R^{1-N} S_N = P_0, \quad (83a,b)$$

has flux (83b) across an hypersphere of area (83a) which is a constant independent of the radius. The potential (81) and field (82a) reduces for $N = 3$ to the point-monopole in space. The line monopole in the plane corresponds to the logarithmic potential (10a-c), and is excluded $N = 2$ from the potential (81), that applies to all other dimensions $N \neq 2$. The field (82a,b) applies to all dimensions including $N = 2$.

4.5. **Potential and field of a multidimensional dipole.**

The potential of a dipole is the second-term (68b) of the multipolar expansion (67):

$$\Phi_1\left(\vec{x}\right) = \sigma_N^{-1} \ P_i x_i \ \left|\vec{x}\right|^{-N} = (P_1/\sigma_N) \ R^{1-N} \cos\theta_1 = [P_1/\sigma_N \ (N-2)] \ R^{1-N} \ P_{1,(N-3)/2} \ (\cos\theta_1) \ , \quad (84a\text{-}d)$$

and: (i) is specified in hyperspherical coordinates with axis along the dipole moment by (84b); (ii) involves (84c) the hyperspherical Legendre polynomial of first degree (25b); (iii) corresponds (84d) to the monopole field (82b) component in the axial direction. The corresponding dipolar field:

$$\vec{U}_1\left(\vec{x}\right) = \nabla\Phi_1 = \sigma_N^{-1} \left(\partial/\partial\vec{x}\right) \left\{\left(\vec{P}_1 \cdot \vec{x}\right) \left|\vec{x}\right|^{-N}\right\} = \sigma_N^{-1} \left[\vec{P}_1 - N\left(P_1 . \vec{x}\right) \vec{x}/R^2\right] R^{-N}, \quad (85)$$

has components parallel (86a) [transverse (86b)] to the axis:

$$U_{1z} = -\ (P_1/\sigma_N) \left(N \ \cos^2\theta_1 - 1\right) R^{-N} \cos\theta_1, \quad (86a)$$
$$U_{1z} = -\ (P_1/\sigma_N) \ N \ R^{-N} \sin\theta_1, \quad (86b)$$

corresponding to hypercylindrical components. The particular case $N = 2$ $(N = 3)$ corresponds to a dipole in the plane (space).

4.6. **Potential and field of a multi-dimensional quadrupole.**

The potential (67) of a multidimensional quadrupole (68c) is given by:

$$\sigma_N\ \Phi_2\left(\vec{x}\right) = \left(N\ P_{ij}\,x_i\,x_j - P_{ii}\,R^2\right)\ R^{-2-N}$$
$$= \left\{N\ \left[P_{11}\ (x_1)^2 + 2\,P_{12}\,x_1\,x_2 + P_{22}\ (x_2)^2\right] - (P_{11}+P_{22})\ R^2\right\}\ R^{-2-N}, \quad (87)$$

choosing the (x_1, x_2)-plane so as to contain the two quadrupole axis. The case of a longitudinal quadrupole (88a,b) simplifies potential (87) to (88c):

$$P_{11} \equiv P_2\,,\ P_{12} = 0 = P_{22}: \quad \Phi_2\left(\vec{x}\right) = (P_2/\sigma_N)\ R^{-N}\ \left(N\cos^2\theta_1 - 1\right),$$
$$= (P_2/\sigma_N)\ \left[R^{-N}/\ (N/2-1)\ \right]\ P_{2,\,(N-3)/2}\ (\cos\ \theta_1)\ , \quad \text{(88a-c)}$$

involving (22a) the hyperspherical Legendre polynomial of second degree (25c). The multidimensional quadrupole potential corresponds to: (i) the third term in the multipolar expansion (67); (ii) the component along the axis of the dipolar field (86a), replacing P_1 by $-\,P_2$; (iii) the generalized Legendre polynomial (25c) of degree two. The corresponding field has (79d,e) hyperspherical (hypercylindrical) components (89a) [(89b)]:

$$\vec{U}_2 = \nabla\Phi_2 = -\,(P_2\ N/\sigma_N)\,R^{-N-1}\left[\vec{e}_R\ \left(N\cos^2\theta_1 - 1\right) + \vec{e}_{\theta_1}\ \sin(2\theta_1)\right], \quad \text{(89a)}$$
$$= -\ (P_2\,N/\sigma_N)\ R^{-N-1}$$
$$\left\{\vec{e}_z\ \cos\theta_1\left[(N+2)\cos^2\theta_1 - 3\right] - \vec{e}_r\ \sin\theta_1\left[(N+2)\cos^2\theta_1 - 1\right]\right\}. \quad \text{(89b)}$$

The particular case $N = 3$ corresponds to an axial quadrupole in space. The potential of an axial multidimensional octupole follows from the axial component.

4.7. **Potential of octupole and other multipoles.**

The potential of an axial octupolar field follows from the axial component of the quadrupolar field (89b) replacing P_1 by $-\,P_3$:

$$\Phi_3\left(\vec{x}\right) = (P_3\,N/\sigma_N)\ R^{-N-1}\ \cos\ \theta_1\ \left[(N+2)\ \cos^2\ \theta_1 - 3\right], \quad (90)$$

in agreement with the spatial case $N = 3$. In the plane (91a,92a) the potential of the axial dipole (84c) [quadrupole (88b)] is (91b) [(92b)]:

$$N = 2 : \Phi_1\,(r,\varphi) = (P_1/2\pi\,r)\cos\varphi = Re\left\{(P_1/2\pi)\,r^{-1}e^{-i\varphi}\right\} = Re\,(P_1/2\pi z)\,, \quad \text{(91a,b)}$$

$$\sigma_2 = 2\,\pi: \quad \Phi_2\ (r,\,\varphi) = \left(P_2/2\,\pi\,r^2\right)\ \left(2\ \cos^2\ \varphi - 1\right) = \left(P_2/2\,\pi\,r^2\right)\ \cos\ (2\varphi)$$
$$= Re\ \left\{(P_2/2\,\pi)\ r^{-2}\ e^{-2i\varphi}\right\} = Re\ \left(P_2/2\pi\ z^2\right). \quad \text{(92a,b)}$$

and is the real part of a simple complex function. Likewise for an octupole (90) in the plane the potential is the real part of the complex function:

$$\Phi_3\ (r,\,\varphi) = Re\ \left(P_3/\pi\ z^3\right) = Re\ \left\{(P_3/\pi)\ r^{-3}\ e^{-3i\varphi}\right\} = \left(P_3/\pi\,r^3\right)\ \cos\ (3\varphi)$$
$$= \left(P_3/\pi\,r^3\right)\ [\cos\ \varphi\ \cos\ (2\varphi) - \sin\ \varphi\ \sin\ (2\varphi)]$$

$$= \left(P_3/\pi\, r^3\right)\ \left[\cos\,\varphi\ \left(2\,\cos^2\varphi - 1\right) - 2\cos\ \varphi\ \left(1 - \cos^2\varphi\right)\right]$$
$$= \left(P_3/\pi\, r^3\right)\ \left(4\ \cos^2\ \varphi - 3\right)\ \cos\,\varphi\,, \tag{93}$$

this agrees with (90) in the plane (91a,92a).

5. Hypersphere theorem and insertion in a uniform field

The insertion of an hypersphere in a uniform field is represented by a dipole, with distinct dipole moments in the cases of zero normal (subsection 5.1) or tangential (subsection 5.4) field. This suggests the hypersphere theorem (subsection 5.2) as the multidimensional extension of circle (sphere) theorem in the plane, involving a reciprocal hyperpotential (subsection 5.3). The sphere at infinity may be used to compare with the asymptotic decay of the potential and field of multipoles (subsection 5.5).

5.1. **Insertion of an hypersphere in an uniform field.**

A uniform field $\vec{U}_\infty$ corresponds to the first term in the potential:

$$\Phi_-\left(\vec{x}\right) = \vec{U}_\infty\,.\,\vec{x} + \left(\vec{P}_1\,.\,\vec{x}\right)\sigma_N^{-1}R^{-N} = \cos\theta_1\left[U_\infty R + \left(P_1/\sigma_N\right)R^{1-N}\right] \tag{94}$$

the second term corresponds to a dipole (84a-d) of moment P_1. The normal component of the field on an hypersphere radius a vanishes (95a):

$$0 = \lim_{R\to a}\ \frac{\partial\Phi_-}{\partial R} =\ \cos\ \theta_1\ \left[U_\infty -\ \left[\left(N-1\right)/\sigma_N\right]\ P_1^-\ a^{-N}\right]\,,$$
$$P_1^- \equiv\ \left[\sigma_N/\ \left(N-1\right)\right]\ U_\infty\ a^N, \qquad \text{(95a,b)}$$

for a dipole moment (95b), that simplifies for a cylinder (96a-c) [sphere (96d-f)]:

$$N = 2,\, \sigma_2 = 2\pi:\ P_1^- = 2\,\pi\,U_\infty\,a^2; \quad N = 3,\ \sigma_3 = 4\pi:\ P_1^- = 2\,\pi\,U_\infty\,a^3. \qquad \text{(96a-f)}$$

in agreement with two (three) dimensional potential theory.

Comparing with the volume (97a) of the hypersphere of radius a:

$$V_N = N^{-1}\,\sigma_N\,a^N:\ P_1^- =\ f_N\,V_N\,U_\infty,\ f_N = N/\left(N-1\right) = 1/\left(1-1/N\right)\,, \qquad \text{(97a-c)}$$

the dipole moment (95b)≡(97b) has a factor (97c) that is: (i) $f_2 = 2$ for a cylinder (96c); (ii) $f_3 = 3/2$ for a sphere (96f); (iii) $f_4 = 4/3$ in four dimensions $N = 4$; (iv) it reduces with increasing dimension tending to a minimum $f_N \to 1$ as $N \to \infty$.

Thus an hypersphere of radius a in an uniform field $\vec{U}$ corresponds to a dipole of moment (95b)≡(97b), related to the volume (97a) of the hypersphere by the factor (97c); hence it equals the product of the volume by the uniform field U, and a factor decreasing with the dimension of the space. The superposition of the dipole with moment (95b) on the uniform field U_∞ leads to a potential (98) and field (99a,b):

$$\Phi_-\ (R\,,\,\theta_1) = U_\infty\ \left[R + \ (N-1)^{-1}\ \ a^N\ R^{1-N}\right]\ \cos\,\theta_1, \tag{98}$$

$$U_R^-\ (R\,,\,\theta_1) = U_\infty\ \left[1 -\ (a/R)^N\right]\ \cos\ \theta_1\,,$$
$$U_{\theta_R}^-\ (R\,,\,\theta_1)\ \ = -\,U_\infty\ \left[1 +\ (a/R)^N\ /\ (N-1)\right]\ \sin\ \theta_1\,; \qquad (99\text{a,b})$$

the radial component vanishes on the hypersphere (100a):

$$U_R^-\ (a,\ \theta_1) = 0, \quad U_{\theta_1}^-\ (a\,,\ \theta_1)\ \ =\ \ (1-1/N)^{-1}\ U_\infty\ \sin\ \theta_1\,, \qquad (100\text{a,b})$$

where the tangential component simplifies to (100b). This agrees with the case of the cylinder (100d) for (100c)

$$N = 2: \quad U^-(a,\varphi) = 2U_\infty \sin\varphi; \quad N = 3 \quad U^-(a,\theta) = (3/2)\ U\ \sin\theta; \quad (100\text{c-f})$$

and sphere (100f) for (100e).

5.2. **Hypersphere theorem for the potential.**

The reciprocal point (101a) with regard to the hypersphere of radius a transforms the potential of a uniform flow (101b) into (101c):

$$S = a^2/R: \ \Phi_0\,(R,\Phi_1) \equiv U_\infty R\cos\theta_1 = \left(U_\infty a^2/S\right)\cos\theta_1 \equiv \Phi_0\left(a^2/S,\theta_1\right); \ (101\text{a-c})$$

the latter should be the potential of a dipole, i.e. the second term in (98)≡(102c):

$$R \equiv \left|\vec{x}\right| \ \ ,\ S \equiv \left|\vec{y}\right|: \quad (N-1)\ \Phi_1\ (R,\,\theta_1)\ \ =\ \ a^N\,U_\infty\,R^{1-N}\ \cos\ \theta_1$$
$$=\ \ (a/R)^{N-2}\ \left(a^2/R\right)\ U_\infty\ \cos\ \theta_1\ \ =\ \ (R/a)^{2-N}\ \Phi_0\ \left(a^2/R,\,\theta_1\right)\ . \ (102\text{a-c})$$

The reciprocal hyperpotential coincides in the plane $N = 2$ (space $N = 3$) with the first circle (second sphere) theorem. This suggests the theorem of the reciprocal hyperpotential: if (103b) is a potential that satisfies the N-dimensional Laplace equation then the reciprocal hyperpotential (103a) is also an harmonic function (103c) and vice-versa:

$$\bar{\Phi}\ (R,\,\theta_1,...,\,\theta_{N-2}\,,\ \varphi) \equiv\ (R/a)^{2-N}\ \Phi\ \left(a^2/R,\,\theta_1,\ldots,\,\theta_{N-2}\,,\,\varphi\right):$$
$$\nabla^2\Phi = 0 \quad \Leftrightarrow \quad \nabla^2\,\bar{\Phi} = 0. \quad (103\text{a-c})$$

From this follows immediately the hypersphere theorem: if Φ is the potential of a field in free space (103b) then (104a) is an harmonic potential (104b) for which the corresponding field is tangent (104c) to the hypersphere of radius a:

$$\Phi_-\,(R,\,\theta_1,...,\theta_{N-2},\,\varphi) \equiv \Phi\ (R,\,\theta_1,\,...,\,\theta_{N-2},\varphi)\ -\ \Phi\ \left(a^2/R,\,\theta_1,\ldots,\,\theta_N,\,\varphi\right)\ (104\text{a})$$
$$\nabla^2\Phi_-\ \ =\ \ 0\ \ =\ \lim_{R\to a}\ \partial\Phi_-/\partial R\,. \quad (104\text{b-c})$$

If the scaling near the origin (105a) corresponds to a finite non-zero field then the reciprocal potential (105b) corresponds to a dipole at infinity:

$$\lim_{R\to 0} R^{-1}\,\Phi\,(R) = 0, \qquad \lim_{R\to\infty} R^{1-N}\,\bar{\Phi}\,(R) = \lim_{R\to 0} R\,\Phi\,\left(a^2/R\right) = 0\,; \qquad \text{(105a,b)}$$

the latter scaling (105b)≡(105c) implies a decay of the field (101d) leading to a zero flux (105e) across an hypersphere of zero radius:

$$R\to\infty:\ \Phi\,(R)\sim R^{1-N},\ \vec{U}\,(R)\sim\ \nabla\Phi\sim\vec{e}_R\ R^{-N},\ U\,(R)\,S_N\sim R^{-1}\,. \quad \text{(105c-e)}$$

This agrees with and generalizes the particular case (101a,b;102a-c) used to infer the reciprocal hyperpotential. All of the preceding results follow from the proof which is made next that the reciprocal potential is an harmonic function.

5.3. **Reciprocal hyperpotential as an harmonic function.**

The proof that the reciprocal hyperpotential (103a) satisfies the N-dimensional Laplace equation (103c) is an extension of the second sphere theorem. The radial derivative of the reciprocal hyperpotential (103a) involves (101a) in:

$$\begin{aligned}
&\partial\bar{\Phi}/\partial R - \ (2-N)\ (R/a)^{1-N}\,a^{-1}\,\Phi \\
&\quad = (\partial/\partial R)\ \left[(R/a)^{2-N}\ \Phi\ \left(a^2/R\right)\right]\ - (2-N)\ (R/a)^{1-N}\ a^{-1}\,\Phi \\
&\quad = \ (R/a)^{2-N}\ (\partial\,\Phi/\partial S)\ (dS/dR) = -\ \left(a^2/R^2\right)\ (R/a)^{2-N}\ \partial\Phi/\partial S \\
&\quad = -\ (R/a)^{-N}\ \partial\Phi/\partial S\,,
\end{aligned} \qquad (106)$$

where have been omitted the variables $(\theta_1, \dots, \theta_{N-2}, \varphi)$ in the N-dimensional Laplacian that are not differentiated. The radial part of the N-dimensional Laplacian (5) involves (106), viz.:

$$\begin{aligned}
&(\partial/\partial R)\ \left[R^{N-1}\,\partial\bar{\Phi}/\partial R\right] = \ (\partial/\partial R)\ \left\{R^{N-1}\ (\partial/\partial R)\ \left[\ (R/a)^{2-N}\ \Phi\ \left(a^2/R\right)\right]\right\} \\
&= (\partial/\partial R)\ \left[\,(2-N)\ a^{N-2}\,\Phi\ -\ R^{-1}\,a^N\,\partial\,\Phi/\partial\,S\right] \\
&= (\partial\Phi/\partial S)\left[(2-N)\,a^{N-2}\,(dS/dR) + R^{-2}a^N\right]\ -\ R^{-1}a^N\left(\partial^2\Phi/\partial^2 S\right)\ (dS/dR) \\
&= \ (\partial\,\Phi/\partial\,S)\ \left[R^{-2}a^N -\ (2-N)\ a^{N-2}\ (a/R)^2\right]\ +\ R^{-1}\,a^N\ (a/R)^2\ \partial^2\,\Phi/\partial\,S^2 \\
&= a^N\ \left[(N-1)\ R^{-2}\ (\partial\,\Phi/\partial\,S) + a^2\,R^{-3}\ \left(\partial^2\,\Phi/\partial\,S^2\right)\right] \\
&= a^{N-4}\ \left[(N-1)\ S^2\,\partial\,\Phi/\partial\,S + S^3\,\partial^2\Phi/\partial\,S^2\right] \\
&\qquad = a^{N-4}\,S^3\ \left\{\,S^{1-N}\ (\partial/\partial\,S)\ \left[\,S^{N-1}\ (\partial\,\Phi/\partial\,S)\right]\right\}\,.
\end{aligned} \qquad (107)$$

It follows that if $\bar{\Phi}$ satisfies the radial part (5) of the N-dimensional Laplacian (107) if Φ does, and vice-versa; the non-radial part of the Laplacian is the same for Φ and $\bar{\Phi}$, so one is an harmonic function if the other is, proving (103a-c).QED.

5.4. **Equipotential hypersphere in an uniform field.**

The introduction of an hypersphere of radius a in uniform field is represented by the potential (94) corresponding to adding a dipole whose moment is given: (i) by (95b) in the case (95a) of zero normal component of the field; (ii) in the case of zero tangential of the field, the hypersphere is an equipotential (108a):

$$0 = \Phi_+ (a, \theta_1) = \cos\theta_1 \left[U_\infty a + \left(P_1^+ / \sigma_N \right) a^{1-N} \right], \quad P_1^+ = -\sigma_N a^N U_\infty, \quad \text{(108a,b)}$$

leading to the dipole moment (108b). Substituting (108b) in (94) specifies the potential (109) [field (110a,b)] for an hypersphere of radius a, that is the equipotential $\Phi_+ = 0$ in an uniform external field:

$$\Phi_+ \ (R, \theta_1) = U_\infty \ R \ \cos \ \theta_1 \ \left[1 - \ (a/R)^N \right] , \quad \text{(109)}$$

$$U_R^+ \ (R, \theta_1) = U_\infty \ \cos \ \theta_1 \ \left[1 + \ (N-1) \ (a/R)^N \right] , \quad \text{(110a)}$$

$$U_\theta^+ \ (R, \theta_1) = - \ U_\infty \ \sin \ \theta_1 \ \left[1 - \ \ (a/R)^N \right] . \quad \text{(110b)}$$

The field is orthogonal to the hypersphere (111a):

$$U_{\theta_1}^+ \ (a, \theta_1) = 0 \, , \quad U_R^+ \ (a, \theta) = N \ U_\infty \ \cos \ \theta_1 \ \equiv \sigma \ (\theta_1) \ , \quad \text{(111a,b)}$$

corresponding to a surface source distribution (111b).

5.5. **Asymptotic decay of potential and fields.**

The Poisson integral in a bounded region (16a,b) simplifies to (13b) in free space if the following asymptotic condition is met an the surface at infinity:

$$\int_{S_\infty} \Phi \ (\partial\Phi / \partial n) \ dS \ = \ 0 \ ; \quad \text{(112)}$$

the latter equation (112) can be met by imposing a suitable asymptotic condition on the potential, e.g assuming that the potential decays asymptotically as an inverse power k of the distance:

$$\Phi \left(\vec{x} \right) \ = \ O \ \left(|\bar{x}| \ ^{-k} \right) , \qquad \partial\Phi \, / \, \partial n \ = \ O \ \left(|\vec{x}|^{-k \ -1} \right) . \quad \text{(113a,b)}$$

The integral (112) may be evaluated in one / two / three / more dimensions for a straight line (114a) / circle in the plane (114b) / sphere in space (114c) / hypersphere in N-dimensional space (114d):

$$N = 1, \quad x \ \in | \ R: \qquad \int \Phi \ \ (d\Phi / dx) \ dx \sim \ O \ \left(|x|^{-2k} \right) \ , \quad \text{(114a)}$$

$$N = 2: \quad \vec{x} \ \in | \ R^2: \qquad \int_{|\vec{x}|=r} \Phi \ \ (\partial\Phi / \partial n) \ ds = \ O \ \left(|\vec{x}|^{-2k} \right) \ , \quad \text{(114b)}$$

$$N = 3, \quad \vec{x} \in | \; R^3 : \qquad \int_{|\vec{x}|=R} \Phi \; (\partial\Phi/\partial n) \; dS = \; O \; \left(|\vec{x}|^{1-2k} \right) \; , \tag{114c}$$

$$N \geq 4, \quad \vec{x} \in | \; R^N : \qquad \int_{|\vec{x}|=R} \Phi \; (\partial\Phi/\partial n) \; dS = \; O \; \left(|\vec{x}|^{N-2k-2} \right) \; , \tag{114d}$$

because the perimeter / area / hyperarea of the circle / sphere/ hypersphere scales as $\sim \; O \; (\, |\vec{x}\,| \,) \; / \; O \; \left(|\vec{x}|^2 \right) \; / \; O \; \left(|\vec{x}|^{N-1} \right)$. The condition (112) is met: (i) on the line (114a) if k > 0; (ii) in the plane (114b) if k > 0 e.g. not for a monopole Φ ˜ log r, but for a dipole or higher order multipoles; (iii) in space (114c) for k > 1 / 2, e.g. for monopoles $\Phi \sim O \left(R^{-1} \right)$, as well as dipoles and higher order multipoles; (iv) in N-dimensional space for all multipoles. The latter statement can be proved as follows: (i) the decay of (114d) at infinity requires (115a,b):

$$N \geq 3 : \qquad \text{k} \; > \; \text{N}/2 - 1; \qquad \Phi \; (\vec{x}) \;\; \sim \;\; O \; \left(|\vec{x}|^{2-N-m} \right) \; , \tag{115a-c}$$

(ii) since the potential of a multipole of order m decays (TABLE I) like (115a), the condition (116a) leads to (116b):

$$k = N + m - 2 : \quad \text{N} + \text{m} - \; 2 \; = \; k \; > \; N/2 \; - 1, \; m \; > \; 1 \; - \; N \; / \; 2 \;\; , \tag{116a-c}$$

(iii) the latter implies (116c); (iv) the latter (116c) is satisfied by any multipole $m = 0, 1, \ldots,$ three or in more dimensions (115a) in agreement with the case (iii) and (iv). The asymptotic condition (116c) also applies $N = 2$ in the plane (114b) for all multipoles except monopole, in agreement with the case (ii). QED.

6. Conclusion

The Laplace equation was solved (section 2) in N-dimensional space with radial symmetry, generalizing to higher dimensions the: (i) one-dimensional step function potential; (ii) two-dimensional logarithmic potential; (iii) three-dimensional inverse distance potential. This specifies the Green function for the N-dimensional Laplace equation, and by convolution with the source term also the solution of the N-dimensional Poisson equation with arbitrary forcing. The latter lead (section 3) to an N-dimensional multipolar expansion, whose coefficients are specified by the hyperspherical Legendre polynomials; the original Legendre polynomials are the particular three-dimensional case. The series expansion and values for particular variable, recurrence and differentiation formulae, the differential equation and other properties are generalized from the Legendre to the hyperspherical Legendre polynomial. The properties of the hyperspherical Legendre polynomials thus specify the potential and field of N-dimensional generalized multipoles (section 4). The generalized N-dimensional dipole may be used to extend the two-dimensional circle and three-dimensional sphere theorems to an N-dimensional hypersphere theorem (section 5) the latter specifies the change in the potential and field due to the introduction of an hypersphere with zero normal or tangential derivative at the surface.

TABLE I

Asymptotic spatial decay of multipoles for the potential (field)

potential (field)	in the plane (1) $N=2$	in space (2) $N=3$	in space of N dimensions (3)
order of multipole	line multipole	point multipole	hypermultipole
$m=0 \Leftrightarrow 2^0=1$ monopole	$\log r\left(r^{-1}\right)$	$R^{-1}\left(R^{-2}\right)$	$R^{2-N}\left(R^{1-N}\right)$
$m=1 \Leftrightarrow 2^1=2$ dipole	$r^{-1}\left(r^{-2}\right)$	$R^{-2}\left(R^{-3}\right)$	$R^{1-N}\left(R^{-N}\right)$
$m=2 \Leftrightarrow 2^2=4$ quadrupole	$r^{-2}\left(r^{-3}\right)$	$R^{-3}\left(R^{-4}\right)$	$R^{-N}\left(R^{-N-1}\right)$
$m=3 \Leftrightarrow 2^3=8$ octupole	$r^{-3}\left(r^{-4}\right)$	$R^{-4}\left(R^{-5}\right)$	$R^{-N}\left(R^{-2-N}\right)$
2^m multipole order m	$r^{-m}\left(r^{-m-1}\right)$	$R^{-1-m}\left(R^{-2-m}\right)$	$R^{2-N-m}\left(R^{1-N-m}\right)$

$$(1): r \equiv \left|x^2+y^2\right|^{1/2};\ (2): R \equiv \left|x^2+y^2+z^2\right|^{1/2};\ (3): R \equiv \left|\sum_{n=1}^{N}(x_n)^2\right|^{1/2}$$

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