Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com
Volume 2 Issue 2(2011), Pages 8-12.

FUNCTIONS RELATED TO BINARY REPRESENTATION OF INTEGERS

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Abstract. A function together with its sum-function is established and a modulo-inequality with an unknown is investigated. The function is to indicate the position of the first 0-bit that occurs from the least significant bit of the binary representation of a positive integer; the sum-function is to sum up the function to an upper limit. Some properties of the function and the sum-function are explored and proved. Particularly, the function is proved to be a solution of the modulo-inequality. The function, the sum-function and the modulo-inequality are all utilizable in computer science.

1. Introduction

Binary representation is to express numbers by using Base 2 that represents numeric values using two symbols, 0 and 1, which are called binary bits. Owing to its straightforward implementation in digital electronic circuitry using logic gates, the binary representation is used internally by all modern computers. Being composed of merely 0 and 1, a binary representation may contain many 0-bits and 1-bits in it. For example, the binary representation of decimal integer 17 is 10001, containing three 0-bits continuously in the middle and two 1-bits at both sides; the binary representation of decimal integer 13 is 1101, containing one 0-bit next to the least significant bit (lsb) and three 1-bits at the other positions. Historical researches on binary representations mainly focused on their arithmetic operations and bit-wise operations and paid less attentions to if there would be any other special trait related with the binary bits. In a recent study on binary trees, it is found that there is an intrinsic relationship between some mathematical problems, which are derived from binary trees, and the distribution of the binary bits ,0 and 1. This article exhibits a few results of the study.

2. Notations and Definitions

In this whole paper, α denotes a non-negative integer and its $(m+1)$-bit binary representation is denoted by $(\alpha_m\alpha_{m-1}...\alpha_1\alpha_0)_2$, namely

$$\alpha = (\alpha_m\alpha_{m-1}...\alpha_1\alpha_0)_2 \tag{2.1}$$

2000 *Mathematics Subject Classification.* 35A07, 35Q53.
Key words and phrases. Binary representation; modulo inequality; binary tree.

Submitted Jul 22, 2011. Published September 15, 2011.
The author is supported by Foshan Burau of Sci.& Tech. under projects 2010C012 and 2011GY006.

where $\alpha_i (i = 0, 1, ..., m)$ are binary number 0 or 1.

Function $z(\alpha)$ is defined to be the position of the first 0-bit that occurs from the lsb of α's binary representation, e.g., $z(0) = z((00000000)_2) = 1$, $z(1) = z((00000001)_2) = 2$,$z(83) = z((01010011)_2) = 3$. Obviously, it holds

$$z(\alpha) = \begin{cases} 1 & \alpha = (\alpha_m \alpha_{m-1} ... \alpha_1 0)_2 \\ s+1 & \alpha = (\alpha_m \alpha_{m-1} ... \alpha_{s+1} 0 \underbrace{1...1}_{s})_2 \end{cases} \tag{2.2}$$

and

$$z(2^k + i) = \begin{cases} z(i) & , 0 \le i < 2^k - 1 \\ z(i) + 1 & , i = 2^k - 1 \end{cases} , k > 0 \tag{2.3}$$

Function $\sigma(\alpha)$ denotes the number of the ones in α's binary representation, namely,

$$\sigma(\alpha) = \sum_{j=0}^{m} \alpha_j \tag{2.4}$$

and function $Z(\alpha)$ is defined by

$$Z(\alpha) = \sum_{0 \le i \le \alpha} z(i) \tag{2.5}$$

The next section will show the relationship among $z(\alpha)$,$\sigma(\alpha)$,$Z(\alpha)$ and the following modulo-inequality with unknown x of positive integer

$$0 \le \alpha \bmod 2^x < 2^{x-1} \tag{2.6}$$

3. Main Results and Proofs

Theorem 3.1. *The three functions $z(\alpha)$, $\sigma(\alpha)$ and $Z(\alpha)$ fit the following identity*

$$Z(\alpha) = 2z(\alpha + 1) - \sigma(\alpha + 1) \tag{3.1}$$

Proof. To prove the theorem, it first needs to prove the following assertions (3.2) to (3.4)

$$Z(2^k - 1) = 2^{k+1} - 1 \tag{3.2}$$

$$Z(2^k) = 2^{k+1} \tag{3.3}$$

$$Z(2^k + i) = \begin{cases} Z(i) + 2^{k+1} - 1 & , 0 \le i < 2^k - 1 \\ 2^{k+2} - 1 & , i = 2^k - 1 \end{cases} , k > 0 \tag{3.4}$$

In fact, by an analysis on the binary representations of the total 2^k numbers from 0 to $2^k - 1$, it is easy to deduce (3.2) according to the following facts:

(1) There are 2^{k-1} numbers that tail with $(0)_2$ and within which each number α fits $z(\alpha) = 1$;
(2) There are 2^{k-2} numbers that tail with $(01)_2$ and within which each number α fits $z(\alpha) = 2$;
(3) There are $2^{k-s-1} (s = 0, 1, ..., k-1)$ numbers that tail with $(0\underbrace{11...1}_{s})_2$ and within which each number α fits $z(\alpha) = s + 1$;
(4) For the unique number $\alpha = 2^k - 1 = (\underbrace{0...0}_{m-k}\underbrace{11...1}_{k})_2$ it holds $z(\alpha) = k + 1$.

Hence it yields

$$Z(2^k-1)=2^{k-1}\times 1+...+2^{k-(s+1)}\times(s+1)+...+2\times(m-1)+2^0\times k+1\times(k+1) \quad (3.5)$$

Multiplying (3.5) by 2 on two sides yields

$$2Z(2^k-1)=2^k\times 1+...+2^{k-s}\times(s+1)+...+2^2\times(k-1)+2\times k+2\times(k+1) \quad (3.6)$$

Subtracting (3.6) by (3.5) results in

$$Z(2^k-1)=2^k+2^{k-1}+2^{k-2}+...+2^{k-s}+...+2^2+2+1=2^{k+1}-1$$

which is exactly (3.2) and directly derives (3.3) owing to

$$Z(2^k)=Z(2^k-1)+z(2^k)=2^{k+1}-1+1=2^{k+1}$$

Since

$$Z(2^k+i)=Z(2^k-1)+z(2^k+0)+...+z(2^k+i), 0\le i<2^k-1$$

it soon yields by referring to (2.3) and (3.2)

$$Z(2^k+i)=Z(2^k-1)+Z(i)=(2^{k+1}-1)+Z(i)$$

This together with the fact that $Z(2^k+(2^k-1))=Z(2^{k+1}-1)=2^{k+2}-1$ immediately validates (3.4).

Next is to prove the identity (3.1). Without loss of generality, let $\alpha=(\alpha_k\alpha_{k-1}...\alpha_{s+1}0\underbrace{1..1}_{s})_2$ and assume $\alpha_k=1$, $\alpha=2^k+(0\alpha_{k-1}...\alpha_{s+1}0\underbrace{1...1}_{s})_2$, then it yields by (3.4)

$$\begin{aligned}Z(\alpha)&=Z(2^k+(0\alpha_{k-1}...\alpha_{s+1}0\underbrace{1...1}_{s})_2)=Z((0\alpha_{k-1}...\alpha_{s+1}0\underbrace{1...1}_{s})_2)+2^{k+1}-1\\&=\alpha_k\times(2^{k+1}-1)+Z((0\alpha_{k-1}...\alpha_{s+1}0\underbrace{1...1}_{s})_2)\end{aligned}$$

Repeating the process on $Z((0\alpha_{k-1}...\alpha_{s+1}0)_2)$ yields

$$\begin{aligned}&Z((0\alpha_{k-1}...\alpha_{s+1}0\underbrace{1...1}_{s})_2)=Z(2^{k-1}+(0\alpha_{k-2}...\alpha_{s+1}0\underbrace{1...1}_{s})_2)\\&=\alpha_{k-1}\times(2^k-1)+Z((0\alpha_{k-2}...\alpha_{s+1}0\underbrace{1...1}_{s})_2)\end{aligned}$$

Consequently, it holds

$$Z((\alpha_k\alpha_{k-1}...\alpha_{s+1}0\underbrace{1...1}_{s})_2)=\sum_{j=k}^{s+1}\alpha_j\times(2^{j+1}-1)+(2^{s+1}-1) \quad (3.7)$$

The formula (3.7) undoubtedly presents a method to calculate $Z(\alpha)$, and actually it is equivalent to the identity (3.1) by the following form

$$Z(\alpha)=2\alpha-\sigma(\alpha)+z(\alpha) \quad (3.8)$$

In fact, if $s=0$, then $z(\alpha)=1$; by denoting $\alpha_0=0$, the (3.7) becomes

$$\begin{aligned}&Z((\alpha_k\alpha_{k-1}...\alpha_1\alpha_0)_2)=\sum_{j=k}^{1}\alpha_j\times(2^{j+1}-1)+1\\&=2\sum_{j=k}^{0}\alpha_j\times 2^j-\sum_{j=k}^{0}\alpha_j+1\\&=2\alpha-\sigma(\alpha)+z(\alpha)\end{aligned}$$

If $s > 0$, then $z(\alpha) = s + 1$; by denoting $\alpha_s = 0$,$\alpha_{s-1} = \alpha_{s-2} = ... = \alpha_0 = 1$, the right side of (3.7) becomes

$$\begin{aligned}
&\sum_{j=k}^{s+1} \alpha_j \times (2^{j+1} - 1) + (2^{s+1} - 1) \\
&= \sum_{j=k}^{s+1} \alpha_j \times 2^{j+1} + \alpha_s \times 2^{s+1} + \sum_{j=s-1}^{0} \alpha_j \times 2^{j+1} - \sum_{j=s-1}^{0} \alpha_j \times 2^{j+1} \\
&\quad - \sum_{j=k}^{s+1} \alpha_j - \alpha_s - \sum_{j=s-1}^{0} \alpha_j + \sum_{j=s-1}^{0} \alpha_j + (2^{s+1} - 1) \\
&= \sum_{j=k}^{0} \alpha_j \times 2^{j+1} - \sum_{j=k}^{0} \alpha_j + s + 1 \\
&= 2\alpha - \sigma(\alpha) + z(\alpha)
\end{aligned}$$

Therefore, either case validates (3.8). Now regarding $Z(\alpha) = Z(\alpha - 1) + z(\alpha)$, the (3.8) becomes

$$Z(\alpha - 1) = 2\alpha - \sigma(\alpha)$$

namely

$$Z(\alpha) = 2(\alpha + 1) - \sigma(\alpha + 1)$$

which validates the theorem 1 and finishes the proof. □

Theorem 3.2. *$z(\alpha)$ is the smallest positive solution of inequality (2.6).*

Proof. To prove the theorem, it is necessary to quote the following modulo identity

$$\alpha \bmod 2^x = \alpha \& (2^x - 1) \tag{3.9}$$

where the symbol & denotes the bitwise AND operation, and the identity was proved in paper [1].

By using the identity (3.9),the inequality (2.6) is rewritten by

$$0 \leq \alpha \& (2^x - 1) < 2^{x-1} \tag{3.10}$$

Note that

$$2^x - 1 = (\underbrace{0...0}_{m-x}\underbrace{1...11}_{x})_2$$

it yields by referring to (2.1) and the definition of bitwise AND

$$\alpha \& (2^x - 1) = (\underbrace{0...0}_{m-x}\underbrace{\alpha_{x-1}...\alpha_1\alpha_0}_{x})_2$$

which says that the inequality (3.10) holds if and only if

$$\alpha_{x-1} = 0$$

namely, x is the position where 0-bit occurs in α's binary representation. Or equivalently, the inequality (2.6) holds at x where 0-bit occurs in α's binary representation. Therefore, $z(\alpha)$ is the smallest positive integer that fits the inequality (2.6), and this finishes the proof of theorem 2. □

4. Applications

A binary tree is a very important data structure in computer science. Finding faster algorithms for storing, traversing and querying data in binary trees has been a research topic for researchers of computer science. A recent study shows that the theorem 1 and the theorem 2 play a key role in a development of algorithms for rapid traversal and query of data in a complete binary trees. Readers who interest in the scope can refer papers [2] and [3] to have the details.

Acknowledgments. The authors would like to thank Foshan Bureau of Science and Technology for her supported projects 2010C012 and 2011GY006.

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