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SOME INEQUALITIES FOR THE RATIO OF GAMMA FUNCTIONS

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Abstract. We prove some inequalities for the ratio $\Gamma(x+\lambda)/\Gamma(x+1)$, $x, \lambda > 0$, where Γ denotes the gamma function. The results are established using a recent method due to Mortici [8]. In some cases our results are completely new. In some others we recover or complete known results.

1. Introduction and background

Many authors studied inequalities for the ratio $\Gamma(x+\lambda)/\Gamma(x+1)$, with $x > 0$ and $0 < \lambda < 1$, where $\Gamma(x)$ denotes the gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt, \qquad x > 0.$$

In particular Gautschi [2] proved the inequalities

$$\frac{1}{(k+1)^{1-\lambda}} < \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < \frac{1}{k^{1-\lambda}}, \qquad 0 < \lambda < 1,\ k = 1, 2, \ldots.$$

Kershaw [4] has given some improvements of these inequalities such as

$$\frac{1}{\left(x - \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}\right)^{1-\lambda}} < \frac{\Gamma(x+\lambda)}{\Gamma(x+1)} < \frac{1}{\left(x + \frac{\lambda}{2}\right)^{1-\lambda}}, \qquad 0 < \lambda < 1,\ x > 0.$$

Lorch [7] established inequalities

$$\frac{1}{(k+\lambda)^{1-\lambda}} < \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < \frac{1}{\left(k + \frac{\lambda}{2}\right)^{1-\lambda}}, \qquad 0 < \lambda < 1,\ k = 1, 2, \ldots, \tag{1.1}$$

useful to prove an interesting bound, of Bernstein type, for the ultraspherical polynomials. In particular he proved that the upper bound in (1.1) holds true also for $\lambda > 2$, while it reverses when $1 < \lambda < 2$.

Laforgia [6], estimating the ratio $\Gamma(k+\lambda)/\Gamma(k+1)$ in terms of the more general function $(k+\alpha)^{\lambda-1}$, proved that the above Lorch's inequalities hold true also for real positive k. A unified treatment of Gautschi-Kershaw type inequalities was given by Giordano et al. [3].

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Another type of inequalities for the ratio $\Gamma(k+\lambda)/\Gamma(k+1)$ involving the psi function $\psi(x) = \Gamma'(x)/\Gamma(x)$ were studied by several authors. Gautschi [2] gave the following inequality

$$\frac{\Gamma(k+\lambda)}{\Gamma(k+1)} > \exp\left[(\lambda-1)\psi(k+1)\right], \qquad 0<\lambda<1, \ \ k=1,2,\ldots.$$

Kershaw [4] has given an improvement of this inequality such as

$$\exp\left[(\lambda-1)\psi\left(x+\frac{\lambda+1}{2}\right)\right] < \frac{\Gamma(x+\lambda)}{\Gamma(x+1)} < \exp\left[(\lambda-1)\psi\left(x+\sqrt{\lambda}\right)\right],$$

for every $x>0$ and $0<\lambda<1$. Recently, Mortici [8] proved that there exist two suitable positive real numbers n_0 and n_1 such that the following inequalities hold true

$$\frac{\Gamma(k+\lambda)}{\Gamma(k+1)} > \exp\left[(1-\lambda)\left(\frac{(1-\lambda)^2}{24(x+\lambda)^2} - \psi\left(x+\frac{\lambda+1}{2}\right)\right)\right], \tag{1.2}$$

$$\frac{\Gamma(k+\lambda)}{\Gamma(k+1)} > \exp\left[(1-\lambda)\left(\frac{(1-\lambda)^2}{24(x+\lambda)^2}\left(1-\frac{\lambda}{x+\lambda}\right) - \psi\left(x+\frac{\lambda+1}{2}\right)\right)\right], \tag{1.3}$$

where the inequality (1.2) holds for every $x \geq n_0$ and $0<\lambda<1$, while the inequality (1.3) holds for every $x \geq n_1$ and $\lambda \in \left(0, \frac{3+4\sqrt{15}}{33}\right)$. He proved also the existence of a third suitable positive real number n_2 such that for every $x \geq n_2$ and $\lambda \in \left(\frac{3+4\sqrt{15}}{33}, 1\right)$ the inequality (1.3) reverses.

The main tools used by Mortici to prove his results are the following two Lemmas [9].

Lemma 1.1. *Let $\{\omega_n\}_{n\geq 1}$ be a sequence convergent to zero and such that*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l \in [-\infty, \infty], \tag{1.4}$$

with $k>1$. Then

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}.$$

This Lemma provides a powerful way to measure the rate of convergence of a sequence. For complete proof of Lemma 1.1, see [9].

Lemma 1.2. *Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

If

$$\ln\frac{a_n b_{n+1}}{b_n a_{n+1}} = \sum_{j=2}^{\infty} \frac{y_j}{n^j},$$

then the following asymptotic series holds

$$a_n \sim b_n \exp\left(\sum_{j=1}^{\infty} \frac{\gamma_j}{n^j}\right), \qquad \text{as } n \to +\infty, \tag{1.5}$$

where the coefficients γ_j are given by the infinite triangular system

$$\gamma_1 - \binom{j-1}{1}\gamma_2 + \cdots + (-1)^j\binom{j-1}{j-2}\gamma_{j-1} = (-1)^j y_j, \qquad j \geq 2. \tag{1.6}$$

In general, as noted in [8], a series of type (1.5) does not converge, but truncated of only a few terms, it provides approximations of any desired accuracy.

In this paper, we prove that the best approximation of the form

$$\frac{\Gamma(x+\lambda)}{\Gamma(x+1)} \approx \frac{1}{(x+\alpha)^{1-\lambda}}, \qquad \text{as } x \to \infty, \tag{1.7}$$

is obtained for $\alpha = \lambda/2$. The results are based on a method due to Mortici [8]. Then, we establish inequalities of type

$$\frac{m_1(x,\lambda)}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}} < \frac{\Gamma(x+\lambda)}{\Gamma(x+1)} < \frac{m_2(x,\lambda)}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}}, \tag{1.8}$$

where

$$m_1(x,\lambda) = \exp\left[-\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\right],$$
$$m_2(x,\lambda) = \exp\left[-\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\left(1-\frac{\lambda}{x}\right)\right].$$

Finally, we give some numerical results which show that the upper and lower bounds (1.8) become closer to the (1.1) ones, for large values of x, and formulate some concluding remarks.

2. Accuracy estimate of approximation

We consider the relative error sequence ω_n, relating to the approximation (1.7), defined by the relation

$$\frac{\Gamma(n+\lambda)}{\Gamma(n+1)} = \frac{\exp(\omega_n)}{(n+\alpha)^{1-\lambda}}, \qquad n = 1, 2, \ldots. \tag{2.1}$$

By means of this sequence we can estimate the accuracy of the approximation of type (1.7). Indeed, by (2.1), we have

$$\omega_n = \ln\left((n+\alpha)^{1-\lambda}\frac{\Gamma(n+\lambda)}{\Gamma(n+1)}\right) \tag{2.2}$$

and, since $\lim_{n\to+\infty}\omega_n = 0$, we can apply Lemma 1.1 to this sequence. Due to the known relation $\Gamma(z+1) = z\Gamma(z)$ we have

$$\begin{aligned}\omega_n - \omega_{n+1} &= \ln\left(\frac{n+1}{n+\lambda}\right) - (1-\lambda)\ln\left(\frac{n+\alpha+1}{n+\alpha}\right)\\ &= \ln\left(\frac{1+\frac{1}{n}}{1+\frac{\lambda}{n}}\right) - (1-\lambda)\ln\left(\frac{1+\frac{\alpha+1}{n}}{1+\frac{\alpha}{n}}\right),\end{aligned}$$

therefore, using this last relation, we can write the difference $\omega_n - \omega_{n+1}$ as a power series into n^{-1} as follows

$$\omega_n - \omega_{n+1} = (1-\lambda)\left[\left(\alpha - \frac{\lambda}{2}\right)\frac{1}{n^2} + \left(-\alpha - \alpha^2 + \frac{\lambda+\lambda^2}{3}\right)\frac{1}{n^3}\right] + O\left(\frac{1}{n^4}\right). \tag{2.3}$$

Now, by Lemma 1.1, we can establish the following

Theorem 2.1. *Let λ be a positive real parameter different by 1 and 2. If $\alpha = \lambda/2$, then the sequence (2.2) converges as n^{-2}, otherwise it converges as n^{-1}.*

Proof. ¿From the formula (2.3), we have

$$\lim_{n\to+\infty} n^k(\omega_n-\omega_{n+1}) = \begin{cases} (1-\lambda)\left(\alpha-\frac{\lambda}{2}\right) & \text{if } k=2 \\ \frac{(1-\lambda)\left(-2\lambda+\lambda^2\right)}{12} & \text{if } k=3 \text{ and } \alpha=\frac{\lambda}{2}. \end{cases}$$

Therefore, if $\alpha=\lambda/2$, by Lemma 1.1 it follows

$$\lim_{n\to+\infty} n^2\omega_n = \frac{(1-\lambda)\left(-2\lambda+\lambda^2\right)}{24}$$

otherwise

$$\lim_{n\to+\infty} n\omega_n = (1-\lambda)\left(\alpha-\frac{\lambda}{2}\right).$$

□

Theorem 2.1 shows that the best approximation of the type (1.7) appears when the maximum rate of convergence of ω_n is obtained, namely in the case $\alpha=\lambda/2$.

3. Preliminary result

In this section, we apply the technique of Lemma 1.2 to construct the asymptotic series of the type (1.5) in the particular case

$$a_n=\frac{\Gamma(n+\lambda)}{\Gamma(n+1)},\qquad b_n=\frac{1}{\left(n+\frac{\lambda}{2}\right)^{1-\lambda}}. \tag{3.1}$$

Lemma 3.1. *By putting the sequences a_n and b_n as in the formula* (3.1)*, the asymptotic series* (1.5) *holds, where the coefficients γ_j are given by the infinite triangular system* (1.6) *whit*

$$y_j=(-1)^j\,\frac{\lambda^j-1+(1-\lambda)\left[\left(\frac{\lambda}{2}+1\right)^j-\left(\frac{\lambda}{2}\right)^j\right]}{j},\qquad j=2,3,\ldots. \tag{3.2}$$

Proof. We have

$$\begin{aligned}
\ln\frac{a_nb_{n+1}}{b_na_{n+1}} &= \ln\left[\frac{n+1}{n+\lambda}\left(\frac{n+\frac{\lambda}{2}}{n+\frac{\lambda}{2}+1}\right)^{1-\lambda}\right] \\
&= \ln\left(\frac{1+\frac{1}{n}}{1+\frac{\lambda}{n}}\right)-(1-\lambda)\ln\left(\frac{1+\frac{1+\lambda/2}{n}}{1+\frac{\lambda/2}{n}}\right) \\
&= \ln\left(1+\frac{1}{n}\right)-\ln\left(1+\frac{\lambda}{n}\right)-(1-\lambda) \\
&\quad\cdot\left[\ln\left(1+\frac{1+\lambda/2}{n}\right)-\ln\left(1+\frac{\lambda/2}{n}\right)\right] \\
&= \sum_{j=1}^{\infty}(-1)^{j-1}\frac{\left(1-\lambda^j\right)}{jn^j}-(1-\lambda)\left[\sum_{j=1}^{\infty}(-1)^{j-1}\frac{\left[(1+\lambda/2)^j-(\lambda/2)^j\right]}{jn^j}\right] \\
&= \sum_{j=2}^{\infty}(-1)^j\,\frac{\lambda^j-1+(1-\lambda)\left[\left(\frac{\lambda}{2}+1\right)^j-\left(\frac{\lambda}{2}\right)^j\right]}{j}\,\frac{1}{n^j}.
\end{aligned}$$

□

Therefore, the infinite triangular system (1.6) becomes

$$\begin{cases} \gamma_1 = 0 \\ \gamma_1 - 2\gamma_2 = \frac{\lambda(\lambda-1)(\lambda-2)}{12} \\ \gamma_1 - 3\gamma_2 + 3\gamma_3 = \frac{\lambda\left(\lambda^2-1\right)(\lambda-2)}{8} \\ \quad\cdots \end{cases} \tag{3.3}$$

and its first three solutions are

$$\begin{cases} \gamma_1 = 0 \\ \gamma_2 = -\frac{\lambda(\lambda-1)(\lambda-2)}{24} \\ \gamma_3 = \frac{\lambda^2(\lambda-1)(\lambda-2)}{24}. \end{cases} \tag{3.4}$$

4. Main results

By suitable truncations of the obtained asymptotic expansion

$$\frac{\Gamma(n+\lambda)}{\Gamma(n+1)} \sim \frac{\exp\left(\sum_{j=1}^{\infty} \frac{\gamma_j}{n^j}\right)}{\left(n+\frac{\lambda}{2}\right)^{1-\lambda}}, \qquad \text{as } n \to +\infty, \tag{4.1}$$

where the coefficients γ_j are solutions of the system (3.3), we can establish now new lower and upper bounds for the ratio $\frac{\Gamma(n+\lambda)}{\Gamma(n+1)}$ by means of the following two theorems.

Theorem 4.1. *For every $x > 0$ and $0 < \lambda < 1$ or $\lambda > 2$ the following lower bound holds*

$$\frac{\Gamma(x+\lambda)}{\Gamma(x+1)} > \frac{\exp\left[-\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\right]}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}}. \tag{4.2}$$

If $1 < \lambda < 2$, the inequality (4.2) reverses.

Theorem 4.2. *There exists a suitable real positive number ν such that, if $\lambda \in \left(0, \frac{3+\sqrt{141}}{33}\right)$, the following lower bound holds*

$$\frac{\Gamma(x+\lambda)}{\Gamma(x+1)} > \frac{\exp\left[-\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\left(1-\frac{\lambda}{x}\right)\right]}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}}, \qquad x > \nu. \tag{4.3}$$

If $\lambda \in (1, 2)$, the inequality (4.3) holds for every $x > 0$.

Moreover, if $\lambda \in \left(\frac{3+\sqrt{141}}{33}, 1\right) \cup (2, +\infty)$, the following upper bound holds

$$\frac{\Gamma(x+\lambda)}{\Gamma(x+1)} < \frac{\exp\left[-\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\left(1-\frac{\lambda}{x}\right)\right]}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}}, \qquad x > 0. \tag{4.4}$$

5. Proofs of main results

Proof. (Theorem 4.1). We consider the function

$$f(x) = \frac{\Gamma(x+\lambda)}{\Gamma(x+1)}\left(x+\frac{\lambda}{2}\right)^{1-\lambda} \exp\left[\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\right]$$

which satisfies the relation

$$\lim_{x\to+\infty} f(x) = 1. \tag{5.1}$$

The inequality (4.2) is proved if we show that $f(x) > 1$ for every $x > 0$ and for $0 < \lambda < 1$ or $\lambda > 2$.

For $x > 0$, let g be the function defined in the following way

$$\begin{aligned} g(x) &= \frac{f(x+1)}{f(x)} \\ &= \left(\frac{x+\lambda}{x+1}\right)\left(\frac{x+\frac{\lambda}{2}+1}{x+\frac{\lambda}{2}}\right)^{1-\lambda} \exp\left[\frac{\lambda(\lambda-1)(\lambda-2)}{24}\left(\frac{1}{(x+1)^2}-\frac{1}{x^2}\right)\right]. \end{aligned}$$

Since $\lim_{x\to+\infty} g(x) = 1$, we will prove that the function g is strictly increasing on $(0, +\infty)$ in such a way that $g(x) < 1$, i.e. $f(x+1) < f(x)$, for every $x > 0$. From this last inequality and by the limit (5.1), it follows that $f(x) > 1$ for every $x > 0$. It is clear that the function g is strictly increasing on $(0, +\infty)$ if the derivate of the function

$$\begin{aligned} \phi(x) &= \ln[g(x)] \\ &= \ln\left(\frac{x+\lambda}{x+1}\right) + (1-\lambda)\ln\left(\frac{x+\frac{\lambda}{2}+1}{x+\frac{\lambda}{2}}\right) + \frac{\lambda(\lambda-1)(\lambda-2)}{24}\left(\frac{1}{(x+1)^2}-\frac{1}{x^2}\right) \end{aligned}$$

is positive on $(0, +\infty)$. Now, after some algebraic calculations, we have

$$\phi'(x) = \frac{\lambda(\lambda-1)(\lambda-2)P_4(x)}{12x^3(1+x)^3(x+\lambda)(2x+\lambda)(2+2x+\lambda)},$$

where

$$\begin{aligned} P_4(x) = {} & 24\lambda x^4 + \left(4+42\lambda+15\lambda^2\right)x^3 + \left(4+26\lambda+21\lambda^2+3\lambda^3\right)x^2 \\ & + \left(6\lambda+11\lambda^2+3\lambda^3\right)x + 2\lambda^2+\lambda^3 > 0, \qquad x > 0. \end{aligned}$$

Since $(\lambda-1)(\lambda-2) > 0$ when $0 < \lambda < 1$ and $\lambda > 2$, the inequality (4.2) is proved. Clearly, if $1 < \lambda < 2$, by using the same method, the reversed inequality is established. □

In a similar way, we can prove inequalities (4.3) and (4.4). First we prove the following

Lemma 5.1. *Let* $P_5(x) = \sum_{i=0}^{5} p_i(\lambda)\, x^i$ *be a polynomial with coefficients*

$$\begin{aligned} p_5(\lambda) &= -2\left(33\lambda^2-6\lambda-4\right) \\ p_4(\lambda) &= -2\left(27\lambda^3+72\lambda^2-8\lambda-8\right) \\ p_3(\lambda) &= -2\left(6\lambda^4+51\lambda^3+70\lambda^2-2\lambda-4\right) \\ p_2(\lambda) &= -2\lambda^2\left(9\lambda^2+44\lambda+35\right) \\ p_1(\lambda) &= -\lambda^2\left(12\lambda^2+37\lambda+14\right) \\ p_0(\lambda) &= -3\lambda^3(\lambda+2). \end{aligned}$$

Then, if $0 < \lambda < \frac{3+\sqrt{141}}{33}$, *the polynomial* $P_5(x)$ *is definitively positive, while, if* $\lambda > \frac{3+\sqrt{141}}{33}$, $P_5(x)$ *is negative for every* $x > 0$.

Proof. The leading coefficient $p_5(\lambda)$ is positive for $0 < \lambda < \frac{3+\sqrt{141}}{33}$ and negative for $\lambda > \frac{3+\sqrt{141}}{33}$. Therefore, it follows that $P_5(x)$ is definitively positive when $0 < \lambda < \frac{3+\sqrt{141}}{33}$. While, in the case $\lambda > \frac{3+\sqrt{141}}{33}$, it is sufficient to show that all coefficients $p_i(\lambda)$ are negative as $p_5(\lambda)$ is. This is clearly true for the coefficients $p_0(\lambda)$, $p_1(\lambda)$ and $p_2(\lambda)$. For $p_4(\lambda)$, we have

$$p_4'(\lambda) = -2\left(81\lambda^2 + 144\lambda - 8\right),$$
$$p_4''(\lambda) = -2\left(162\lambda + 144\right).$$

Since $p_4''(\lambda) < 0$, for every $\lambda > 0$, it follows that $p_4'(\lambda)$ decreasing on $(0, +\infty)$. But $p_4'\left(\frac{3+\sqrt{141}}{33}\right) < 0$ and consequently $p_4'(\lambda) < 0$ on $\left(\frac{3+\sqrt{141}}{33}, +\infty\right)$. Finally, since $p_4(\lambda)$ decreasing on $\left(\frac{3+\sqrt{141}}{33}, +\infty\right)$ and $p_4\left(\frac{3+\sqrt{141}}{33}\right) < 0$, it follows that also $p_4(\lambda) < 0$ on $\left(\frac{3+\sqrt{141}}{33}, +\infty\right)$.

Similarly, we have that $p_3(\lambda) < 0$ on $\left(\frac{3+\sqrt{141}}{33}, +\infty\right)$, indeed

$$p_3'(\lambda) = -2\left(24\lambda^3 + 153\lambda^2 + 140\lambda - 2\right),$$
$$p_3''(\lambda) = -2\left(72\lambda^2 + 306\lambda + 140\right),$$

$p_3''(\lambda) < 0$ for every $\lambda > 0$, $p_3'\left(\frac{3+\sqrt{141}}{33}\right) < 0$, $p_3\left(\frac{3+\sqrt{141}}{33}\right) < 0$. □

Proof. (Theorem 4.2). We consider the function

$$f(x) = \frac{\Gamma(x+\lambda)}{\Gamma(x+1)}\left(x + \frac{\lambda}{2}\right)^{1-\lambda} \exp\left[\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\left(1 - \frac{\lambda}{x}\right)\right]$$

which tends to 1 as $x \to +\infty$, and let g be the function defined, for $x > 0$, in the following way

$$\begin{aligned} g(x) &= \frac{f(x+1)}{f(x)} \\ &= \left(\frac{x+\lambda}{x+1}\right)\left(\frac{x+\frac{\lambda}{2}+1}{x+\frac{\lambda}{2}}\right)^{1-\lambda} \exp\left[\frac{\lambda(\lambda-1)(\lambda-2)}{24}\left(\frac{1-\frac{\lambda}{x+1}}{(x+1)^2} - \frac{1-\frac{\lambda}{x}}{x^2}\right)\right]. \end{aligned}$$

Since $\lim_{x\to+\infty} g(x) = 1$, also in this case the results of this theorem are proved studying the monotonicity of the function g. In particular, the inequality (4.3) holds if g is definitively strictly increasing, while the inequality (4.4) holds if g is strictly decreasing on $(0, +\infty)$.

To study the monotonicity of the function g, we consider again the derivate of the function

$$\begin{aligned} \phi(x) = \ln[g(x)] &= \ln\left(\frac{x+\lambda}{x+1}\right) \\ &+ (1-\lambda)\ln\left(\frac{x+\frac{\lambda}{2}+1}{x+\frac{\lambda}{2}}\right) + \frac{\lambda(\lambda-1)(\lambda-2)}{24}\left(\frac{1-\frac{\lambda}{x+1}}{(x+1)^2} - \frac{1-\frac{\lambda}{x}}{x^2}\right). \end{aligned}$$

After some algebraic calculations, we have

$$\phi'(x) = \frac{\lambda(\lambda-1)(\lambda-2)P_5(x)}{24x^4(1+x)^4(x+\lambda)(2x+\lambda)(2+2x+\lambda)},$$

where $P_5(x)$ is the polynomial studied in Lemma 5.1.

¿From the study of sign of the polynomial $P_5(x)$ and by the sign of the factor $(\lambda-1)(\lambda-2)$, we can conclude that, if $\lambda \in \left(0, \frac{3+\sqrt{141}}{33}\right)$, there exist a positive real number ν such that the function g is strictly increasing on (ν, ∞), while, if $\lambda \in (1, 2)$,

g is strictly increasing for every $x > 0$. Moreover, if $\lambda \in \left(\frac{3+\sqrt{141}}{33}, 1\right) \cup (2, +\infty)$, g is strictly decreasing for every $x > 0$. Consequently, the inequalities (4.3) and (4.4) are proved. □

6. Numerical results and concluding remarks

In this concluding section we report first two tables with some numerical results, obtained by means of the computer algebra system Mathematica ©, which show that the inequalities proved in Theorem 4.1 and 4.2 improve the ones given in formula (1.1) for large values of x. The numerical values reported on these tables have been rounded off at the fifteenth decimal place.

- For $\lambda = 0.4 < \frac{3+\sqrt{141}}{33}$

	$x = 10$	$x = 100$
$\frac{1}{(x+\lambda)^{1-\lambda}}$	0.195440219941994	0.039727327161912
$\frac{\exp\left[-\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\left(1-\frac{\lambda}{x}\right)\right]}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}}$	0.197429073365366	0.039768910048359
$\frac{\Gamma(x+\lambda)}{\Gamma(x+1)}$	0.197429092003520	0.039768910048773

- For $\lambda = 0.7 > \frac{3+\sqrt{141}}{33}$

	$x = 10$	$x = 100$
$\frac{1}{(x+\lambda)^{1-\lambda}}$	0.491116886025469	0.250663534301190
$\frac{\exp\left[-\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\right]}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}}$	0.495984944815014	0.250925208069497
$\frac{\Gamma(x+\lambda)}{\Gamma(x+1)}$	0.495988778365127	0.250925210061790
$\frac{\exp\left[-\frac{\lambda(\lambda-1)(\lambda-2)}{24x^2}\left(1-\frac{\lambda}{x}\right)\right]}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}}$	0.495988894110860	0.250925210067489
$\frac{1}{\left(x+\frac{\lambda}{2}\right)^{1-\lambda}}$	0.496041366311398	0.250925493497084

Now, some concluding remarks are possible considering the asymptotic expansion (4.1) in the particular case $\lambda = 3/2$. In this case the coefficients (3.4) become

$$\begin{cases} \gamma_1 = 0 \\ \gamma_2 = \frac{1}{64} \\ \gamma_3 = -\frac{3}{128}. \end{cases} \tag{6.1}$$

For every integer n, the left-hand side of (4.1) can be written as follows

$$\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} = \left(n+\frac{1}{2}\right)\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!} = \left(n+\frac{1}{2}\right)\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\ \sqrt{\pi}}{n!}$$
$$= \left(n+\frac{1}{2}\right)\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2n\ (2n-2)\cdots 4\cdot 2}\ \sqrt{\pi}, \tag{6.2}$$

while the right-hand side becomes

$$\frac{\exp\left(\sum_{j=1}^{\infty}\frac{\gamma_j}{n^j}\right)}{\left(n+\frac{3}{4}\right)^{-1/2}} = \left(n+\frac{3}{4}\right)^{1/2}\sum_{i=0}^{\infty}\frac{1}{i!}\left(\sum_{j=1}^{\infty}\frac{\gamma_j}{n^j}\right)^i$$
$$= \left(n+\frac{3}{4}\right)^{1/2}\left[1+\left(\frac{\gamma_1}{n}+\frac{\gamma_2}{n^2}+\frac{\gamma_3}{n^3}+\cdots\right)+\frac{1}{2}\left(\frac{\gamma_1}{n}+\frac{\gamma_2}{n^2}+\frac{\gamma_3}{n^3}+\cdots\right)^2+\cdots\right]$$
$$= \left(n+\frac{3}{4}\right)^{1/2}\left[1+\frac{\gamma_1}{n}+\left(\frac{\gamma_1^2}{2}+\gamma_2\right)\frac{1}{n^2}+\left(\gamma_3+\gamma_1\gamma_2+\frac{\gamma_1^3}{6}\right)\frac{1}{n^3}+\cdots\right]. \tag{6.3}$$

Therefore, from (4.1), (6.2) and (6.3), substituting the values (6.1) into this last expression, we have

$$\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2n\ (2n-2)\cdots 4\cdot 2}\ \sim \frac{1}{\sqrt{\pi}}\left(1+\frac{1}{64\ n^2}-\frac{3}{128\ n^3}+\cdots\right)\frac{\left(n+\frac{3}{4}\right)^{1/2}}{\left(n+\frac{1}{2}\right)}, \tag{6.4}$$

as $n \to +\infty$. Now, since

$$\frac{\left(n+\frac{3}{4}\right)^{1/2}}{\left(n+\frac{1}{2}\right)} = \frac{1}{\sqrt{n}}\frac{\left(1+\frac{3}{4n}\right)^{1/2}}{\left(1+\frac{1}{2n}\right)} = \frac{1}{\sqrt{n}}\left(1+\frac{3}{8n}-\frac{9}{128n^2}+\cdots\right)$$
$$\cdot\left(1-\frac{1}{2n}+\frac{1}{4n^2}+\cdots\right) = \frac{1}{\sqrt{n}}\left(1-\frac{1}{8n}-\frac{1}{128n^2}+\cdots\right),$$

the asymptotic formula (6.4) becomes

$$\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2n\ (2n-2)\cdots 4\cdot 2}\ \sim \frac{1}{\sqrt{n\pi}}\left(1-\frac{1}{8\ n}+\frac{1}{128\ n^2}+\cdots\right), \tag{6.5}$$

as $n \to +\infty$.

In this way we obtained the same result established by Tricomi and Erdélyi [11, formula (22)] from which it follows the following approximation formula for π (see also [11, formula (23)])

$$\pi = \frac{1}{n}\left[\frac{2n\ (2n-2)\cdots 4\cdot 2}{(2n-1)(2n-3)\cdots 3\cdot 1}\left(1-\frac{1}{8\ n}+\frac{1}{128\ n^2}+O\left(\frac{1}{n^3}\right)\right)\right]. \tag{6.6}$$

But we can do more. Indeed, in this particular case $\lambda = 3/2$, we can apply the results of the theorems 4.1 and 4.2 to obtain a lower and upper bounds for π. Thus, from the inequalities (4.2) and (4.3) we have, for every integer n and $\lambda = 3/2$,

$$(4n+3)\left[\frac{(2n)!!}{(2n+1)!!}\right]^2\exp\left(\frac{1}{32\ n^2}\left(1-\frac{3}{2\ n}\right)\right)$$
$$< \pi < (4n+3)\left[\frac{(2n)!!}{(2n+1)!!}\right]^2\exp\left(\frac{1}{32\ n^2}\right),$$

namely

$$\exp\left(-\frac{3}{64\ n^3}\right) < \frac{\pi}{(4n+3)\left[\frac{(2n)!!}{(2n+1)!!}\right]^2 \exp\left(\frac{1}{32\ n^2}\right)} < 1. \tag{6.7}$$

We can also consider another lower bound in (6.7). Indeed, by considering the series expansion of the function $\exp\left(-\frac{3}{64\ n^3}\right)$, we can write

$$1 - \frac{3}{64\ n^3} < \frac{\pi}{(4n+3)\left[\frac{(2n)!!}{(2n+1)!!}\right]^2 \exp\left(\frac{1}{32\ n^2}\right)} < 1 \tag{6.8}$$

or

$$1 - \frac{3}{64\ n^3} + \frac{9}{8192\ n^6} - \frac{27}{1572864\ n^9} < \frac{\pi}{(4n+3)\left[\frac{(2n)!!}{(2n+1)!!}\right]^2 \exp\left(\frac{1}{32\ n^2}\right)} < 1$$

and so on. We report, in the following table, some numerical results (with a rounding off at the fifteenth decimal place) relating to the lower bound in (6.8) for different values of n.

$n = 20$	$n = 200$	$n = 2000$
0.999994140625000	0.999999994140625	0.999999999994141

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