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ON FINITE STRICT CO-SINGULARITY OF VOLTERRA OPERATORS ON NON-REFLEXIVE SPACE

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Abstract. Finite strict co-singularity of Volterra operators $V_a : L^1[0,1+a] \to C[0,1]$, where $0 \le a < \infty$, defined by $(V_a f)(x) = \int_0^{(1+a)x} f(t)dt$, $x \in [0,1]$, $t \in [0,1+a]$ is explored by calculating Mityagin numbers. Also, Approximation, Gelfand, Kolmogorov, Bernstein, and Isomorphism s-numbers are obtained for these operators.

1. Background and Main Results

Compactness of operators is one of the most studied properties. But not all operators are compact. Due to the significance in various branches of Mathematics, compact operators have been extensively studied (see [4, 5, 6, 11, 14, 15]). In most cases, the use of s-numbers is in measuring the degree of compactness of operators but not always. For example, the Bernstein numbers (defined in Section 2) are used to study the finite strict singularity (definition given in Section 2) which is a weaker property than compactness (see [3]). There is another weaker property than compactness called strict co-singularity (defined in Section 2) which is associated with the strict singularity of the operator's dual. Mityagin numbers is one of the best ways to explore its finite strict co-singularity (defined in Section 2) of operators. Properties of some non-compact operators which are not far away from being compact have also been explored. One such operator is the Volterra operator defined by

$$(Vf)(t) = \int_0^t f(s)ds, \ (0 \le t \le 1) \text{ for } f \in L^1(0,1) \tag{1.1}$$

between the spaces L^1 and $C[0,1]$. This operator is not compact but possesses a desired property called strict singularity (defined in Section 2) (see [2, 10]). For this operator, Bakşi *et. al.* obtained exact values of Approximation, Gelfand,

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Kolmogorov and Isomorphism numbers in [1]. In present work we introduce a generalization of the Volterra operator defined in (1.1) by extending the domain of definition to $[0, 1 + a]$, where $a \geq 0$, and estimate the values of the Approximation, Gelfand, Kolmogorov, Bernstein, Mityagin, Isomorphism s-numbers for the generalized operator.
It can be easily seen that operator V_a is not compact. Take a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $L^1[0, 1 + a]$, where $f_n(t) = \frac{n+1}{(1+a)^{n+1}} t^n$ for $t \in [0, 1 + a]$. Then image of this sequence under V_a is $\{x^{n+1}\}_{n=1}^{\infty}$ having no weakly convergent subsequence in $C[0, 1]$.
In the following we state the main results.

Theorem 1.1. *For the Volterra operator $V_a : L^1[0, 1 + a] \to C[0, 1]$, the following are obtained*

$$a_n(V_a) = c_n(V_a) = d_n(V_a) = \frac{1}{2} \ for \ n \geq 2. \tag{1.2}$$

Theorem 1.2. *For the Volterra operator $V_a : L^1[0, 1 + a] \to C[0, 1]$, the following are obtained*

$$b_n(V_a) = \mathfrak{m}_n(V_a) = i_n(V_a) = \frac{1}{2n - 1} \ for \ n \in \mathbb{N}. \tag{1.3}$$

The structure of rest of the paper is as follows. Section 2 contains notations and definitions of rudiments. In Section 3, the norm of the operator V_a is calculated and auxiliary results are proved. The proofs of main results are presented in Section 4.

2. Elementary Material

2.1. **Normed Spaces.** Let X, Y be two Banach spaces. By B_X, S_X we denote respectively the closed unit ball and the unit sphere of X. The set of all bounded linear operators from X to Y is denoted by $B(X, Y)$. For $T \in B(X, Y)$, the operator norm of T will be denoted by $\|T\|_{op}$. We write $dim X$ for the dimension of the space X. For a closed subspace M of X, the quotient space is defined by

$$X/M = \{t + M : t \in X\}$$

with the quotient norm

$$\|[t]\|_{X/M} = \inf_{m \in M} \|t - m\|_X,$$

where $[t]$ denotes the element of X/M given by $[t] = t + M = \{t + m : m \in M\}$. For brevity, we write $\|t\|_{X/M}$ instead of writing $\|[t]\|_{X/M}$.

$L^1[0, 1 + a]$ defines the Lebesgue space of all Lebesgue integrable functions on $[0, 1 + a]$ taking values in $\mathbb{R}$ and identified almost everywhere. The norm of $f \in L^1[0, 1 + a]$ is defined by

$$\|f\|_{L^1} = \int_0^{1+a} |f(u)| \, du, \ u \in [0, 1 + a].$$

$C[0, 1]$ will denote the space of real valued continuous functions on $[0, 1]$ under the norm

$$\|f\|_\infty = \sup_{0 \leq x \leq 1} |f(x)| \, .$$

A Banach space X is said to have lifting property if for any closed subspace N of an arbitrary Banach space Y, every operator $T : X \to Y/N$ admits a lifting $\widehat{T}$ such that $\|\widehat{T}\| \leq (1 + \epsilon) \|T\|$ for every $\epsilon > 0$. The space L^1 has lifting property, for instance, see [15, p. 36]. For short, the space $L^1[0, 1+a]$ will be written as L^1 only.

2.2. **Definitions.**

Definition 2.1. *An operator T acting between Banach spaces X and Y, is said to be finitely strictly singular if for every $\epsilon > 0$ there correspond $N_\epsilon \in \mathbb{N}$ such that for every subspace $M \subseteq X$ with $\dim M \geq N_\epsilon$, there exists u in S_M such that $\|T(u)\| \leq \epsilon$.*

Definition 2.2. *Let T be an operator acting between Banach spaces X and Y and $\mathcal{M}$ be an arbitrary infinite co-dimensional closed subspace of Y. We say that T is strictly co-singular, if for every such $\mathcal{M}$, the map νT, where $\nu : Y \to Y/\mathcal{M}$, is a quotient map having non-closed range.*

Definition 2.3. *An operator T acting between Banach spaces X and Y, is said to be finitely strictly cosingular if the sequence of its Mityagin numbers tends to zero* [7].

Definition 2.4. *For Banach spaces X and Y and an operator $T \in B(X,Y)$, we associate a sequence $s_n(T)$ of scalars satisfying the following properties:*

(S1) *Monotonicity:* $\|T\| = s_1(T) \geq s_2(T) \geq s_3(T) \geq ... \geq 0$,
(S2) $s_n(T+S) \leq s_n(T) + \|S\|$ *for every* $S \in B(X,Y)$,
(S3) *Ideal Property:* $s_n(B \circ T \circ A) \leq \|B\| s_n(T) \|A\|$ *for every* $A \in B(Z_1, X)$ *and* $B \in B(Y, Z_2)$,
(S4) *Norming Property:* $s_n(Id : \ell_n^2 \to \ell_n^2) = 1$,
(S5) *Rank Property:* $s_n(T) = 0$ *whenever rank* $T < n$.

Then $s_n(T)$ is called the n-th s-number of T. The number $s_n(T)$ is called the n-th strict s-number of T when the following condition

(S6) $s_n(Id : E \to E) = 1$ *for every Banach space E of* $\dim E = n$,

is considered in place of (S4).

The s-numbers have varied definitions in literature. Initially, A. Pietsch formulated the definition of s-numbers (see [12]) which makes use of condition (S6). Later, the definition was modified so that a larger class of s-numbers (such as Chang, Hilbert, Weyl numbers etc.) can be covered. A detailed study of s-numbers can be found in [12, 13] or [8].

For $T \in B(X,Y)$ and $n \in \mathbb{N}$, we define the n-th Approximation, Gelfand, Kolmogorov, Bernstein, Mityagin and Isomorphism numbers by

$$a_n(T) = \inf_{\substack{F \in B(X,Y) \\ rank\, F < n}} \|T - F\|,$$
$$c_n(T) = \inf_{\substack{M \subseteq X \\ codim\, M < n}} \sup_{x \in B_M} \|Tx\|_Y,$$
$$d_n(T) = \inf_{\substack{N \subseteq Y \\ dim\, N < n}} \sup_{x \in B_X} \|Tx\|_{Y/N},$$
$$b_n(T) = \sup_{\substack{M \subseteq X \\ dim\, M \geq n}} \inf_{x \in S_M} \|Tx\|_Y,$$
$$\mathfrak{m}_n(T) = \sup_{\substack{N \subseteq Y \\ codim\, N \geq n}} \sup\{\alpha \geq 0 : \alpha B_{Y/N} \subseteq (\pi_N \circ T) B_X\},$$

where $\pi_N : Y \to Y/N$ is a canonical surjection of closed subspace N of Y.

$$i_n(T) = \sup_{dim\,(E)\geq n} \|P\|^{-1}\|Q\|^{-1},$$

respectively, where E is Banach space and $P \in B(Y,E)$, $Q \in B(E,X)$ such that $P \circ T \circ Q$ defines identity map on E. The above s-numbers are connected through some inequalities, which are bounded below by Isomorphism numbers and bounded above by Approximation numbers. To be concrete, for $T \in B(X,Y)$ and $n \in \mathbb{N}$, the following relation is obtained

$$i_n(T) \leq b_n(T) \leq min\{c_n(T), d_n(T)\} \leq max\{c_n(T), d_n(T)\} \leq a_n(T).$$

In addition, if the space X possesses lifting property, then Approximation numbers coincide with Kolmogorov numbers, for every $n \in \mathbb{N}$, see ([8, 13]).

3. Auxiliary Results

Proposition 3.1. *The norm of the operator V_a is 1. i.e. $\|V_a\|_{op} = 1$.*

Proof. For any $f \in L^1$, we have

$$\|V_a f\|_\infty = \sup_{0\leq x\leq 1} |(V_a f)(x)| = \sup_{0\leq x\leq 1} \left| \int_0^{(1+a)x} f(t)dt \right| \leq \sup_{0\leq x\leq 1} \int_0^{(1+a)x} |f(t)|dt.$$
$$\leq \int_0^{1+a} |f(t)|dt = \|f\|_{L^1},$$

and from where we obtain $\|V_a f\|_\infty \leq 1$ for $f \in B_{L^1}$.
For the equality, consider the function $f = \frac{1}{1+a}\chi_{[0,1+a]} \in L^1$. Then we have $\|f\|_{L^1} = 1$ and $\|V_a f\|_\infty$

$$= \sup_{0\leq x\leq 1} |(V_a f)(x)| = \sup_{0\leq x\leq 1} \left| \int_0^{(1+a)x} f(t)dt \right| = \sup_{0\leq x\leq 1} \left| \int_0^{(1+a)x} \frac{1}{1+a}\chi_{[0,1+a]}(t)dt \right|$$
$$= \sup_{0\leq x\leq 1} \frac{1}{1+a} \left| \int_0^{(1+a)x} \chi_{[0,1+a]}(t)dt \right| = \sup_{0\leq x\leq 1} \left| \frac{(1+a)x}{1+a} \right| = 1, \text{ since } a \geq 0.$$

Thus, the operator norm is

$$\|V_a\|_{op} = \sup_{\|f\|_{L^1}\leq 1} \|V_a f\|_\infty = 1.$$

□

Lemma 3.2. *For $n \geq 2$, the following estimation for Kolmogorov numbers is obtained*

$$d_n\left(V_a : L^1 \to C[0,1]\right) \leq \frac{1}{2}.$$

Proof. In view of monotonicity property of s-numbers, it is sufficient to prove that $d_2(V_a) \leq \frac{1}{2}$. For a closed subspace N of $C[0,1]$, we have

$$d_2(V_a) = \inf_{\substack{N\subseteq C[0,1] \\ dim\,N<2}} \sup_{f\in B_{L^1}} \|V_a f\|_{C[0,1]/N} \leq \sup_{f\in B_{L^1}} \|V_a f\|_{C[0,1]/\mathbb{R}}$$
$$= \sup_{f\in B_{L^1}} \inf_{k\in\mathbb{R}} \|V_a f - k\|_\infty.$$

For every $\epsilon > 0$ there is an $f \in B_{L^1}$ such that

$$d_2(V_a) - \epsilon \leq \inf_{k\in\mathbb{R}} \|V_a f - k\|_\infty$$
$$d_2(V_a) - \epsilon \leq \inf_{k\in\mathbb{R}} \sup_{0\leq x\leq 1} |(V_a f - k)(x)| \leq \sup_{0\leq x\leq 1} |(V_a f - k)(x)| .$$

On considering $k = \frac{1}{2}\int_0^{1+a} f(t)dt$, the following is obtained

$$\begin{aligned}
d_2(V_a) - \epsilon &\leq \sup_{0\leq x\leq 1} \left| \int_0^{(1+a)x} f(t)dt - \frac{1}{2}\int_0^{1+a} f(t)dt \right| \\
&= \sup_{0\leq x\leq 1} \left| \int_0^{(1+a)x} f(t)dt - \frac{1}{2}\int_0^{(1+a)x} f(t)dt - \frac{1}{2}\int_{(1+a)x}^{1+a} f(t)dt \right| \\
&= \sup_{0\leq x\leq 1} \left| \frac{1}{2}\int_0^{(1+a)x} f(t)dt - \frac{1}{2}\int_{(1+a)x}^{1+a} f(t)dt \right| \\
&= \frac{1}{2}\sup_{0\leq x\leq 1} \left| \int_0^{(1+a)x} f(t)dt - \int_{(1+a)x}^{1+a} f(t)dt \right| \\
&\leq \frac{1}{2}\sup_{0\leq x\leq 1} \left[\left| \int_0^{(1+a)x} f(t)dt \right| + \left| \int_{(1+a)x}^{1+a} f(t)dt \right| \right] \\
&\leq \frac{1}{2}\sup_{0\leq x\leq 1} \left[\int_0^{(1+a)x} |f(t)|dt + \int_{(1+a)x}^{1+a} |f(t)|dt \right] \\
&= \frac{1}{2}\int_0^{(1+a)} |f(t)|dt = \frac{1}{2}\|f\|_{L^1} \leq \frac{1}{2}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, therefore we get $d_2(V_a) \leq \frac{1}{2}$ and hence $d_n(V_a) \leq \frac{1}{2}$ for all $n \in \mathbb{N} \setminus \{1\}$. □

Lemma 3.3. *For $n \geq 2$, the Gelfand numbers of V_a are estimated as*

$$c_n\left(V_a : L^1 \to C[0,1]\right) \geq \frac{1}{2}.$$

Proof. By definition of Gelfand numbers, corresponding to each $\epsilon > 0$, one can have a subspace $M \subseteq L^1$ with codimension $M < n$ such that

$$c_n(V_a) + \epsilon \geq \sup_{f\in B_M} \|V_a f\|_\infty. \tag{3.1}$$

To prove the lemma, we define functions $f_j = \frac{2^{j+1}}{1+a}\chi_{I_j}$, where $I_j = \left(\frac{1+a}{2^{j+1}}, \frac{1+a}{2^j}\right)$. Then for $j \in \mathbb{N}$,

$$\begin{aligned}
\|f_j\|_{L^1} &= \int_0^{1+a} |f_j(t)|dt = \int_0^{1+a} \left| \frac{2^{j+1}}{1+a}\chi_{I_j} \right| dt = \frac{2^{j+1}}{1+a}\int_0^{1+a} |\chi_{I_j}|dt \\
&= \frac{2^{j+1}}{1+a}\left[\int_0^{\frac{1+a}{2^{j+1}}} |\chi_{I_j}|dt + \int_{\frac{1+a}{2^{j+1}}}^{\frac{1+a}{2^j}} |\chi_{I_j}|dt + \int_{\frac{1+a}{2^j}}^{1+a} |\chi_{I_j}|dt \right] \\
&= \frac{2^{j+1}}{1+a}\left[\frac{1+a}{2^j} - \frac{1+a}{2^{j+1}} \right] = 1,
\end{aligned}$$

and for distinct j and k, $\|f_j - f_k\|_{L^1}$

$$
\begin{aligned}
&= \int_0^{1+a} |(f_j - f_k)(t)|dt = \int_0^{1+a} \left| \left(\frac{2^{j+1}}{1+a}\chi_{I_j} - \frac{2^{k+1}}{1+a}\chi_{I_k} \right)(t) \right| dt \\
&= \frac{1}{1+a} \int_0^{1+a} \left| \left(2^{j+1}\chi_{I_j} - 2^{k+1}\chi_{I_k}\right)(t) \right| dt \\
&= \frac{1}{1+a} \Bigg[\int_0^{\frac{1+a}{2^{k+1}}} \left| \left(2^{j+1}\chi_{I_j} - 2^{k+1}\chi_{I_k}\right)(t) \right| dt + \int_{\frac{1+a}{2^{k+1}}}^{\frac{1+a}{2^k}} \left| \left(2^{j+1}\chi_{I_j} - 2^{k+1}\chi_{I_k}\right)(t) \right| dt \\
&+ \int_{\frac{1+a}{2^k}}^{\frac{1+a}{2^{j+1}}} \left| \left(2^{j+1}\chi_{I_j} - 2^{k+1}\chi_{I_k}\right)(t) \right| dt + \int_{\frac{1+a}{2^{j+1}}}^{\frac{1+a}{2^j}} | \left(2^{j+1}\chi_{I_j} - 2^{k+1}\chi_{I_k}\right)(t)|dt \\
&+ \int_{\frac{1+a}{2^j}}^{1+a} \left| \left(2^{j+1}\chi_{I_j} - 2^{k+1}\chi_{I_k}\right)(t) \right| dt \Bigg] \text{ (for } j < k) \\
&= \frac{1}{1+a} \left[\left|-2^{k+1}\right| \left(\frac{1+a}{2^k} - \frac{1+a}{2^{k+1}} \right) + \left|2^{j+1}\right| \left(\frac{1+a}{2^j} - \frac{1+a}{2^{j+1}} \right) \right] \\
&= 2,
\end{aligned}
$$

and

$$
\begin{aligned}
\|V_a f_j - V_a f_k\|_\infty &= \sup_{0\le x\le 1} |(V_a f_j - V_a f_k)(x)| = \sup_{0\le x\le 1} |V_a f_j(x) - V_a f_k(x)| \\
&= \sup_{0\le x\le 1} \left| \int_0^{(1+a)x} f_j(t)dt - \int_0^{(1+a)x} f_k(t)dt \right| \\
&= \sup_{0\le x\le 1} \left| \int_0^{(1+a)x} \frac{2^{j+1}}{1+a}\chi_{I_j}(t)dt - \int_0^{(1+a)x} \frac{2^{k+1}}{1+a}\chi_{I_k}(t)dt \right| \\
&= \frac{1}{1+a} \sup_{0\le x\le 1} \left| \int_0^{(1+a)x} 2^{j+1}\chi_{I_j}(t)dt - \int_0^{(1+a)x} 2^{k+1}\chi_{I_k}(t)dt \right|.
\end{aligned}
$$

If we assume $j > k$, then either $(1+a)x \in I_j$ or $(1+a)x \in I_k$. Therefore, we have

$$\|V_a f_j - V_a f_k\|_\infty = 1.$$

Now adopting the procedure as in [1, Lemma 3.2], we obtain $c_n(V_a) \ge \frac{1}{2}$ for all $n \ge 2$. □

Lemma 3.4. *For* $n \in \mathbb{N}$, $b_n\left(V_a : L^1 \to C[0,1]\right) \le \frac{1}{2n-1}$.

Proof. By definition of n-th Bernstein numbers, for every $\epsilon_1 > 0$ there is a closed subspace M of L^1 with $dim M \ge n$ such that

$$b_n(V_a) - \epsilon_1 \le \inf_{f\in S_M} \|V_a f\|_\infty.$$

For $g \in M$, let $f = \frac{g}{\|g\|_{L^1}}$. Then $f \in S_M$ and we have

$$\|V_a g\|_\infty \ge \|g\|_{L^1}\left(b_n(V_a) - \epsilon_1\right). \tag{3.2}$$

Let M be an n-dimensional subspace of L^1. Then by linearity and injectivity of V_a, $V_a(M)$ is an n-dimensional subspace of $C[0,1]$. By [9, Proposition 1.4], for any $\epsilon_2 > 0$ there exists $g \in M$ such that $\|V_a(g)\|_\infty \le 1 + \epsilon_2$, and an n-tuple of points $t_1 < t_2 < ... < t_n$ in $[0,1]$ such that $V_a(g)(t_k) = (-1)^k,\ 1 \le k \le n$. We define a function $\eta : [0,1] \to [0,1+a]$ by $\eta(x) = (1+a)x$. Then clearly η is a homeomorphism and corresponding to each point $t_k \in]0,1[$, we have $\eta(t_k) \in]0,1+a[$ and $0 < \eta(t_1) < \eta(t_2) < ... < \eta(t_n) < 1+a$. For the calculation of $b_n(V_a)$, we calculate the norm of $g \in M$ as $\|g\|_{L^1}$

$$
\begin{aligned}
&= \int_0^{1+a} |g(t)|\, dt \\
&= \int_0^{\eta(t_1)} |g(t)|\, dt + \int_{\eta(t_1)}^{\eta(t_2)} |g(t)|\, dt + ... + \int_{\eta(t_{n-1})}^{\eta(t_n)} |g(t)|\, dt \\
&+ \int_{\eta(t_n)}^{1+a} |g(t)|\, dt \\
&\geq \left| \int_0^{\eta(t_1)} g(t)dt \right| + \left| \int_{\eta(t_1)}^{\eta(t_2)} g(t)dt \right| + ... + \left| \int_{\eta(t_{n-1})}^{\eta(t_n)} g(t)dt \right| \\
&+ \left| \int_{\eta(t_n)}^{1+a} g(t)dt \right| \\
&\geq \left| \int_0^{\eta(t_1)} g(t)dt \right| + \left| \int_{\eta(t_1)}^{\eta(t_2)} g(t)dt \right| + ... + \left| \int_{\eta(t_{n-1})}^{\eta(t_n)} g(t)dt \right| \\
&= \left| \int_0^{\eta(t_1)} g(t)dt \right| + \sum_{k=1}^{n-1} \left| \int_{\eta(t_k)}^{\eta(t_{k+1})} g(t)dt \right| \\
&= \left| \int_0^{\eta(t_1)} g(t)dt \right| + \sum_{k=1}^{n-1} \left| \int_0^{\eta(t_{k+1})} g(t)dt - \int_0^{\eta(t_k)} g(t)dt \right| \\
&= \left| \int_0^{(1+a)t_1} g(t)dt \right| + \sum_{k=1}^{n-1} \left| \int_0^{(1+a)t_{k+1}} g(t)dt - \int_0^{(1+a)t_k} g(t)dt \right| \\
&= |V_a(g)(t_1)| + \sum_{k=1}^{n-1} |V_a(g)(t_{k+1}) - V_a(g)(t_k)| \\
&= |(-1)| + \sum_{k=1}^{n-1} \left|(-1)^{k+1} - (-1)^k\right| = 1 + 2(n-1).
\end{aligned}
$$

Therefore, we obtain

$$\|g\|_{L^1} \geq 2n - 1.$$

Now by (3.2), for every $g \in M$, we have

$$1 + \epsilon_2 \geq \|V_a g\|_\infty \geq (b_n(V_a) - \epsilon_1)\|g\|_{L^1} \geq (b_n(V_a) - \epsilon_1)(2n-1),$$

which implies that

$$\frac{1+\epsilon_2}{2n-1} \geq b_n(V_a) - \epsilon_1,$$

and, hence, by arbitrariness of ϵ_1 and ϵ_2, we get

$$b_n(V_a) \leq \frac{1}{2n-1}.$$

□

Lemma 3.5. *For $n \in \mathbb{N}$ and $0 \le a < \infty$, $\mathfrak{m}_n\big(V_a : L^1[0,1+a] \to C[0,1]\big) \le \frac{1}{2n-1}$.*

Proof. By ([12, page 210]), we have

$$\mathfrak{m}_{\mathfrak{n}}(T) = \sup_{\substack{N\subset Y\\ codim\, N=n}} \sup\{\alpha \ge 0 : \alpha B_{Y/N} \subset (\pi_N \circ T)B_X\}.$$

Therefore, for every $\epsilon > 0$, there exists a subspace N of C with codimension n such that

$$\mathfrak{m}_{\mathfrak{n}}(V_a) - \epsilon \le \sup\{\alpha \ge 0 : \alpha B_{C/N} \subset (\pi_N \circ V_a)B_{L^1}\}. \tag{3.3}$$

One can observe that $V_a(B_{L^1}) \not\subset N$, because if $V_a(f) \in N$ for all $f \in B_{L^1}$, then $(\pi_N \circ V_a)B_{L^1} \not\supset \alpha B_{C/N}$ for any $\alpha \ge 0$. It insures that there exists some $f \in L^1$ such that $V_a f \notin N$, we denote such functions by f_N. Now there exists a subspace Z of $C[0,1]$ with $dim Z = n$ such that $C[0,1] = N \oplus Z$. We write $V_a f_N = 0 + z$ for $f_N \in L^1$ and $z \in Z$. By linearity and injectivity of V_a, $V_a|_Z^{-1}$ exists. Consider the subspace $V_a^{-1}Z = \{f \in L^1 : V_a f \notin N \setminus \{0\}\}$ of L^1. Then $dim V_a^{-1}Z = n$. Then for all such $\alpha \ge 0$ as in (3.3) and for all $f \in S_{V_a^{-1}Z}$, there exists an $h \in (\alpha+\delta)S_{C/N} \cap \pi_N V_a B_{L^1}$ with $\delta \ge 0$ such that $\|h\|_{C/N} = \alpha + \delta$ and $h = V_a f + N$ for some $f \in V_a|_Z^{-1}$. Then we have $\|V_a f + N\|_{C/N} = \alpha + \delta$, from where it follows that $\|V_a f + N\|_{C/N} \ge \alpha$ for all $f \in S_{V_a^{-1}Z}$. Then one obtains

$$\|V_a f\|_\infty \ge \|V_a f + N\|_{C/N} \ge \alpha \text{ for all } f \in S_{V_a^{-1}Z}.$$

This gives

$$\|V_a f\|_\infty \ge \alpha \|f\|_{L^1} \text{ for all } f \in V_a^{-1}Z.$$

Then by adopting the procedure of the proof of Lemma 3.4, we obtain

$$\alpha \le \frac{1}{2n-1},$$

which by artbitrariness of ϵ, yields $\mathfrak{m}_{\mathfrak{n}}(V_a) \le \frac{1}{2n-1}$ for all $n \in \mathbb{N}$. This proves the lemma. $\square$

Lemma 3.6. *For $n \in \mathbb{N}$, $i_n\big(V_a : L^1 \to C[0,1]\big) \ge \frac{1}{2n-1}$.*

Proof. Consider the Banach space $l^1_{w,n}$, which is n-dimensional weighted subspace of sequence space l^1. For $\mathbf{x} = \{x_k\}_{k=1}^n \in l^1_{w,n}$, the norm is defined by

$$\|\{x_k\}\|_{k=1}^n{}_{l^1_{w,n}} = \sum_{k=1}^n w_k\,|x_k|\,,$$

where $w_k = 2$ for $1 \le k \le n-1$, and $w_n = 1$. For computing the isomorphism numbers, we construct maps $P : C[0,1] \to l^1_{w,n}$ and $Q : l^1_{w,n} \to L^1$ such that the tower

$$\ell^1_{w,n} \xrightarrow{Q} L^1[0,1+a] \xrightarrow{V_a} C[0,1] \xrightarrow{P} \ell^1_{w,n}$$

reduces to the identity map on $l^1_{w,n}$. We define the map $P : C[0,1] \to l^1_{w,n}$ by

$$(Pf)_k = \frac{2n-1}{1+a} f\left(\frac{2k-1}{2n-1}\right) \text{ for } 1 \le k \le n,$$

and towards the construction of the map Q we divide the unit interval $[0,1]$ into $2n-1$ sub-intervals denoted by $\xi_1, \xi_2, ..., \xi_{2n-1}$, where $\xi_k = \left[\frac{k-1}{2n-1}, \frac{k}{2n-1}\right]$ for $1 \le k \le 2n-1$. Now by the map η defined in proof of Lemma 3.4, corresponding to each sub-interval ξ_k, we have sub-interval I_k of the interval $[0,1+a]$ defined by

$I_k = \left[\frac{k-1}{2n-1}(1+a), \frac{k}{2n-1}(1+a)\right]$ for $1 \leq k \leq 2n-1$. Hence, corresponding to every partition of $[0,1]$ we obtain a unique partition of $[0,1+a]$, such that $\eta(x) \in I_k$ whenever $x \in \xi_k$. The map $Q : l^1_{w,n} \to L^1$ is defined by

$$Q\big(\{x_k\}_{k=1}^n\big) = \sum_{k=1}^{n-1} x_k \left(\chi_{I_{2k-1}} - \chi_{I_{2k}}\right) + x_n \chi_{I_{2n-1}}.$$

One observes that

$$\begin{aligned}\big((V_a \circ Q)\{x_k\}_{k=1}^n\big)(x) &= \big(V_a\big(Q(\{x_k\}_{k=1}^n)\big)\big)(x) \\ &= \int_0^{(1+a)x} \big(Q\{x_k\}_{k=1}^n\big)(t)dt \\ &= \begin{cases} x_k\big(x - \frac{2k-2}{2n-1}(1+a)\big) \text{ for } x \in \xi_{2k-1} \\ -x_k\big(x - \frac{2k}{2n-1}(1+a)\big) \text{ for } x \in \xi_{2k}. \end{cases}\end{aligned}$$

Therefore for $x \in \xi_{2k-1}$, we have

$$\begin{aligned}(P(V_a(B = Q\{x_k\}_{k=1}^n))(x))_k &= x_k \frac{2n-1}{1+a}\left(\frac{2k-1}{2n-1}(1+a) - \frac{2k-2}{2n-1}(1+a)\right) \\ &= x_k,\end{aligned}$$

and for $x \in \xi_{2k}$, one obtains

$$\begin{aligned}(P(V_a(Q\{x_k\}_{k=1}^n))(x))_k &= -x_k \frac{2n-1}{1+a}\left(\frac{2k}{2n-1}(1+a) - \frac{2k}{2n-1}(1+a)\right) \\ &= x_k,\end{aligned}$$

proving that $P \circ V_a \circ Q = I_{l^1_{w,n}}$. It is easily seen the maps P and Q are bounded operators. Their norms are computed as under.

$$\|Q\|_{op} = \sup_{\|\{x_k\}\|_{k=1}^n = 1} \|Q\left(\{x_k\}_{k=1}^n\right)\|_{L^1}, \tag{3.4}$$

where

$$\|Q\left(\{x_k\}_{k=1}^n\right)\|_{L^1} = \int_0^{1+a} \left|\left(Q\{x_k\}_{k=1}^n\right)(t)\right| dt$$

and by definition of Q, we have

$$\left(Q\{x_k\}_{k=1}^n\right)(t) = \begin{cases} x_k \text{ for } t \in I_{2k-1},\ 1 \leq k \leq n, \\ -x_k \text{ for } t \in I_{2k},\ 1 \leq k \leq n-1. \end{cases}$$

Therefore, $\|Q\left(\{x_k\}_{k=1}^n\right)\|_{L^1}$

$$
\begin{aligned}
&= \int_0^{1+a} \left|\left(Q\{x_k\}_{k=1}^n\right)(t)\right| dt \\
&= \int_0^{(1+a)(2n-1)^{-1}} \left|\left(Q\{x_k\}_{k=1}^n\right)(t)\right| dt \\
&+ \int_{(1+a)(2n-1)^{-1}}^{2(1+a)(2n-1)^{-1}} \left|\left(Q\{x_k\}_{k=1}^n\right)(t)\right| dt \\
&+ ... + \int_{(k-1)(1+a)(2n-1)^{-1}}^{k(1+a)(2n-1)^{-1}} \left|\left(Q\{x_k\}_{k=1}^n\right)(t)\right| dt + ... \\
&+ \int_{(2n-2)(1+a)(2n-1)^{-1}}^{1+a} \left|\left(Q\{x_k\}_{k=1}^n\right)(t)\right| dt \\
&= |x_1|\,|I_1| + |-x_1|\,|I_2| + ... + |x_n|\,|I_{2n-1}|,
\end{aligned}
$$

where the length of interval I_k, $1 \le k \le 2n-1$, is $|I_k| = \frac{1+a}{2n-1}$ and then we have

$$
\begin{aligned}
\|Q\left(\{x_k\}_{k=1}^n\right)\|_{L^1} &= \frac{1+a}{2n-1}\left[\sum_{k=1}^{n-1} 2|x_k| + x_n\right] = \frac{1+a}{2n-1}\left[\sum_{k=1}^{n-1} w_k|x_k|\right] \\
&= \frac{1+a}{2n-1}\|\{x_k\}_{k=1}^n\|_{l^1_{w,n}}.
\end{aligned}
$$

Therefore, by (3.4), we obtain $\|Q\|_{op} = \frac{1+a}{2n-1}$.
Next,

$$
\begin{aligned}
\|P\|_{op} &= \sup_{\|f\|_\infty \le 1} \|\{(Pf)_k\}_{k=1}^n\|_{l^1_{w,n}} = \sup_{\|f\|_\infty \le 1} \sum_{k=1}^n w_k \left|(Pf)_k\right| \\
&= \sup_{\|f\|_\infty \le 1} \sum_{k=1}^n w_k \left|\frac{2n-1}{1+a} f\left(\frac{2k-1}{2n-1}\right)\right| \\
&= \sup_{\|f\|_\infty \le 1} \left|\frac{2n-1}{1+a}\right| \sum_{k=1}^n w_k \left|f\left(\frac{2k-1}{2n-1}\right)\right|. \qquad (3.5)
\end{aligned}
$$

Now, for $f \in C[0,1]$ with $\|f\|_\infty \le 1$, we have $|f(x)| \le \|f\|_\infty \|x\| \le 1$. On using this into (3.5), one gets

$$
\|P\|_{op} \le \left|\frac{2n-1}{1+a}\right| \sum_{k=1}^n w_k \le \left|\frac{2n-1}{1+a}\right| (2(n-1)+1) \le \frac{(2n-1)^2}{1+a}.
$$

Let's take the function $f \in C[0,1]$, where $f(x) = 1$. Then obviuosly $\|f\|_\infty = 1$ and

$$
\begin{aligned}
\|Pf\|_{l^1_{w,n}} = \|\{(Pf)_k\}_{k=1}^n\|_{l^1_{w,n}} &= \sum_{k=1}^n w_k \left|\frac{2n-1}{1+a} f\left(\frac{2k-1}{2n-1}\right)\right| \\
&= \frac{(2n-1)^2}{1+a},
\end{aligned}
$$

and hence supremum is attained. Therefore, the operator norm is $\|P\|_{op} = \frac{(2n-1)^2}{1+a}$. Then by the definition of isomorphism numbers

$$i_n(V_a) = \sup_{dim\, E \geq n} \|P\|^{-1}\|Q\|^{-1} \geq \|P\|^{-1}\|Q\|^{-1} = \frac{1+a}{(2n-1)^2}.\frac{2n-1}{1+a} = \frac{1}{2n-1}.$$

□

4. Proof of Main Results

We now prove the main results.

Proof of Theorem 1.1. We know that the space L^1 has the lifting property, therefore by Lemma 3.2, we have $a_n(V_a) = d_n(V_a) \leq \frac{1}{2}$ for all $n \geq 2$. Since approximation numbers are biggest among all the s-numbers, by Lemma 3.3, we obtain $\frac{1}{2} \leq c_n(V_a) \leq a_n(V_a) \leq \frac{1}{2}$ for all $n \geq 2$ and hence $a_n(V_a) = c_n(V_a) = \frac{1}{2}$ for all $n \geq 2$. Again by lifting property, we obtain $d_n(V_a) = a_n(V_a) = \frac{1}{2}$ for all $n \geq 2$, proving (1.2). □

Proof of Theorem 1.2. By the fact that amongst all strict s-numbers, isomorphism numbers are smallest, Lemmas 3.4 and 3.6 give us $\frac{1}{2n-1} \leq i_n(V_a) \leq b_n(V_a) \leq \frac{1}{2n-1}$ for all $n \in \mathbb{N}$, and by Lemmas 3.5 and 3.6, we obtain $\frac{1}{2n-1} \leq i_n(V_a) \leq \mathfrak{m}_n(V_a) \leq \frac{1}{2n-1}$ for all $n \in \mathbb{N}$, from where we get $b_n(V_a) = \mathfrak{m}_n(V_a) = i_n(V_a) = \frac{1}{2n-1}$ for all $n \in \mathbb{N}$. This establishes (1.3). □

From [7], we give following remark.

Remark. *Following the Theorem 1.2, the convergence of $b_n(V_a)$ and $\mathfrak{m}_n(V_a)$ towards zero implies finite strict singularity and finite strict co-singularity of the operator respectively.*

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