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ROUGH $\mathcal{I}$-CONVERGENCE OF SEQUENCES IN 2-NORMED SPACES

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Abstract. In this paper, we have introduced the notions of rough $\mathcal{I}$-convergence of sequences and defined $\mathcal{I}$-cluster point of a sequence and proved some results associated with this notion in 2-normed spaces. Also, we have defined rough $\mathcal{I}$-limit set of a sequence and shown that this set is closed and convex. Further, we have defined the notion of rough $\mathcal{I}$-Cauchy sequence in the same space.

1. Introduction

Since sequence convergence plays a very important role in the fundamental theory of mathematics, there are many convergence concepts in summability theory, classical measure theory, approximation theory and probability theory and the relationships between them are discussed. In 1951, the idea of ordinary convergence of real sequences was extended to statistical convergence of real sequences independently by Fast [12], Steinhaus [29] and Schoenberg [30]. After long 50 years, in 2000 Kostyrko et al. [19] introduced the concept of $\mathcal{I}$-convergence of sequences as a generalization of statistical convergence where $\mathcal{I}$ is an ideal of subsets of the set of natural numbers. Since then this idea has been nurtured by several authors in different directions [4, 7, 21, 22, 33, 28].

In 2001, Phu [24] first introduced the notion of rough convergence of sequences in finite dimensional normed spaces and in the same paper he investigated that r-limit set is bounded, closed and convex and some interesting results were studied by Phu [24, 25]. In 2003, Phu [26] extended this concept to infinite dimensional normed spaces. Later, this notion was extended into rough statistical convergence [1], rough ideal convergence [9, 27] and this idea was studied by many authors in different directions and different spaces as in [2, 5, 10, 17, 18, 20]. The reader may refer to the textbooks [6] and [23] for summability theory, sequence spaces and related topics.

The concept of 2-normed spaces was introduced and studied by S. Gähler, a German Mathematician who worked at German Academy of Science, Berlin, in a series of paper in German language in Mathematische Nachrichten, see for example

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references [15, 16]. This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. of USA in 1969 entitled 2-Banach spaces [34]. In the same year Gähler published another paper on this theme in the same journal [15]. A. H. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with S. Gähler and S. C. Gupta [16] of 1975 also provide valuable results related to the theme of this paper. Results upto 1977 were summarized in the survey paper by A. H. Siddiqi [31]. For more details, the readers may refer to the books [13, 11]. Arslan and Dündar[2, 3] have studied the notions of rough convergence and rough statistical convergence in 2-normed spaces. So, nowadays, in the light of various and growing applications of ideals it is very natural to extend the interesting notions of rough statistical convergence in 2-normed spaces to ideal version of the convergence in the same space. In this paper, we investigate some results in the most general possible form which are ideal analogues to the results of [2, 3].

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of reals respectively. First we recall some basic definitions and notations.

Definition 2.1. [19] *A family $\mathcal{I}$ of subsets of a non empty set Y is said to be an ideal in Y if*

(1) $\emptyset \in \mathcal{I}$;
(2) $A, B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$;
(3) $A \in \mathcal{I}$ *and* $B \subset A$ *implies* $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called non trivial if $Y \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$. A non trivial ideal $\mathcal{I}$ is called admissible if $\{\{x\} : x \in Y\} \subset \mathcal{I}$.

Definition 2.2. [19] *A non empty family $\mathcal{F}$ of subsets of a non empty set Y is called a filter in Y if the following properties hold:*

(1) $\emptyset \notin \mathcal{F}$;
(2) $A, B \in \mathcal{F}$ *implies* $A \cap B \in \mathcal{F}$;
(3) $A \in \mathcal{F}$ *and* $A \subset B$ *implies* $B \in \mathcal{F}$.

Lemma 2.1. *(see*[19]*) If $\mathcal{I} \subset 2^Y$ is a non trivial ideal then the class $\mathcal{F}(\mathcal{I}) = \{Y \setminus A : A \in \mathcal{I}\}$ is a filter on Y which is called filter associated with the ideal $\mathcal{I}$.*

Definition 2.3. *(see* [3]*) Let K be a subset of the set of positive integers $\mathbb{N}$ and let us denote the set $\{k \in K : k \leq n\}$ by $K(n)$. Then the natural density of K is $\delta(K) = \lim_{n\to\infty} \frac{|K(n)|}{n}$, where $|K(n)|$ denotes the number of elements in $K(n)$.*

Definition 2.4. [14] *Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies the following conditions:*

(1) $\|x, y\| = 0$ *if and only if x and y are linearly dependent in X;*
(2) $\|x, y\| = \|y, x\|$ *for all x, y in X;*
(3) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ *for all α in $\mathbb{R}$ and for all x, y in X;*
(4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ *for all x, y, z in X.*

The pair $(X, \|.,.\|)$ is then called a 2-normed space.

Example 2.1. *Let $X = \mathbb{R}^2$. Define $\|\cdot,\cdot\|$ on $\mathbb{R}^2$ by $\|x, y\| = |x_1y_2 - x_2y_1|$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then $(X, \|\cdot,\cdot\|)$ is a 2-normed space.*

Definition 2.5. *Let L be an element in a 2-normed space $(X, \|\cdot,\cdot\|)$. Then for each non zero $z \in X$, closed ball $\overline{B_r(L)}$ and open ball $B_r(L)$ of radius $r > 0$ with centered L are defined as $\overline{B_r(L)} = \{y \in X : \|y - L, z\| \leq r\}$ and $B_r(L) = \{y \in X : \|y - L, z\| < r\}$.*

Definition 2.6. *(see[2]) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim_{n\to\infty} \|x_n - L, z\| = 0$, for every $z \in X$. In such a case L is called limit of $\{x_n\}_{n\in\mathbb{N}}$ and we write $x_n \xrightarrow{\|\cdot,\cdot\|} L$.*

Definition 2.7. [32] *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a 2-normed space X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ and $z \in X$ the set $\{n \in \mathbb{N} : \|x_n - x, z\| \geq \varepsilon\}$ belongs to $\mathcal{I}$. In this case x is called $\mathcal{I}$-limit of $\{x_n\}_{n\in\mathbb{N}}$ and we write $x_n \xrightarrow{\mathcal{I}} x$.*

Definition 2.8. [2] *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $(X, \|\cdot,\cdot\|)$ 2-normed linear space and r be a non negative real number. $\{x_n\}_{n\in\mathbb{N}}$ is said to be rough convergent (r-convergent) to L if*

$$\forall\varepsilon > 0,\ \exists\ n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow \|x_n - L, z\| < r + \varepsilon.$$

In this case L is called rough limit (r-limit) of $\{x_n\}_{n\in\mathbb{N}}$ and we write $x_n \xrightarrow{r} L$. In general r-limit of a sequence is no more unique for $r > 0$. So we consider the so-called r-limit set of $\{x_n\}_{n\in\mathbb{N}}$ denoted by $LIM_2^r(x_n)$ and defined by $LIM_2^r(x_n) = \{L \in X : x_n \xrightarrow{r} L\}$.

Definition 2.9. [3] *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be rough statistically convergent (r_2st-convergent) to L, provided the set $\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}$ has natural density zero, for every $\varepsilon > 0$ and each non zero $z \in X$.*

3. Main Results

Throughout the paper $\mathcal{I}$ stands for a non trivial admissible ideal in $\mathbb{N}$ and X denotes a 2-nomed space unless otherwise stated. First we introduce the definition of rough $\mathcal{I}$-convergence in 2-normed spaces.

Definition 3.1. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and r be a non negative real number. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be rough $\mathcal{I}$-convergent to $\xi \in X$ if for every $\varepsilon > 0$ and each non zero $z \in X$ the set $\{n \in \mathbb{N} : \|x_n - \xi, z\| \geq r + \varepsilon\} \in \mathcal{I}$. In this case ξ is called rough $\mathcal{I}$-limit ($r - \mathcal{I}$-limit) of $\{x_n\}_{n\in\mathbb{N}}$ with respect to the 2-norm $\|\cdot,\cdot\|$ and we write $x_n \xrightarrow{r-\mathcal{I}} \xi$.*

Remark 3.1. (i) *Here r is called roughness degree of rough $\mathcal{I}$-convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$. If we put $r = 0$ in the above definition then the notion of rough $\mathcal{I}$-convergence coincides with the notion of ordinary $\mathcal{I}$-convergence with respect to the 2-norm $\|\cdot,\cdot\|$. So in this regard the whole discussion is on the fact $r > 0$.*
(ii) *Let $\mathcal{I}_f$ be the class of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal. Now, if $\mathcal{I} = \mathcal{I}_f$ then rough $\mathcal{I}$-convergence agrees with rough convergence with respect to the 2-norm $\|\cdot,\cdot\|$.*
(iii) *If we take $\mathcal{I}_\delta$ as the class of all subsets of $\mathbb{N}$ whose natural density are zero.*

Then $\mathcal{I}_\delta$ will be a non-trivial admissible ideal. If $\mathcal{I} = \mathcal{I}_\delta$ then rough $\mathcal{I}$-convergence agrees with rough statistical convergence with respect to the 2-norm $\|\cdot,\cdot\|$.

It may happen that $\mathcal{I}$-convergence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is not assured but there may exist a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X such that it is $\mathcal{I}$-convergent and satisfies the condition $\{n \in \mathbb{N} : \|x_n - y_n, z\| \geq r\} \in \mathcal{I}$ for each non zero $z \in X$. Then $\{x_n\}_{n\in\mathbb{N}}$ is rough $\mathcal{I}$-convergent to the same $\mathcal{I}$-limit of $\{y_n\}_{n\in\mathbb{N}}$. Indeed, since for every $\varepsilon > 0$ and each non zero $z \in X$ $\{n \in \mathbb{N} : \|x_n - \xi, z\| \geq r + \varepsilon\} \subseteq \{n \in \mathbb{N} : \|x_n - y_n, z\| \geq r\} \cup \{n \in \mathbb{N} : \|y_n - \xi, z\| \geq \varepsilon\}$.

In general, the rough $\mathcal{I}$-limit of a sequence is not unique in a 2-normed space which can be described by the following example. So we consider the set of all rough $\mathcal{I}$-limits of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X denoted by $\mathcal{I} - LIM_2^r(x_n)$ and defined by $\mathcal{I} - LIM_2^r(x_n) = \{\xi \in X : x_n \xrightarrow{r-\mathcal{I}} \xi\}$. So we have a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be rough $\mathcal{I}$-convergent if $\mathcal{I} - LIM_2^r(x_n) \neq \emptyset$.

Example 3.1. *Let $X = \mathbb{R}^2$ equipped with the 2-norm $\|\cdot,\cdot\|$ defined by Example 2.1. Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ which contains all those subsets of $\mathbb{N}$ having natural density zero. Let us define $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_n = \begin{cases} ((-1)^n, 0), & \text{if } n \neq i^2 (i \in \mathbb{N}) \\ (n, n), & \text{otherwise} \end{cases}$. Then for $r \geq 1$, $\mathcal{I} - LIM_2^r(x_n) = \overline{B_r((-1,0))} \cap \overline{B_r((1,0))}$, since for $\xi \in \overline{B_r((-1,0))} \cap \overline{B_r((1,0))}$, where $\overline{B_r(x_0)} = \{y \in X : \|y - x_0, z\| \leq r\}$ for each non zero $z \in X$, we have $\{n \in \mathbb{N} : \|x_n - \xi, z\| \geq r + \varepsilon\} \subset \{1^2, 2^2, 3^2, \ldots, i^2, \ldots\}$. Since the later set of this inclusion has natural density zero, $\{n \in \mathbb{N} : \|x_n - \xi, z\| \geq r + \varepsilon\} \in \mathcal{I}$. And if $r < 1$ then $\mathcal{I} - LIM_2^r(x_n) = \emptyset$. Also, we have $LIM_2^r(x_n) = \emptyset$ for any r.*

Remark 3.2. *From the Example 3.1, we see $\mathcal{I} - LIM_2^r(x_n) \neq \emptyset$ does not imply $LIM_2^r(x_n) \neq \emptyset$. But $LIM_2^r(x_n) \neq \emptyset$ always implies $\mathcal{I} - LIM_2^r(x_n) \neq \emptyset$ since a finite subset of $\mathbb{N}$ belongs to $\mathcal{I}$. So, we conclude $LIM_2^r(x_n) \subseteq \mathcal{I} - LIM_2^r(x_n)$.*

Theorem 3.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$ such that $x_n \xrightarrow{r_1-\mathcal{I}} \xi$. If $r_1 < r_2 (r_1, r_2 > 0)$ then $x_n \xrightarrow{r_2-\mathcal{I}} \xi$.*

Proof. The proof directly follows the Definition 3.1. □

Remark 3.3. *From the above theorem we can conclude $\mathcal{I} - LIM_2^{r_1}(x_n) \subset \mathcal{I} - LIM_2^{r_2}(x_n)$.*

We take r as a positive real number in the sequal unless otherwise stated.

Definition 3.2. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-bounded with respect to the 2-norm $\|\cdot,\cdot\|$ if there exists a number $G > 0$ such that for each non zero $z \in X$, the set $\{n \in \mathbb{N} : \|x_n, z\| \geq G\} \in \mathcal{I}$.*

We have seen in [2] that if $\{x_n\}$ in X is bounded then $LIM_2^r(x_n) \neq \emptyset$ and hence $\mathcal{I} - LIM_2^r(x_n) \neq \emptyset$. Now we find out a relationship between $\mathcal{I}$-boundedness of a sequence and its rough $\mathcal{I}$-limit set.

Theorem 3.2. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$. Then $\{x\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded if and only if there exists some $r > 0$ such that $\mathcal{I} - LIM_2^r(x_n) \neq \emptyset$.*

Proof. First suppose that $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$ bounded sequence. Then there exists a number $G > 0$ such that for each non zero $z \in X$, $\{n \in \mathbb{N} : \|x_n, z\| \geq G\} \in \mathcal{I}$.

Let $A = \{n \in \mathbb{N} : \|x_n, z\| < G\}$ and $r = sup\{\|x_n, z\| : n \in A\}$. So for $n \in A$ and for each non zero $z \in X$, $\|x_n, z\| \leq r \Rightarrow \|x_n - \theta, z\| < r + \varepsilon$ for any $\varepsilon > 0$, where θ is the zero vector of X. Therefore $\{n \in \mathbb{N} : \|x_n - \theta, z\| \geq r + \varepsilon\} \in \mathcal{I}$. So $\theta \in \mathcal{I} - LIM_2^r(x_n)$ and hence $\mathcal{I} - LIM_2^r(x_n) \neq \emptyset$.

Conversely suppose that $\mathcal{I} - LIM_2^r(x_n) \neq \emptyset$ for some $r > 0$ and $\xi \in \mathcal{I} - LIM_2^r(x_n)$. Let $\varepsilon > 0$ be given. Then for each non zero $z \in X$, $\{n \in \mathbb{N} : \|x_n - \xi, z\| \geq r + \varepsilon\} \in \mathcal{I}$. Let $M = \sup\{\|\xi, z\| : z \in X\}$. Since $\|x_n, z\| \leq \|x_n - \xi, z\| + \|\xi, z\| \leq \|x_n - \xi, z\| + M$. Therefore $\{n \in \mathbb{N} : \|x_n, z\| \geq r + \varepsilon + M\} \subseteq \{n \in \mathbb{N} : \|x_n - \xi, z\| \geq r + \varepsilon\}$. Suppose $G = r + \varepsilon + M$. Therefore $\{n \in \mathbb{N} : \|x_n, z\| \geq G\} \in \mathcal{I}$. This shows that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded. □

Theorem 3.3. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$. Then we have $diam(\mathcal{I} - LIM_2^r(x_n)) \leq 2r$. In general, $diam(\mathcal{I} - LIM_2^r(x_n))$ has no smaller bound.*

Proof. First suppose that $diam(\mathcal{I} - LIM_2^r(x_n)) > 2r$. Then there exist $\xi_1, \xi_2 \in \mathcal{I} - LIM_2^r(x_n)$ such that for each non zero $z \in X$, $\|\xi_1 - \xi_2, z\| > 2r$. Let $0 < \varepsilon < \frac{\|\xi_1 - \xi_2, z\|}{2} - r$. Suppose $A = \{n \in \mathbb{N} : \|x_n - \xi_1, z\| \geq r + \varepsilon\}$ and $B = \{n \in \mathbb{N} : \|x_n - \xi_2, z\| \geq r + \varepsilon\}$. Therefore $A, B \in \mathcal{I}$. Now, for each non zero $z \in X$ and $n \in A^c \cap B^c$ we have $\|\xi_1 - \xi_2, z\| \leq \|x_n - \xi_1, z\| + \|x_n - \xi_2, z\| < r + \varepsilon + r + \varepsilon = 2(r + \varepsilon) < \|\xi_1 - \xi_2, z\|$, which is a contradiction. Therefore $diam(\mathcal{I} - LIM_2^r(x_n)) \leq 2r$.

Now, for the second part we suppose a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to ξ. Then for every $\varepsilon > 0$ and each non zero $z \in X$, $\{n \in \mathbb{N} : \|x_n - \xi, z\| \geq \varepsilon\} \in \mathcal{I}$. Now, let $A = \{n \in \mathbb{N} : \|x_n - \xi, z\| \geq \varepsilon\}$ and $\beta \in \overline{B_r(\xi)}$. So, for $n \in A^c$ and each non zero $z \in X$ we have $\|x_n - \beta, z\| \leq \|x_n - \xi, z\| + \|\xi - \beta, z\| < \varepsilon + r$. Therefore $\{n \in \mathbb{N} : \|x_n - \beta, z\| \geq r + \varepsilon\} \in \mathcal{I}$ i.e. $\beta \in \mathcal{I} - LIM_2^r(x_n)$ and as a result, we write $\mathcal{I} - LIM_2^r(x_n) = \overline{B_r(\xi)}$. This shows that upper bound $2r$ of the diameter of the set $\mathcal{I} - LIM_2^r(x_n)$ can not be reduced anymore. This completes the proof. □

Now we will state the algebraic characterization of rough $\mathcal{I}$-convergence of sequences in 2-normed spaces.

Theorem 3.4. *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences in a 2-normed space $(X, \|\cdot,\cdot\|)$. Then, for some roughness degree $r > 0$ the following statements hold:*

(1) *If $x_n \xrightarrow{r-\mathcal{I}} \xi$ and $\alpha(\neq 0) \in \mathbb{R}$ then $\alpha x_n \xrightarrow{r-\mathcal{I}} \alpha\xi$*
(2) *If $x_n \xrightarrow{r-\mathcal{I}} \xi$ and $y_n \xrightarrow{r-\mathcal{I}} \beta$ then $x_n + y_n \xrightarrow{r-\mathcal{I}} \xi + \beta$.*

Proof. The proof is easy. So, we omit details. □

We will discuss on some topological and geometrical properties of rough $\mathcal{I}$-limit set of a sequence.

Theorem 3.5. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $X, \|\cdot,\cdot\|$. Then the set $\mathcal{I} - LIM_2^r(x_n)$ is closed.*

Proof. Let η be an arbitrary limit point of $\mathcal{I} - LIM_2^r(x_n)$. Then for every $\varepsilon > 0$, $B_{\frac{\varepsilon}{2}}(\eta) \cap \mathcal{I} - LIM_2^r(x_n) \neq \emptyset$. Let $x_* \in B_{\frac{\varepsilon}{2}}(\eta) \cap \mathcal{I} - LIM_2^r(x_n)$. Then for each non zero $z \in X$, the set $M = \{n \in \mathbb{N} : \|x_n - x_*, z\| \geq r + \frac{\varepsilon}{2}\} \in \mathcal{I}$ and $\|x_* - \eta, z\| < \frac{\varepsilon}{2}$. Now for $n \in M^c$, $\|x_n - \eta, z\| \leq \|x_n - x_*, z\| + \|x_* - \eta, z\| < r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon$. Therefore $\{n \in \mathbb{N} : \|x_n - \eta, z\| \geq r + \varepsilon\} \in \mathcal{I}$. Hence $\eta \in \mathcal{I} - LIM_2^r(x_n)$. This completes the proof. □

Theorem 3.6. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $X, \|\cdot,\cdot\|$. Then for some $r>0$ the set $\mathcal{I}-LIM_2^r(x_n)$ is convex.*

Proof. Let $\xi_1, \xi_2 \in \mathcal{I}-LIM_2^r(x_n)$. Then for every $\varepsilon>0$ and each non zero $z\in X$, $A=\{n\in\mathbb{N} : \|x_n-\xi_1, z\| \geq r+\varepsilon\} \in \mathcal{I}$ and $B=\{n\in\mathbb{N} : \|x_n-\xi_2, z\| \geq r+\varepsilon\} \in \mathcal{I}$. Now, for $n\in A^c\cap B^c$ and for each $\alpha\in[0,1]$, $\|x_n-[(1-\alpha)\xi_1+\alpha\xi_2], z\| = \|(1-\alpha)(x_n-\xi_1)+\alpha(x_n-\xi_2), z\| \leq (1-\alpha)\|x_n-\xi_1, z\| + \alpha\|x_n-\xi_2, z\| < (1-\alpha)(r+\varepsilon)+\alpha(r+\varepsilon) = r+\varepsilon$. Therefore $\{n\in\mathbb{N} : \|x_n-[(1-\alpha)\xi_1+\alpha\xi_2], z\| \geq r+\varepsilon\} \subset A\cup B \in \mathcal{I}$. This gives $(1-\alpha)\xi_1+\alpha\xi_2 \in \mathcal{I}-LIM_2^r(x_n)$ i.e. $\mathcal{I}-LIM_2^r(x_n)$ is a convex set. □

Theorem 3.7. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is rough $\mathcal{I}$-convergent to $\xi\in X$ if and only if there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X such that $y_n \xrightarrow{\mathcal{I}} \xi$ and $\|x_n-y_n, z\| \leq r$ for $n\in\mathbb{N}$ and each non zero $z\in X$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ is rough $\mathcal{I}$-convergent to ξ. Then, for each non zero $z\in X$ we have

$$\mathcal{I}-\limsup \|x_n-\xi, z\| \leq r. \tag{3.1}$$

Now, for each non zero $z\in X$ we define a sequence $\{y_n\}_{n\in\mathbb{N}}$ by

$$y_n = \begin{cases} \xi & \text{if } \|x_n-\xi, z\| \leq r \\ x_n + r\frac{\xi-x_n}{\|x_n-\xi, z\|} & \text{otherwise.} \end{cases}$$

i.e. we have

$$\|y_n-\xi, z\| = \begin{cases} 0 & \text{if } \|x_n-\xi, z\| \leq r \\ \|x_n-\xi, z\| - r & \text{otherwise.} \end{cases}$$

Thus, from the definition of y_n, we can write $\|x_n-y_n, z\| \leq r$ for $n\in\mathbb{N}$ and also, using 3.1 we have $\mathcal{I}-\limsup \|y_n-\xi, z\| = 0$ i.e. $y_n \xrightarrow{\mathcal{I}} \xi$.

Conversely, suppose that $y_n \xrightarrow{\mathcal{I}} \xi$ and $\|x_n-y_n, z\| \leq r$ for $n\in\mathbb{N}$ and each non zero $z\in X$. Then for every $\varepsilon>0$ and each non zero $z\in X$ the set $M=\{n\in\mathbb{N} : \|y_n-\xi, z\| \geq \varepsilon\} \in \mathcal{I}$. Now for $n\in M^c$, $\|x_n-\xi, z\| \leq \|x_n-y_n, z\| + \|y_n-\xi, z\| < r+\varepsilon$. Hence $\{n\in\mathbb{N} : \|x_n-\xi, z\| \geq r+\varepsilon\} \in \mathcal{I}$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ is rough $\mathcal{I}$-convergent to ξ. This completes the proof. □

Now we introduce $\mathcal{I}$-cluster point of a sequence in 2-normed spaces.

Definition 3.3. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$. A point $x_*\in X$ is said to be an $\mathcal{I}$-cluster point of $\{x_n\}_{n\in\mathbb{N}}$ if for every $\varepsilon>0$ and each non zero $z\in X$, the set $\{n\in\mathbb{N} : \|x_n-x_*, z\| < \varepsilon\} \notin \mathcal{I}$. The set of all $\mathcal{I}$-cluster points of $\{x_n\}_{n\in\mathbb{N}}$ with respect to the 2-norm $\|\cdot,\cdot\|$ is denoted as $\Lambda_2^{x_n}(\mathcal{I})$.*

Theorem 3.8. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$. For an arbitrary $\beta\in\Lambda_2^{x_n}(\mathcal{I})$ and each non zero $z\in X$ we have $\|\xi-\beta, z\| \leq r$ for all $\xi\in\mathcal{I}-LIM_2^r(x_n)$.*

Proof. If possible, let there exist $\beta\in\Lambda_2^{x_n}(\mathcal{I})$ and $\xi\in\mathcal{I}-LIM_2^r(x_n)$ such that for each non zero $z\in X$, $\|\xi-\beta, z\| > r$. Choose $\varepsilon = \frac{\|\xi-\beta, z\|-r}{2}$. Therefore for each non zero $z\in X$, $A_1=\{n\in\mathbb{N} : \|x_n-\beta, z\| < \varepsilon\} \notin \mathcal{I}$ and $A_2=\{n\in\mathbb{N} : \|x_n-\xi, z\| \geq r+\varepsilon\} \in \mathcal{I}$. Now for $n\in A_1$ we have $\|x_n-\xi, z\| \geq \|\xi-\beta, z\| - \|x_n-\beta, z\| > 2\varepsilon+r-\varepsilon = r+\varepsilon$. This shows that $n\in A_2$. Therefore $A_1\subset A_2$. Since $A_2\in\mathcal{I}$

so it would happen $A_1 \in \mathcal{I}$, but it leads to a contradiction. This completes the proof. $\square$

Theorem 3.9. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is $\mathcal{I}$-convergent to ξ if and only if $\mathcal{I} - LIM_2^r(x_n) = \overline{B_r(\xi)}$.*

Proof. The necessity part has been already proved in Theorem 3.3. For the sufficiency, let $\mathcal{I} - LIM_2^r(x_n) = \overline{B_r(\xi)} \neq \emptyset$. Then by Theorem 3.2, $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded. Suppose the sequence $\{x_n\}_{n\in\mathbb{N}}$ has another $\mathcal{I}$-cluster point β different from ξ. Let $\xi + \frac{r}{\|\xi-\beta,z\|}(\xi - \beta) = \eta$. Then the point η satisfies $\|\eta - \beta, z\| = (\frac{r}{\|\xi-\beta,z\|} + 1)\|\xi - \beta, z\| = r + \|\xi - \beta, z\| > r$. Now, Since $\beta \in \Lambda_2^{x_n}(\mathcal{I})$, by Theorem 3.8 we have $\eta \notin \mathcal{I} - LIM_2^r(x_n)$. But this is absurd as $\|\eta - \xi, z\| = r$ and $\mathcal{I} - LIM_2^r(x_n) = \overline{B_r(\xi)}$. Therefore ξ is the unique $\mathcal{I}$-cluster point of $\{x_n\}_{n\in\mathbb{N}}$. So $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to ξ. This completes the proof. $\square$

Theorem 3.10. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$. Then the following statements hold:*

(1) *If $\beta \in \Lambda_2^{x_n}(\mathcal{I})$ then $\mathcal{I} - LIM_2^r(x_n) \subseteq \overline{B_r(\beta)}$.*
(2) *$\mathcal{I} - LIM_2^r(x_n) = \bigcap_{\beta\in\Lambda_2^{x_n}(\mathcal{I})} \overline{B_r(\beta)} = \{x_0 \in X : \Lambda_2^{x_n}(\mathcal{I}) \subseteq \overline{B_r(x_0)}\}$.*

Proof. (1) Let $\xi \in \mathcal{I} - LIM_2^r(x_n)$ and $\beta \in \Lambda_2^{x_n}(\mathcal{I})$. Then for each non zero $z \in X$ and by Theorem 3.8 we have $\|\xi - \beta, z\| \leq r$. This implies that $\xi \in \overline{B_r(\beta)}$. Therefore $\mathcal{I} - LIM_2^r(x_n) \subseteq \overline{B_r(\beta)}$.

(2) Using Part (1), we have $\mathcal{I} - LIM_2^r(x_n) \subseteq \bigcap_{\beta\in\Lambda_2^{x_n}(\mathcal{I})} \overline{B_r(\beta)}$. Let $\eta \in \bigcap_{\beta\in\Lambda_2^{x_n}(\mathcal{I})} \overline{B_r(\beta)}$. Then for each non zero $z \in X$, we have $\|\eta - \beta, z\| \leq r$, for all $\beta \in \Lambda_2^{x_n}(\mathcal{I})$. This shows that $\Lambda_2^{x_n}(\mathcal{I}) \subseteq \overline{B_r(\eta)}$. Also, $\bigcap_{\beta\in\Lambda_2^{x_n}(\mathcal{I})} \overline{B_r(\beta)} \subseteq \{x_0 \in X : \Lambda_2^{x_n}(\mathcal{I}) \subseteq \overline{B_r(x_0)}\}$. Now assume $\eta \notin \mathcal{I} - LIM_2^r(x_n)$. Then there exists an $\varepsilon > 0$ such that for each non zero $z \in X$, we have $\{n \in \mathbb{N} : \|x_n - \eta, z\| \geq r + \varepsilon\} \notin \mathcal{I}$, which gives there exists an $\mathcal{I}$-cluster point β of the sequence $\{x_n\}_{n\in\mathbb{N}}$ with $\|\eta - \beta, z\| \geq r + \varepsilon$. Therefore $\Lambda_2^{x_n}(\mathcal{I}) \nsubseteq \overline{B_r(\eta)}$ and $\eta \notin \{x_0 \in X : \Lambda_2^{x_n}(\mathcal{I}) \subseteq \overline{B_r(x_0)}\}$. This gives $\{x_0 \in X : \Lambda_2^{x_n}(\mathcal{I}) \subseteq \overline{B_r(x_0)}\} \subseteq \mathcal{I} - LIM_2^r(x_n)$. Therefore $\mathcal{I} - LIM_2^r(x_n) = \bigcap_{\beta\in\Lambda_2^{x_n}(\mathcal{I})} \overline{B_r(\beta)} = \{x_0 \in X : \Lambda_2^{x_n}(\mathcal{I}) \subseteq \overline{B_r(x_0)}\}$. This completes the proof. $\square$

Theorem 3.11. *Let $\{x_n\}_{n\in\mathbb{N}}$ be an $\mathcal{I}$-bounded sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$. Then $\Lambda_2^{x_n}(\mathcal{I}) \subseteq \mathcal{I} - LIM_2^r(x_n)$ where $r = diam(\Lambda_2^{x_n}(\mathcal{I}))$.*

Proof. Let $y \notin \mathcal{I} - LIM_2^r(x_n)$. Then there exists an $\varepsilon > 0$ such that for each non zero $z \in X$, the set $\{n \in \mathbb{N} : \|x_n - y, z\| \geq r + \varepsilon\} \notin \mathcal{I}$. Again, since the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded, there exists another $\mathcal{I}$-cluster point $y_1 \in X$ such that $\|y - y_1, z\| > r + \frac{\varepsilon}{2}$ for each non zero $z \in X$. Therefore $y \notin \Lambda_2^{x_n}(\mathcal{I})$ and hence the result follows. $\square$

Now we define rough $\mathcal{I}$-Cauchy sequences in 2-normed spaces and investigate some important results in the same spaces.

Definition 3.4. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is said to be rough $\mathcal{I}$-Cauchy sequence of roughness degree $\rho > 0$ if for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\{n \in \mathbb{N} : \|x_n - x_m, z\| \geq \rho + \varepsilon\} \in \mathcal{I}$ for each non zero $z \in X$. Also, we call ρ as a $\mathcal{I}$-Cauchy degree of $\{x_n\}_{n\in\mathbb{N}}$ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ is called a ρ-$\mathcal{I}$-Cauchy sequence in X.*

Proposition 3.12. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a ρ-$\mathcal{I}$-Cauchy sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$ and $\rho_0 > \rho$. Then ρ_0 is also a $\mathcal{I}$-Cauchy degree of $\{x_n\}_{n\in\mathbb{N}}$.*

Proposition 3.13. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is $\mathcal{I}$-bounded if and only if there exists a $\rho > 0$ such that $\{x_n\}_{n\in\mathbb{N}}$ is a ρ-$\mathcal{I}$-Cauchy sequence in X.*

Theorem 3.14. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$. Then $\mathcal{I} - LIM_2^r(x_n) \neq \emptyset$ if and only if for every $\rho \geq 2r$, $\{x_n\}_{n\in\mathbb{N}}$ is a ρ-$\mathcal{I}$-Cauchy sequence.*

Proof. Let $\xi \in \mathcal{I} - LIM_2^r(x_n)$. Then for every $\varepsilon > 0$ and each non zero $z \in X$, the set $A = \{n \in \mathbb{N} : \|x_n - \xi, z\| \geq r + \frac{\varepsilon}{2}\} \in \mathcal{I}$. So, $A^c = \{n \in \mathbb{N} : \|x_n - \xi, z\| < r + \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$. Then there exists a positive integer $m \in A^c$ such that $\|x_m - \xi, z\| < r + \frac{\varepsilon}{2}$, for each non zero $z \in X$. Now for $n \in A^c$, we have

$$\|x_n - x_m, z\| = \|x_n - \xi + \xi - x_m, z\| \leq \|x_n - \xi, z\| + \|x_m - \xi, z\| = r + \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} = 2r + \varepsilon.$$

Therefore $\{n \in \mathbb{N} : \|x_n - x_m, z\| \geq 2r + \varepsilon\} \in \mathcal{I}$. Hence, by Proposition 3.12 for every $\rho \geq 2r$, $\{x_n\}_{n\in\mathbb{N}}$ is a ρ-$\mathcal{I}$-Cauchy sequence.

Conversely suppose that ρ is a $\mathcal{I}$-Cauchy degree of $\{x_n\}_{n\in\mathbb{N}}$ for every $\rho \geq 2r > 0$. Then, by Proposition 3.13 the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded. So, by Theorem 3.2, $\{x_n\}_{n\in\mathbb{N}}$ is rough $\mathcal{I}$-convergent with roughness degree $\rho > 0$. This completes the proof. □

4. Conclusions and future developments

In this paper, we have introduced and discussed on rough $\mathcal{I}$-convergence of sequences in 2-normed spaces. Later on, by using these ideas as discussed in this paper, one can extend this notion to different forms of rough convergence using the concept of ideals and natural density in the same space such as rough $\mathcal{I}$-convergence for difference sequences, rough $\mathcal{I}_2$-convergence of double sequences, rough $\mathcal{I}$-statistical convergence of sequences, rough $\mathcal{I}_2$-lacunary statistical convergence of double sequences and so on and consequently, more investigations, generalizations and applications about these types of convergence can be revealed to us, although similar results may occur in some cases.

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