Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 14 Issue 3(2023), Pages 1-6
https://doi.org/10.54379/jiasf-2023-3-1

THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY RESULTING FROM THE FUNCTION CONTAINING e^x AND x^e

YU KONAGA, YUSUKE NISHIZAWA, RYUJI SAITO, RIKI SATO

Abstract. In this paper, we improve the arithmetic-geometric mean inequality by using the function containing e^x and x^e.

1. Introduction

For real positive numbers $a_1, a_2, \dots a_n$, we define the arithmetic mean A and the geometric mean G by

$$A = \frac{1}{n}\sum_{i=1}^{n} a_i \quad and \quad G = \left(\prod_{i=1}^{n} a_i\right)^{1/n}.$$

Schaumberger [13] proved the arithmetic-geometric mean inequality $A \geq G$ by using the inequality $e^x \geq x^e$ for $x > 0$. After that, Hassani [5] improved the arithmetic-geometric mean inequality by using the inequality $e^x \geq x^e + e^{-2}(x-e)^2$ for $x > 0$. Hassani's argument is as follows. We consider the function

$$F(x) = e^x - x^e - e^{-2}(x-e)^2$$

for $x > 0$, then the function $F(x) \geq 0$ with equality if and only if $x = e$. For $1 \leq i \leq n$, we put $x = a_i e/G$ in $F(x) \geq 0$ and the following inequality

$$e^{\frac{a_i e}{G}} \geq \left(\frac{a_i e}{G}\right)^e + \left(\frac{a_i}{G} - 1\right)^2$$

holds, with equality if and only if $a_i e/G = e$. So we can get

$$\begin{aligned} e^{ne\frac{A}{G}} &\geq \prod_{i=1}^{n}\left(\left(\frac{a_i e}{G}\right)^e + \left(\frac{a_i}{G} - 1\right)^2\right) \geq \left(\prod_{i=1}^{n}\frac{a_i e}{G}\right)^e + \prod_{i=1}^{n}\left(\frac{a_i}{G} - 1\right)^2 \\ &= e^{ne}\left(1 + \frac{1}{e^{ne}}\prod_{i=1}^{n}\left(\frac{a_i}{G} - 1\right)^2\right). \end{aligned}$$

2000 *Mathematics Subject Classification.* 26D15, 26E60.
Key words and phrases. Arithmetic mean-Geometric mean inequality.

Submitted July 21, 2012. Published August 19, 2023.
Communicated by Feng Qi.

Hence, we obtain

$$ne\frac{A}{G} \geq ne + \ln\left(1 + \frac{1}{e^{ne}}\prod_{i=1}^{n}\left(\frac{a_i}{G} - 1\right)^2\right)$$

and

$$A \geq G + \frac{G}{ne}\ln\left(1 + \frac{1}{e^{ne}}\prod_{i=1}^{n}\left(\frac{a_i}{G} - 1\right)^2\right).$$

Thus, we can get the inequality $A \geq G + \mathcal{R} \geq G$, where

$$\mathcal{R} = \frac{G}{ne}\ln\left(1 + \frac{1}{e^{ne}}\prod_{i=1}^{n}\left(\frac{a_i}{G} - 1\right)^2\right) \geq 0\,.$$

Recently, Hassani et al.[6] used the function family

$$F(x) = e^x - x^e - e^{\eta x/e-2}(x-e)^2$$

to further improve the arithmetic-geometric mean inequality, where $\eta = 2, 3, 4, 5$. In [3], many proofs of the arithmetic-geometric mean inequality are posed. Moreover, new proofs and refinements of the arithmetic mean-geometric mean inequality are known, see also, [1, 2, 4, 7, 8, 9, 10, 11, 12]. In this paper, we improve the arithmetic-geometric mean inequality by using a function containing e^x and x^e, which is different from the functions used by Hassani [5], [6]. We obtain the following refinements.

Theorem 1.1. *For any real positive numbers* $a_1, a_2, \ldots, a_n$ *and* $\alpha > 0$ *such that* $e^{2+\alpha} - 2e^{\alpha}\alpha - 2e^{1+\alpha}\alpha + e^{\alpha}\alpha^2 + 2\alpha^{1+e} = 0$*, the inequality*

$$A - G \geq \mathcal{S}_e$$

holds, with equality if and only if $a_1 = a_2 = \cdots = a_n$*, where*

$$\mathcal{S}_e = \frac{G}{ne}\sum_{i=1}^{n}\ln\left(1 + ce^2\left(\frac{a_i}{G} - 1\right)^2\right) \geq 0 \quad \text{and} \quad c = \frac{1}{2\alpha - e^2 + 2e\alpha - \alpha^2}\,.$$

Theorem 1.2. *For any real positive numbers* $a_1, a_2, \ldots, a_n$ *and* $\alpha > 0$ *such that* $e^{2+\alpha} - 2e^{\alpha}\alpha - 2e^{1+\alpha}\alpha + e^{\alpha}\alpha^2 + 2\alpha^{1+e} = 0$*, the inequality*

$$A - \frac{eG}{\alpha}\ln\alpha \geq \mathcal{S}_\alpha$$

holds, with equality if and only if $a_1 = a_2 = \cdots = a_n$*, where*

$$\mathcal{S}_\alpha = \frac{G}{n\alpha}\sum_{i=1}^{n}\ln\left(1 + c\left(\frac{a_i\alpha}{G} - e\right)^2\right) \geq 0 \quad \text{and} \quad c = \frac{1}{2\alpha - e^2 + 2e\alpha - \alpha^2}\,.$$

Note that according to the numerical computation, it can be seen that $\alpha \cong 4.08175$ and $c \cong 0.158618$.

2. Lemmas

We need some lemmas for the proofs of Theorems 1.1 and 1.2.

Lemma 2.1. *For $x>0$, we define*

$$F_1(x)=e^{2+x}-2e^xx-2e^{1+x}x+e^xx^2+2x^{1+e}\,.$$

Then, there exists the real number $4<\alpha<4.1$ such that $F_1(x)>0$ for $0<x<e$, $x>\alpha$ and $F_1(x)<0$ for $e<x<\alpha$.

Proof. We set

$$f(x)=\ln\left(e^{2+x}+e^xx^2+2x^{1+e}\right)-\ln\left(2e^xx+2e^{1+x}x\right)\,.$$

Obviously, $f(e)=0$ and the derivative of $f(x)$ is

$$f'(x)=\frac{(x-e)(e^{1+x}+e^xx-2x^{1+e})}{x(e^{2+x}+e^xx^2+2x^{1+e})}\,.$$

We consider the sign of $e^{1+x}+e^xx-2x^{1+e}$ and set

$$g(x)=\ln\left(e^{1+x}+e^xx\right)-\ln\left(2x^{1+e}\right)=x+\ln(e+x)-\ln 2-(1+e)\ln x\,.$$

Obviously, $g(e)=0$ and the derivative of $g(x)$ is

$$g'(x)=\frac{-e-e^2+x^2}{x(e+x)}\,.$$

Here, we have $g'(x)<0$ for $0<x<\sqrt{e+e^2}\cong 3.1792$ and $g'(x)>0$ for $x>\sqrt{e+e^2}$. From $\lim\limits_{x\to 0+0}g(x)=\infty$, $\lim\limits_{x\to\infty}g(x)=\infty$ and $g\left(\sqrt{e+e^2}\right)=-0.0400965$, there exists only one real number x_0 such that $g(x)>0$ for $0<x<e$, $x>x_0$ and $g(x)<0$ for $e<x<x_0$. Hence, $f'(x)<0$ for $0<x<e$, $e<x<x_0$ and $f'(x)>0$ for $x>x_0$. Since we have $\lim\limits_{x\to 0+0}f(x)=\infty$, $f(e)=0$, $f(4)\cong -0.000386011$ and $f(41/10)\cong 0.000102653$, there exists the real number $4<x_1<41/10$ such that $f(x)>0$ for $0<x<e$, $x>x_1$ and $f(x)<0$ for $e<x<x_1$. Thus, the proof of Lemma 2.1 is complete. □

Lemma 2.2. *For $x>0$ and $\alpha>0$ with $e^{2+\alpha}-2e^\alpha\alpha-2e^{1+\alpha}\alpha+e^\alpha\alpha^2+2\alpha^{1+e}=0$, we define*

$$F_2(x)=e^x-\left(1+c\,(x-e)^2\right)x^e\,,$$

where

$$c=\frac{1}{2\alpha-e^2+2e\alpha-\alpha^2}\,.$$

Then, the inequality $F_2(x)\geq 0$ holds for $x>0$, with equality if and only if $x=e,\alpha$.

Proof. We set

$$f(x)=\ln e^x-\ln\left(\left(1+c\,(x-e)^2\right)x^e\right)=x-\ln\left(1+c\,(x-e)^2\right)-e\ln x$$

and the derivative of $f(x)$ is

$$f'(x)=1-\frac{e}{x}-\frac{2c(x-e)}{1+c(x-e)^2}=\frac{(x-e)g(x)}{x\left(1+c(e-x)^2\right)}\,,$$

where

$$g(x)=1+ce^2-2cx-2cex+cx^2\,.$$

By Lemma 2.1, we have

$$\frac{1}{2\left(\frac{41}{10}\right)-e^2+2e\left(\frac{41}{10}\right)-4^2}<c<\frac{1}{2\cdot 4-e^2+2e\cdot 4-\left(\frac{41}{10}\right)^2}$$

and $0.140828 < c < 0.180271$. Thus, $g(x)$ is convex downward for $x > 0$. From

$$g\left(1+e\pm\sqrt{1-\frac{1}{c}+2e}\right)=0\,,$$

we have $g(x) > 0$ for $0 < x < x_0$, $x > x_1$ and $g(x) < 0$ for $x_0 < x < x_1$, where

$$x_0=1+e-\sqrt{1-\frac{1}{c}+2e}\quad\text{and}\quad x_1=1+e+\sqrt{1-\frac{1}{c}+2e}\,.$$

By $c = 1/(2\alpha - e^2 + 2e\alpha - \alpha^2)$, we have

$$\begin{aligned}x_0&=1+e-\sqrt{1-(2\alpha-e^2+2e\alpha-\alpha^2)+2e}=1+e-\sqrt{(1+e-\alpha)^2}\\&=1+e-(-1-e+\alpha)=2+2e-\alpha>2+2e-\frac{41}{10}\cong 3.33656\end{aligned}$$

and

$$\begin{aligned}x_1&=1+e+\sqrt{1-(2\alpha-e^2+2e\alpha-\alpha^2)+2e}=1+e+\sqrt{(1+e-\alpha)^2}\\&=1+e+(-1-e+\alpha)=\alpha\,.\end{aligned}$$

Therefore, we obtain $f'(x) > 0$ for $e < x < x_0$, $x > x_1$ and $f'(x) < 0$ for $0 < x < e$, $x_0 < x < x_1$. Hence, $f(x)$ is strictly increasing for $e < x < x_0$, $x > x_1$ and strictly decreasing for $0 < x < e$, $x_0 < x < x_1$. From $e^{2+\alpha}-2e^{\alpha}\alpha-2e^{1+\alpha}\alpha+e^{\alpha}\alpha^2+2\alpha^{1+e} = 0$, we have $\alpha^e = e^{\alpha} + e^{1+\alpha} - e^{2+\alpha}/(2\alpha) - e^{\alpha}\alpha/2$ and

$$\begin{aligned}f(x_1)=f(\alpha)&=\ln\frac{e^{\alpha}}{\left(1+c\,(\alpha-e)^2\right)\alpha^e}\\&=\ln\frac{e^{\alpha}}{\left(1+c\,(\alpha-e)^2\right)\left(e^{\alpha}+e^{1+\alpha}-\frac{e^{2+\alpha}}{2\alpha}-\frac{e^{\alpha}\alpha}{2}\right)}\\&=\ln\frac{e^{\alpha}}{\left(1+c\,(\alpha-e)^2\right)\left(\frac{e^{\alpha}(2\alpha-e^2+2e\alpha-\alpha^2)}{2\alpha}\right)}\\&=\ln\frac{e^{\alpha}}{\left(1+c\,(\alpha-e)^2\right)\cdot\frac{e^{\alpha}}{2\alpha c}}=\ln\frac{2\alpha}{\frac{1}{c}+(\alpha-e)^2}=0\,.\end{aligned}$$

By $f(e) = 0$, we can get $f(x) \geq 0$ for $x > 0$, with equality if and only if $x = e, \alpha$. Therefore, the proof of Lemma 2.2 is complete. □

3. Proofs of Theorems

Proof of Theorem 1.1. For $1 \leq i \leq n$, we put $x = a_i e/G$ in $F_2(x) \geq 0$ and the following inequality

$$e^{\frac{a_i e}{G}}\geq\left(1+c\left(\frac{a_i e}{G}-e\right)^2\right)\left(\frac{a_i e}{G}\right)^e$$

holds, with equality if and only if $a_i e/G = e$. We have

$$\begin{aligned}
e^{\frac{a_1+a_2+\cdots a_n}{G}e} &\geq \prod_{i=1}^{n}\left(1+ce^2\left(\frac{a_i}{G}-1\right)^2\right)\cdot\left(\frac{a_1\cdot a_2\cdot\cdots\cdot a_n\cdot e^n}{G^n}\right)^e,\\
e^{\frac{nA}{G}e} &\geq \prod_{i=1}^{n}\left(1+ce^2\left(\frac{a_i}{G}-1\right)^2\right)\cdot e^{ne},\\
\frac{nA}{G}e &\geq \ln\left(\prod_{i=1}^{n}\left(1+ce^2\left(\frac{a_i}{G}-1\right)^2\right)\right)+ne,\\
\frac{A}{G} &\geq \frac{1}{ne}\sum_{i=1}^{n}\ln\left(1+ce^2\left(\frac{a_i}{G}-1\right)^2\right)+1,\\
A-G &\geq \frac{G}{ne}\sum_{i=1}^{n}\ln\left(1+ce^2\left(\frac{a_i}{G}-1\right)^2\right)=\mathcal{S}_e\,.
\end{aligned}$$

Here, it is clear that $\mathcal{S}_e \geq 0$. Also, if $A-G=\mathcal{S}_e$ then $a_i/G=1$ for $1\leq i\leq n$, so we have $a_1=a_2=\cdots=a_n$. Therefore, the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. For $1\leq i\leq n$, we put $x=a_i\alpha/G$ in $F_2(x)\geq 0$ and the following inequality

$$e^{\frac{a_i\alpha}{G}} \geq \left(1+c\left(\frac{a_i\alpha}{G}-e\right)^2\right)\left(\frac{a_i\alpha}{G}\right)^e$$

holds, with equality if and only if $a_i\alpha/G=\alpha$. We have

$$\begin{aligned}
e^{\frac{a_1+a_2+\cdots a_n}{G}\alpha} &\geq \prod_{i=1}^{n}\left(1+c\left(\frac{a_i\alpha}{G}-e\right)^2\right)\cdot\left(\frac{a_1\cdot a_2\cdot\cdots\cdot a_n\cdot \alpha^n}{G^n}\right)^e,\\
e^{\frac{nA}{G}\alpha} &\geq \prod_{i=1}^{n}\left(1+c\left(\frac{a_i\alpha}{G}-e\right)^2\right)\alpha^{ne},\\
\frac{nA}{G}\alpha &\geq \ln\left(\prod_{i=1}^{n}\left(1+c\left(\frac{a_i\alpha}{G}-e\right)^2\right)\right)+ne\ln\alpha,\\
\frac{A}{G} &\geq \frac{1}{n\alpha}\sum_{i=1}^{n}\ln\left(1+c\left(\frac{a_i\alpha}{G}-e\right)^2\right)+\frac{e}{\alpha}\ln\alpha,\\
A-\frac{eG}{\alpha}\ln\alpha &\geq \frac{G}{n\alpha}\sum_{i=1}^{n}\ln\left(1+c\left(\frac{a_i\alpha}{G}-e\right)^2\right)=\mathcal{S}_\alpha\,.
\end{aligned}$$

Here, it is clear that $\mathcal{S}_\alpha\geq 0$. Also, if $A-\frac{eG}{\alpha}\ln\alpha=\mathcal{S}_\alpha$ then $a_i/G=1$ for $1\leq i\leq n$, that is $a_1=a_2=\cdots=a_n$. Therefore, the proof of Theorem 1.2 is complete. □

Acknowledgments

The authors would like to thank the referees for his/her comments that helped us improve this article.

References

[1] J. M. Aldaz, A refinement of the inequality between arithmetic and geometric means, *J. Math. Inequal.* , **2** (2008), 473–477. https://dx.doi.org/10.7153/jmi-02-42.

[2] H. Alzer, A new refinement of the arithmetic mean geometric mean inequality, *Rocky Mountain J. Math.* , **27** (1997), 663–667.

[3] P. S. Bullen, D. S. Mitrinovic, P. M. Vasic, Means and Their Inequalities, Mathematics and its Applications (East European Series), vol. 31, D. Reidel Publishing Co., Dordrecht, 1988.

[4] D. I. Cartwright and M. J.Field, A refinement of the arithmetic mean geometric mean inequality, *Proc. Amer. Math. Soc.* , **71** (1978), 36–38.

[5] M. Hassani, On the arithmetic-geometric mean inequality, *Tamkang J. Math.* , **44** (2013), 453–456. https://doi.org/10.5556/j.tkjm.44.2013.1418.

[6] M. Hassani, V. Krasniqi and Y. Nishizawa, Refinements of Arithmetic mean-Geometric mean inequality associated with the number e, *J. Inequal. Spec. Funct.* , **12** (2021), no.1, 23–35.

[7] A. McD. Mercer, Bounds for $A-G$, $A-H$, $G-H$ and a family of inequalities of Ky Fan's type, using a general method, *J. Math. Anal. Appl.* , **243** (2000), 163–173.

[8] P. R. Mercer, Refined arithmetic, geometric and harmonic mean inequalities, *Rocky Mountain J. Math.* , **33** (2003), no. 4, 1459–1464.

[9] D. S. Mitrinovic and P. M. Vasic, Analytic Inequalities, Springer-Verlag, New York, 1970.

[10] F. Qi, X.-J. Zhang, and W.-H. Li, Lévy-Khintchine representation of the geometric mean of many positive numbers and applications, *Math. Inequal. Appl.* , **17** (2014), no. 2, 719–729. http://dx.doi.org/10.7153/mia-17-53.

[11] F. Qi, X.-J. Zhang, and W.-H. Li, An integral representation for the weighted geometric mean and its applications, *Acta Math. Sin.* , (Engl. Ser.) **30** (2014), no. 1, 61–68. https://doi.org/10.1007/s10114-013-2547-8.

[12] F. Qi and D. Lim, Integral representations of bivariate complex geometric mean and their applications, *J. Comput. Appl. Math.* , **330** (2018), 41–58. https://doi.org/10.1016/j.cam.2017.08.005.

[13] N. Schaumberger, The AM-GM Inequality via $x^{1/x}$, *CollegeMath. J.* , **20**(1989), 320.

Faculty of Education, Saitama University, Shimo-okubo 255, Sakura-ku, Saitama-city, Saitama, Japan

E-mail address: y.konaga.459@ms.saitama-u.ac.jp

Faculty of Education, Saitama University, Shimo-okubo 255, Sakura-ku, Saitama-city, Saitama, Japan

E-mail address: ynishizawa@mail.saitama-u.ac.jp

Faculty of Education, Saitama University, Shimo-okubo 255, Sakura-ku, Saitama-city, Saitama, Japan

E-mail address: r.saito.343@ms.saitama-u.ac.jp

Faculty of Education, Saitama University, Shimo-okubo 255, Sakura-ku, Saitama-city, Saitama, Japan

E-mail address: r.sato.426@ms.saitama-u.ac.jp