Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 14 Issue 2(2023), Pages 1-12
https://doi.org/10.54379/jiasf-2023-2-1

A NOTE ON THE GENERALIZED CONTRACTION CLASSES AND COMMON FIXED POINTS IN NORMED SPACES

MASOOMEH HOSSEINI FARAHI, MAHMOUD HASSANI, AND REZA ALLAHYARI

Abstract. We introduce a new class of generalized contractive mapping to establish a common fixed point theorem in normed spaces. Our results improved some known fixed-point theorems in the literature.

1. introduction and preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is a contraction if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y),$$

for any $x, y \in X$.
If X is complete, then every contraction on X has a unique fixed point that can be derived as the limit of iteration of the contraction at some point of X, which is known as the Banach contraction principle.
In 1997, Alber and Guerre-Delabriere [3] generalized the notion of contraction as follows:
A mapping $T : X \to X$ is a ϕ-weak contraction if there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

for every $x, y \in X$.

The following classes of functions are essential in studying fixed point theorems.

- (i) $\psi : [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$.
- (ii) $\phi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\phi(t) = 0$ if and only if $t = 0$.
- (iii) $\theta : [0, \infty) \to [0, \infty)$ is a continuous function with $\theta(t) = 0$ if and only if $t = 0$.

2000 *Mathematics Subject Classification.* 37C25, 47H09, 47H10, 26E25.
Key words and phrases. Normed space; complete b-metric spaces; multi-valued mappings; fixed point.

Submitted January 8, 2023. Published April 24, 2023.
Communicated by R.Agarwal.

Throughout the paper, Ψ is the set of all functions ψ satisfying (i), Φ is the set of all functions ϕ satisfying (ii) and Θ is the set of all functions θ satisfying (iii).

In [3], the authors have shown that every single-valued ϕ-weak contraction on a Hilbert space has a unique fixed point. Rhoades [18] showed that most parts of the results of [3] are true for any Banach space. He also proved the following generalization of the Banach contraction principle:

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a ϕ-weak contraction on X, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$, then T has a unique fixed point.*

Dutta and Choudhury [13] proved the following extension of Theorem 1.1.

Theorem 1.2. *Let (X, d) be a complete metric space and let the map $T : X \to X$ satisfies the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \qquad (x, y \in X),$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Doric [12] generalized Theorem 1.2 as follows:

Theorem 1.3. *Let (X, d) be a complete metric space and let the map $T : X \to X$ satisfies the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

for any $x, y \in X$, where M is given by

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(Tx, y))\},$$

and

- *(i) $\psi : [0, \infty) \to [0, \infty)$ is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,*
- *(ii) $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$.*
  Then T has a unique fixed point.

Fixed point theorems for multi-valued operators using the Hausdorff metric were initiated by Nadler [16] in 1969. The concept of a b-metric space was introduced by Bakhtin [5] and later used by Czerwik [9]. After that, several interesting results about the existence of fixed points for single-valued and multi-valued operators in b-metric spaces have been obtained (see, e.g., [1, 2, 6, 7, 10, 14, 15, 17, 19]).

In 2012, Bota et al. [4] proved the following theorem in complete b-metric spaces:

Theorem 1.4. *Let (X, d) be a complete b-metric space and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\phi(t) = 0$ if and only if $t = 0$. Suppose that $T : X \to X$, $S : X \to CB(X)$, where $CB(X)$ denotes the family of all nonempty closed bounded subsets of X, are such that for all $x, y \in X$*

$$H(\{Tx\}, Sy) \leq M(x, y) - \phi(M(x, y))$$

where

$$M(x,y) = max\{d(x,y), D(x,Tx), D(y,Sy), \frac{1}{2s}(D(x,Sy) + D(y,Tx))\},$$

then T and S have a unique common fixed point in X.

This paper presents a new common fixed point theorem for multi-valued and single-valued operators on complete normed spaces.
Our results generalize some well-known common fixed point theorems given by Zhang and Song [20], Rhoades [18], Ćirić [8], Daffer and Kaneko [11], and Aydi, Bota, Karapinar, and Moradi [4].

In the sequel, we recall some well-known facts which will be needed later. Throughout this paper, $\mathbb{R}$ denotes the real line, and $\mathbb{N}$ is the set of all natural numbers.

Definition 1.5. *Let $X \subseteq \mathbb{R}$ be a vector space. A nonnegative function $\|\cdot\| : X \to \mathbb{R}^+$ is called a norm provided that, for all $x, y \in X$, the following conditions hold:*

i) $\|x\| = 0$ *implies* $x = 0$*;*
ii) $\|\lambda x\| = |\lambda| \, \|x\|$ *for every* $\lambda \in \mathbb{R}$*;*
iii) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a normed space.

Definition 1.6. *Let X be a normed vector space.*

(*i*) *A sequence $\{x_n\}$ in X is called convergent if there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.*
(*ii*) *A sequence $\{x_n\}$ in X is called Cauchy if $\|x_n - x_m\| \to 0$ as $n, m \to \infty$.*
(*iii*) *A normed vector space X is said to be complete if every Cauchy sequence in X converges.*
(*iv*) *A set $B \subset X$ is said to be closed if for any sequence $\{x_n\}$ in B which $\{x_n\}$ is convergent to $z \in X$, we have $z \in B$.*

Proposition 1.7. *In a normed vector space, the following assertions hold:*

(*i*) *Let $(X, \|\cdot\|)$ be a normed vector space. Let $\{x_n\}$ be a sequence in $(X, \|\cdot\|)$. Then $\{x_n\}$ can have at most one limit.*
(*ii*) *Every convergent sequence is Cauchy in any normed linear space.*

Let X be a normed vector space, and let $CB(X)$ be the family of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, we define

$$H(A,B) = \max\{\rho(A,B), \rho(B,A)\},$$

where

$$\rho(A,B) = \sup\{D(a,B), a \in A\}, \qquad \rho(B,A) = \sup\{D(b,A), b \in B\}$$

with

$$D(a,C) = \inf\{\|a - x\|, x \in C\}, \qquad (C \in CB(X)).$$

The following result follows directly from these concepts.

Lemma 1.8. *Let X be a normed vector space.*
For any $A, B, C \in CB(X)$ and any $x, y \in X$, we have the following assertions:

(*i*) $D(x,A) = 0 \Leftrightarrow x \in \bar{A} = A$,
(*ii*) $D(x,B) \leq \|x - b\|$ *for any* $b \in B$,
(*iii*) $\rho(A,B) \leq H(A,B)$,
(*iv*) $D(x,B) \leq H(A,B)$ *for all* $x \in A$,

(v) $H(A, A) = 0$,
(vi) $H(A, B) = H(B, A)$,
(vii) $H(A, C) \leq H(A, B) + H(B, C)$,
(viii) $D(x, A) \leq \|x - y\| + D(y, A)$.
(ix) *for every* $\alpha > 0$, $b \in B$, *there exists* $a \in A$ *such that*

$$\|a - b\| \leq H(A, B) + \alpha.$$

2. Main result

This section demonstrates a common fixed point theorem for a new class of generalized contractive mapping in normed spaces.

Theorem 2.1. *Let* X *be a complete normed vector space and* $\psi \in \Psi$, $\phi \in \Phi$, *and* $\theta \in \Theta$. *Consider the maps* $T : X \to X$, $S : X \to CB(X)$ *where* S *is a multi-valued map and a constant* $L > 0$ *be such that the inequality*

$$\psi(H(\{Tx\}, Sy)) \leq \psi(M(x, y)) - \phi(\theta(M(x, y))) + L\psi(N(x, y)) \tag{2.1}$$

holds for all $x, y \in X$, *where*

$$M(x, y) = \max\{\|x - y\|, D(x, Tx), D(y, Sy), \frac{1}{2}[D(x, Sy) + D(y, Tx)]\},$$
$$N(x, y) = \min\{D(x, Tx), D(y, Ty), D(x, Sy), D(y, Tx)\}.$$

Then S *and* T *have a unique common fixed point in* X, *that is, there exists* $z \in X$ *such that* $z = Tz$ *and* $z \in Sz$.

Proof. It is easy to show that $x = y$ is a common fixed point of T and S if and only if $M(x, y) = 0$. Thus we suppose that for all $x, y \in X$, we have $M(x, y) > 0$.
We will complete the proof through the following steps:
Step 1: Let $x_0 \in X$ and $x_1 \in Sx_0$. Set $x_2 = Tx_1$.
By choosing $\alpha = \frac{\phi(\theta(M(x_2,x_1)))}{2}$ in Lemma 1.8, there exists $x_3 \in Sx_2$ such that

$$\|x_3 - x_2\| \leq H(\{Tx_1\}, Sx_2) + \frac{\phi(\theta(M(x_2, x_1)))}{2}.$$

We let $x_4 = Tx_3$. Analogously, one can find $x_5 \in Sx_4$ such that

$$\|x_5 - x_4\| \leq H(\{Tx_3\}, Sx_4) + \frac{\phi(\theta(M(x_4, x_3)))}{2}.$$

Inductively, we let $x_{2n} = Tx_{2n-1}$, and by lemma 1.8, there exists $x_{2n+1} \in Sx_{2n}$ such that

$$\begin{aligned}\|x_{2n+1} - x_{2n}\| &\leq H(\{Tx_{2n-1}\}, Sx_{2n}) + \frac{\phi(\theta(M(x_{2n}, x_{2n-1})))}{2} \\ &\leq H(\{Tx_{2n-1}\}, Sx_{2n}) + \frac{\phi(\theta(M(x_{2n}, x_{2n-1})))}{2}.\end{aligned}$$

Since ψ is nondecreasing, we have

$$\begin{aligned}\psi(\|x_{2n+1} - x_{2n}\|) &= \psi(D\{Tx_{2n-1}\}, x_{2n+1})) \\ &\leq \psi(H(\{Tx_{2n-1}\}, Sx_{2n})) \\ &\leq \psi(H(\{Tx_{2n-1}\}, Sx_{2n})).\end{aligned} \tag{2.2}$$

Thus

$$\psi(\|x_{2n+1} - x_{2n}\|) \leq \psi(H(\{Tx_{2n-1}\}, Sx_{2n})) + \frac{\phi(\theta(M(x_{2n}, x_{2n-1})))}{2}.$$

From (2.1) we get that

$$\psi(\|x_{2n+1}-x_{2n}\|) \leq \psi(M(x_{2n},x_{2n-1})) - \frac{\phi(\theta(M(x_{2n},x_{2n-1})))}{2} + L\psi(N(x_{2n},x_{2n-1})). \tag{2.3}$$

Step2: We show that $\lim_{n\to+\infty}\|x_n - x_{n+1}\| = 0$.
For any $n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_{2n-1}-x_{2n}\| \leq & M(x_{2n-1},x_{2n}) \\ = & \max\{\|x_{2n-1}-x_{2n}\|, D(x_{2n-1},Tx_{2n-1}), D(x_{2n},Sx_{2n}) \\ & , \frac{1}{2}[D(x_{2n-1},Sx_{2n}) + D(x_{2n},Tx_{2n-1})]\} \\ \leq & \max\{\|x_{2n-1}-x_{2n}\|, \|x_{2n-1}-x_{2n}\|, \|x_{2n}-x_{2n+1}\| \\ & , \frac{1}{2}\|x_{2n-1}-x_{2n+1}\|\} \\ \leq & \max\{\|x_{2n-1}-x_{2n}\|, \|x_{2n}-x_{2n+1}\| \\ & , \frac{1}{2}[(\|x_{2n-1}-x_{2n}\| + \|x_{2n}-x_{2n+1}\|)]\} \\ = & \max\{\|x_{2n-1}-x_{2n}\|, \|x_{2n}-x_{2n+1}\|\}.\end{aligned}$$

If $\max\{\|x_{2n-1}-x_{2n}\|, \|x_{2n}-x_{2n+1}\|\} = \|x_{2n}-x_{2n+1}\|$, from (2.3) and using the fact that $N(x_{2n},x_{2n-1}) = 0$, we have

$$\begin{aligned}\psi(\|x_{2n+1}-x_{2n}\|) \leq & \psi(M(x_{2n},x_{2n-1})) - \frac{\phi(\theta(M(x_{2n},x_{2n-1})))}{2} \\ & + L\psi(N(x_{2n},x_{2n-1}))) \\ \leq & \psi(\|x_{2n}-x_{2n+1}\|) - \frac{\phi(\theta(M(x_{2n},x_{2n-1})))}{2}.\end{aligned}$$

So $\frac{\varphi(\theta(M(x_{2n},x_{2n-1})))}{2} = 0$, that is, $M(x_{2n},x_{2n-1}) = 0$, which is a contradiction. Hence $\max\{\|x_{2n-1}-x_{2n}\|, \|x_{2n}-x_{2n+1}\|\} = \|x_{2n-1}-x_{2n}\|$. Then $M(x_{2n-1},x_{2n}) = \|x_{2n-1}-x_{2n}\|$ for each $n \geq 1$. We have

$$\|x_{2n}-x_{2n+1}\| \leq \|x_{2n-1}-x_{2n}\|. \tag{2.4}$$

Similar to the process of (2.2), we get also

$$\psi(\|x_{2n+1}-x_{2n+2}\|) \leq \psi(H(\{Tx_{2n+1}\}, Sx_{2n})).$$

By using (2.1) we have

$$\psi(\|x_{2n+1}-x_{2n+2}\|) \leq \psi(M(x_{2n+1},x_{2n})) - \phi(\theta(M(x_{2n+1},x_{2n}))) + L\psi(N(x_{2n+1},x_{2n})), \tag{2.5}$$

where

$$\begin{aligned}M(x_{2n+1},x_{2n}) =&\max\{\|x_{2n+1}-x_{2n}\|,D(x_{2n+1},Tx_{2n+1}),D(x_{2n},Sx_{2n})\\&,\frac{1}{2}[D(x_{2n+1},Sx_{2n})+D(x_{2n},Tx_{2n+1})]\}\\ \leq&\max\{\|x_{2n+1}-x_{2n}\|,\|x_{2n+1}-x_{2n+2}\|,\|x_{2n}-x_{2n+1}\|\\&,\frac{1}{2}[\|x_{2n+1}-x_{2n+1}\|+\|x_{2n}-x_{2n+2}\|]\}\\ \leq&\max\{\|\{dx_{2n+1}-x_{2n}\|,\|x_{2n+1}-x_{2n+2}\|\\&,\frac{\|x_{2n}-x_{2n+1}\|+\|x_{2n+1}-x_{2n+2}\|}{2}\}\\ =&\max\{\|x_{2n+1}-x_{2n}\|,\|x_{2n+1}-x_{2n+2}\|\}\\ =&\|x_{2n+1}-x_{2n}\|.\end{aligned}$$

If $\max\{\|x_{2n+1}-x_{2n}\|,\|x_{2n+1}-x_{2n+2}\|\}=\|x_{2n+1}-x_{2n+2}\|$, from (2.5) and using the fact $N(x_{2n+1},x_{2n})=0$ we have

$$\begin{aligned}\psi(\|x_{2n+1}-x_{2n+2}\|)\leq&\psi(M(x_{2n+1},x_{2n}))-\phi(\theta(M(x_{2n+1},x_{2n})))\\ <&\psi(M(x_{2n+1},x_{2n}))\\ =&\psi(\|x_{2n+1}-x_{2n+2}\|).\end{aligned}$$

Thus

$$\psi(\|x_{2n+1}-x_{2n+2}\|)<\psi(\|x_{2n+1}-x_{2n+2}\|),$$

which is a contradiction. Therefore

$$M(x_{2n+1},x_{2n})=\|x_{2n+1}-x_{2n}\|$$

and

$$\|x_{2n+1}-x_{2n+2}\|\leq\|x_{2n+1}-x_{2n}\|. \tag{2.6}$$

From (2.4) and (2.6), we get that

$$\|x_n-x_{n+1}\|\leq\|x_{n-1}-x_n\|,\qquad \forall n\geq 0.$$

Thus $\{\|x_n-x_{n+1}\|;\ n\in\mathbb{N}\}$ is a non-increasing sequence of positive numbers. Hence, there is $l\geq 0$ such that

$$\lim_{n\to\infty}M(x_n,x_{n+1})=\lim_{n\to\infty}\|x_n-x_{n+1}\|=l\geq 0.$$

We show that $l=0$. On the contrary, suppose $l>0$. We know $\phi(\theta(l))>0$ from (2.3) and taking limits as $n\to\infty$, since ϕ is lower semi-continuous, we get

$$\begin{aligned}\psi(l)\leq&\psi(l)-\frac{\phi(\theta(l))}{2}+L\psi(0)\\ <&\psi(l)\end{aligned}$$

this is a contradiction, thus $l=0$. So we have

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0. \tag{2.7}$$

Step 3: We will prove that $\{x_n\}$ is a Cauchy sequence. Because of (2.7), it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence.

Suppose $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon>0$, for which we can find two subsequences $\{x_{2mi}\}$, $\{x_{2ni}\}$ of $\{x_{2n}\}$ such that ni is the smallest index, for which

$$ni>mi>i,\qquad \|x_{2mi}-x_{2ni}\|\geq\varepsilon.$$

This means that

$$\|x_{2mi} - x_{2ni-2}\| < \varepsilon. \tag{2.8}$$

By using the triangular inequality, we get

$$\varepsilon \leq \|x_{2mi} - x_{2ni}\| \leq \|x_{2mi} - x_{2mi+1}\| + \|x_{2mi+1} - x_{2n_i}\|.$$

By taking the upper limits as $i \to \infty$, we get

$$\varepsilon \leq \lim_{i\to\infty} \sup \|x_{2mi+1} - x_{2ni}\|. \tag{2.9}$$

On the other hand, we have

$$\|x_{2mi} - x_{2ni-1}\| \leq \|x_{2mi} - x_{2ni-2}\| + \|x_{2ni-2} - x_{2ni-1}\|.$$

Using (2.8) and taking the upper limit as $i \to \infty$, we get

$$\lim_{i\to\infty} \sup \|x_{2mi} - x_{2ni-1}\| \leq \varepsilon. \tag{2.10}$$

Again, using the triangular inequality, we have

$$\varepsilon \leq \|x_{2mi} - x_{2ni}\| \leq \|x_{2mi} - x_{2ni-1}\| + \|x_{2ni-1} - x_{2ni}\|.$$

By taking the upper limit as $i \to \infty$, we get

$$\varepsilon \leq \lim_{i\to\infty} \sup \|x_{2mi} - x_{2ni-1}\|. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\varepsilon \leq \limsup_{i\to\infty} \|x_{2mi} - x_{2ni-1}\| \leq \varepsilon. \tag{2.12}$$

Again, using the triangular inequality, we have

$$\begin{aligned}\varepsilon \leq \|x_{2mi} - x_{2ni}\| &\leq \|x_{2mi} - x_{2ni-2}\| + \|x_{2ni-2} - x_{2ni}\| \\ &\leq \|x_{2mi} - x_{2ni-2}\| + \|x_{2ni-2} - x_{2ni-1}\| + \|x_{2ni-1} - x_{2ni}\|.\end{aligned}$$

By taking the upper limit as $i \to \infty$, using (2.8) we obtain

$$\varepsilon \leq \limsup_{i\to\infty} \|x_{2mi} - x_{2ni}\| \leq \varepsilon. \tag{2.13}$$

Again, using the triangular inequality, we infer that

$$\|x_{2mi+1} - x_{2ni-1}\| \leq \|x_{2mi+1} - x_{2mi}\| + \|x_{2mi} - x_{2ni-1}\|.$$

By using (2.10) and taking the upper limit as $i \to \infty$, we reach to

$$\limsup_{i\to\infty} \|x_{2mi+1} - x_{2ni-1}\| \leq \varepsilon. \tag{2.14}$$

Again, using the triangular inequality, we get

$$\begin{aligned}\varepsilon \leq \|x_{2mi} - x_{2ni}\| &\leq \|x_{2mi} - x_{2mi+1}\| + \|x_{2mi+1} - x_{2ni}\| \\ &\leq \|x_{2mi} - x_{2mi+1}\| + \|x_{2mi+1} - x_{2ni-1}\| + \|x_{2ni-1} - x_{2ni}\|.\end{aligned}$$

By taking the upper limit as $i \to \infty$, and using (2.14) we have

$$\varepsilon \leq \limsup_{i\to\infty} \|x_{2mi+1} - x_{2ni-1}\| \leq \varepsilon. \tag{2.15}$$

From (2.1) and similar to the process (2.2) we get

$$\begin{aligned}\psi(\|x_{2mi+1} - x_{2ni}\|) \leq &\psi(H(Sx_{2mi}, \{Tx_{2ni-1}\})) \\ &\leq \psi(M(x_{2mi}, x_{2ni-1})) - \phi(\theta(M(x_{2mi}, x_{2ni-1})) \\ &\quad + L\psi(N(x_{2mi}, x_{2ni-1})),\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}M(x_{2mi},x_{2ni-1}) =& \max\{\|x_{2mi}-x_{2ni-1}\|, D(x_{2mi},Tx_{2mi}), D(x_{2ni-1},Sx_{2ni-1})\\ &, \frac{1}{2}[D(x_{2mi},Sx_{2ni-1})+D(x_{2ni-1},Tx_{2mi})]\}\\ \le& \max\{\|x_{2mi}-x_{2ni-1}\|, \|x_{2mi}-x_{2mi+1}\|, \|x_{2ni}-x_{2ni-1}\|\\ &, \frac{1}{2}[\|x_{2mi}-x_{2ni}\|+\|x_{2mi+1}-x_{2ni-1}\|]\}.\end{aligned}$$

Taking the upper limit and using (2.10), (2.13) and (2.14), we have

$$\begin{aligned}\limsup_{i\to\infty} M(x_{2mi},x_{2ni-1}) \le& \max\{\limsup_{i\to\infty}\|x_{2mi}-x_{2ni-1}\|\\ &, \limsup_{i\to\infty}\|x_{2mi}-x_{2mi+1}\|, \limsup_{i\to\infty}\|x_{2ni}-x_{2ni-1}\|\\ &, \frac{\limsup_{i\to\infty}\|x_{2mi}-x_{2ni}\|+\limsup_{i\to\infty}\|x_{2mi+1}-x_{2ni-1}\|}{2}\}\\ \le& \max\{\varepsilon,0,0,\}=\varepsilon,\end{aligned}$$

and using (2.12), (2.13), and (2.15) we have

$$\min\left\{\varepsilon, \frac{\varepsilon+\varepsilon}{2}\right\}=\varepsilon$$

we get

$$\varepsilon \le \limsup_{i\to\infty} M(x_{2mi},x_{2ni-1}) \le \varepsilon$$

and

$$\varepsilon \le \liminf_{i\to\infty} M(x_{2mi},x_{2ni-1}) \le \varepsilon \tag{2.17}$$

and

$$\begin{aligned}N(x_{2mi},x_{2ni-1}) =& \min\{D(x_{2mi},Tx_{2mi}), D(x_{2ni-1},Tx_{2ni-1}),\\ & D(x_{2mi},Sx_{2ni-1}), D(x_{2ni-1},Tx_{2mi})\}.\end{aligned} \tag{2.18}$$

From (2.18), $\limsup_{i\to\infty} N(x_{2mi},x_{2ni-1})=0$.

Now taking the upper limit as $i\to\infty$ in (2.16) and using (2.9) and (2.18) we have

$$\begin{aligned}\psi(\varepsilon)=\psi(1\cdot\varepsilon) \le& \psi(\limsup_{i\to\infty}\|x_{2mi+1}-x_{2ni}\|)\\ \le& \psi(\limsup_{i\to\infty} M(x_{2mi},x_{2ni-1}))\\ & -\phi(\theta(\liminf_{i\to\infty} M(x_{2mi},x_{2ni-1})))\\ \le& \psi(\varepsilon)-\phi(\theta(\liminf_{i\to\infty} M(x_{2mi},x_{2ni-1})))\end{aligned}$$

which implies that

$$\phi(\theta(\liminf_{i\to\infty} M(x_{2mi},x_{2ni-1})))=0.$$

Thus

$$\liminf_{i\to\infty} M(x_{2mi},x_{2ni-1})=0$$

which is in contradiction with (2.17).

Hence $\{x_n\}$ is a Cauchy sequence in X.

Step 4: As $\{x_n\}$ is a Cauchy sequence and X is a complete normed space, there exists $u \in X$ such that

$$\lim_{n\to\infty} \|x_n - u\| = 0.$$

We show that $u = Tu$ and $u \in Su$. Similar to the process (2.2)

$$\begin{aligned} \psi(D(x_{2n+2}, Su) &\leq \psi(H(\{Tx_{2n+1}\}, Su)) \\ &\leq \psi(M(x_{2n+1}, u)) - \phi(\theta(M(x_{2n+1}, u))) \\ &\quad + L\psi(N(x_{2n+1}, u)), \end{aligned} \tag{2.19}$$

$$\begin{aligned} D(u, Su) \leq & M(x_{2n+1}, u) \\ &= \max\{\|x_{2n+1} - u\|, \|x_{2n+1} - x_{2n+2}\|, D(u, Su) \\ &\quad , \frac{1}{2}[D(x_{2n+1}, Su) + \|u - x_{2n+2}\|]\}. \end{aligned} \tag{2.20}$$

By using the triangular inequality, we have

$$\|x_{2n+1} - u\| \leq \|x_{2n+1} - x_{2n}\| + \|x_{2n} - u\|.$$

As

$$\lim_{n\to\infty} \|x_{2n+1} - u\| = 0 \text{ and } \lim_{n\to\infty} \|x_{2n+2} - u\| = 0. \tag{2.21}$$

Again, using the triangular inequality, we get

$$D(x_{2n+1}, Su) \leq (\|x_{2n+1} - u\| + D(u, Su)).$$

By using (2.21)

$$\lim_{n\to\infty} D(x_{2n+1}, Su) \leq D(u, Su).$$

By taking limit from (2.20)

$$\begin{aligned} D(u, Su) &\leq \lim_{n\to\infty} M(x_{2n+1}, u) \\ &\leq \max\{D(u, Su), \frac{D(u, Su)}{2}\} = D(u, Su), \\ &\lim_{n\to\infty} M(x_{2n+1}, u) = D(u, Su). \end{aligned} \tag{2.22}$$

As

$$D(u, Su) \leq [\|u - x_{2n+2}\| + D(x_{2n+2}, Su)],$$

by taking the upper limit as $n \to \infty$, we have

$$D(u, Su) \leq \limsup_{n\to\infty} D(x_{2n+2}, Su) \tag{2.23}$$

and $\lim_{n\to\infty} N(x_{2n+1}, u) = 0$. Because ψ is continuous, by using (2.19), (2.22), (2.23) we have

$$\psi(D(u, Su)) \leq \psi(D(u, Su)) - \phi(\theta(D(u, Su))).$$

Then $\phi(\theta(D(u, Su))) = 0$. Thus $D(u, Su) = 0$.

We conclude that $u \in Su$. From (2.1) we have

$$\begin{aligned}\psi(D(Tu,u)) \leq& \psi(D(Tu,u))\\ \leq& \psi(s^2 H(\{Tu\},Su))\\ \leq& \psi(M(u,u)-\phi(\theta(M(u,u)))\\ &+L\psi(N(u,u)).\end{aligned} \tag{2.24}$$

As

$$\begin{aligned}M(u,u) &= max\{\|u-u\|, D(u,Tu), D(u,Su), \frac{1}{2}[D(u,Su)+D(u,Tu)]\}\\ &= max\{D(u,Tu), \frac{D(u,Tu)}{2}\} = D(u,Tu).\end{aligned}$$

Moreover as $N(u,u)=0$ from (2.24) we have

$$\psi(D(Tu,u)) \leq \psi(D(Tu,u)) - \phi(\theta(D(Tu,u))) + 0.$$

Then $\phi(\theta(D(Tu,u)))=0$. Thus $D(u,Tu)=0 \Rightarrow u=Tu$.

Step 5: Now, we show that the fixed point is unique. Suppose that z is another fixed point of S and T, i.e., $z=Tz$ and $z \in Sz$, then

$$\begin{aligned}\psi(\|u-z\|) =& \psi(D(Tu,z)) \leq \psi(D(Tu,z)) \leq \psi(H(\{Tu\},Sz))\\ \leq& \psi(M(u,z)) - \phi(\theta(M(u,z)) + L\psi(N(u,z)),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\|u-z\| \leq M(u,z) =& \max\{\|u-z\|, D(u,Tu), D(z,Sz)\\ &, \frac{1}{2}[D(u,Sz)+D(z,Tu)]\}.\end{aligned}$$

Hence

$$M(u,z) = \|u-z\|.$$

Moreover, $N(u,z)=0$, from (2.25) we have

$$\psi(\|u-z\|) \leq \psi(\|u-z\|) - \phi(\theta(\|u-z\|)).$$

Then $\phi(\theta(\|u-z\|))=0$. Thus $\|u-z\|=0$, i.e. $u=z$.

□

Here are some examples of functions and maps that satisfy the conditions of Theorem 2.1.

Example 2.2. *Let X be a Hilbert space, $T: X \to X$ be a nonexpansive map (i.e., $\|Tx-Ty\| \leq \|x-y\|$ for all $x,y \in X$), and $S: X \to CB(X)$ be a firmly nonexpansive map (i.e., for any $x \in X$, the set $\{y \in X: \ y \in Sy\}$ is nonempty and closed, and $\|y-x\|^2 + \|Sy-Sx\|^2 \geq \|y-Sx\|^2 + \|x-Sx\|^2$ for all $y \in Sy$). In this casem we can choose $\psi(t)=t$, $\phi(t)=t/2$, and $\theta(t)=t/2$. The constant L can be chosen to be 1. Then the inequality in Theorem 2.1 holds, and the map T and the multi-valued map S have a unique common fixed point.*

Example 2.3. *Let $X = C[0,1]$ be the space of continuos functions on $[0,1]$, equipped with the sup norm, and let $T: X \to X$ be the Volterra operator defined by $(Tf)(x) = \int_0^x f(t)\,dt$ for $f \in X$ and $x \in [0,1]$. Let $S: X \to CB(X)$ be the set-valued map defined by $(Sf)(x) = \{f(x)\}$ for $f \in X$ and $x \in [0,1]$. In this case, we can choose $\psi(t)=t$, $\phi(t)=\sqrt{t}/2$, and $\theta(t)=\sqrt{t}/2$. The constant L can be choosen to be $1/\sqrt{2}$. Then the inequality in Theorem 2.1 holds, and the map T and the multi-valued map S have a unique common fixed point.*

Example 2.4. *Let $X = L^p([0,1])$ be the space of Lebesgue measurable functions on $[0,1]$, that are integrable to the p-th power, equipped with the L^p norm where $1 < p < \infty$. Let $T : X \to X$ be the integral operator defined by $(Tf)(x) = \int_0^1 K(x,t) f(t)\,dt$ for $f \in X$ and $x \in [0,1]$, where $K(x,t)$ is a Kernel function satisfying certain conditions. Let $S : X \to CB(X)$ be the set-valued map defined by $(Sf)(x) = \left\{ g \in X : \ g(x) = \sup_{t\in[0,1]} f(t) \right\}$ for $f \in X$ and $x \in [0,1]$. In this case, we can choose $\psi(t) = t^p$, $\phi(t) = t^{p/2}$, and $\theta(t) = t^{1/p}$. The constant L can be choosen to be $1/\sqrt{p}$. Then the inequality in Theorem 2.1 holds, and the map T and the multi-valued map S have a unique common fixed point.*

Acknowledgement

We thank the anonymous reviewer for carefully reading our manuscript and for insightful comments and suggestions.

References

[1] H. Afshari, H. Aydi, and E. Karapinar, Existence of fixed points of set-valued mappings in b-metric spaces, *East Asian Math. J.* Volume **32,** issue 3, (2016), 319-332, DOI : 10.7858/eamj.2016.024.

[2] U. Aksoy, E. Karapinar, and I. M. Erhan, Fixed points of generalized α-admissible contractions on b-metric spaces with an application to boundary value problems, *J. Nonlinear Convex Anal.* Volume **17,** Number 6, (2016), 1095-1108.

[3] Y. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, *New Res. Oper. Theory: Adv. Appl.* Volume **98**, (1997), 7-22.

[4] H. Aydi, M-F. Bota, E. Karapinar, and S. Moradi, A common fixed point for weak ϕ-contractions on b-metric spaces, *Fixed point Theory.* Volume **13**, Number 2, (2012), 337-346.

[5] I. A. Bakhtin, The contraction mapping principle in quasi-metric spaces, *Funct. Anal. Unianowsk Gos. Ped. Inst.* Volume **30**, (1989), 26-37.

[6] V. Berinde, Generalized contractions in quasi-metric spaces, *Seminar on Fixed Point Theory.* Volume **3**, (1993), 3-9.

[7] M. Bota, C. Chifu, and E. Karapinar, Fixed point theorems for generalized (α-ψ)-Ciric-type contractive multivalued operators in b-metric spaces, *J. Nonlinear Sci. Appl.* Volume **9**, Issue 3, (2016), 1165-1177.

[8] Lj. B. Ćirić, A generalization of Banachs contraction principle, *Proc. Amer. Math. Soc.* Volume **45**, (1974), 267-273.

[9] S. Czerwik, K. Dlutek, and S. L. Singh, Round-off stability of iteration procedures for operators in b-metric spaces, *J. Natur. Phys. Sci.* Volume **11**, (1997), 87-94.

[10] S. Czerwik, K. Dlutek, and S. L. Singh, Round-off stability of iteration procedures for set-valued operators in b-metric spaces, *J. Natur. Phys. Sci.* Volume **15**, (2001), 1-8.

[11] P. Z. Daffer and H. Kaneko, Fixed points of generalized contractive Multi-valued mappings, *J. Math. Anal. Appl.* Volume **192**, (1995), 655-666.

[12] D. Doric, Common fixed point for generalized (ψ, ϕ)-weak contractions, *Appl. Math. Lett.* Volume **22**, (2009), 1896-1900.

[13] P. N. Dutta and B. S. Choudhury, A generalization of contraction principle in metric spaces, *Fixed Point Theory Appl.* Article ID 406368, (2008).

[14] S. Gulyaz, On some α-admissible contraction mappings on Branciari b-metric spaces , *Adv. Theory Nonlinear Anal. Appl.* Volume **1**, Number1, (2017), 1-13, Article Id : 2017:1:1.

[15] M. S. Khan, M. Swaleh, and S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.* Volume **30**, (1984), 1-9.

[16] S. B. Nadler, Multivalued contraction mappings, *Pacific J. Math.* Volume **30**, (1969), 475-488.

[17] O. Popescu, Fixed point for (ψ, ϕ)-weak contractions, *Appl. Math. Lett.* Volume **24**, (2011), 1-4.
[18] B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Anal.* Volume **47**, (2001), 2683- 2693.
[19] S. L. Singh, S. Czerwik, K. Krol, and A. Singh, Coincidences and fixed points of hybrid contractions, *Tamsui Oxf. J. Math. Sci.* Volume **24**, Number 4, (2008), 401-416.
[20] Q. Zhang and Y. Song, Fixed point theory for generalized ϕ-weak contractions, *Appl. Math. Lett.* Volume **22**, (2009), 75-78.

Masoomeh Hosseini Farahi
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: farahimasoome@gmail.com

Mahmoud Hassani
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: hassani@mshdiau.ac.ir

Reza Allahyari
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: rezaallahyari@mshdiau.ac.ir