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ASYMPTOTIC EXPRESSIONS AND FORMULAS FOR FINITE SUMS OF POWERS OF BINOMIAL COEFFICIENTS INVOLVING SPECIAL NUMBERS AND POLYNOMIALS

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Abstract. The main objective in this paper is to study on special numbers and polynomials that contain finite sums of powers of binomial coefficients. By using generating function methods, some formulas and relations related to these numbers and the Apostol-Bernoulli and Apostol-Euler numbers of negative higher order, the Bernoulli and Euler numbers, the Stirling type numbers, the combinatorial numbers, the Bell polynomials, the Fubini type polynomials, and the Legendre polynomials are presented. Moreover, asymptotic expressions of the finite sums of powers of binomial coefficients for these numbers are given. Some numeric values of these asymptotic expressions are illustrated by the tables. Finally, some inequalities for these numbers are given.

1. Introduction

The binomial coefficients and finite sums including higher powers of binomial coefficients have been frequently used in many areas of mathematics and other applied sciences. In order to study finite sums involving binomial coefficients, there are many different techniques. Among these techniques, generating functions are commonly used ones, and Simsek [40] gave the following novel generating functions for sums of finite sums of powers of binomial coefficients:

$$\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\lambda^{k}e^{tk}=\sum_{m=0}^{\infty}y_{6}\left(m,n;\lambda,p\right)\frac{t^{m}}{m!}, \tag{1.1}$$

where $n,p\in\mathbb{N}=\{1,2,3,...\}$ and $\lambda\in\mathbb{R}$ (or $\mathbb{C}$).

Here and in the following let $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of integers, real numbers and complex numbers, respectively. Also let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

The numbers $y_6\left(m,n;\lambda,p\right)$ are computed by the following formula:

$$y_{6}\left(m,n;\lambda,p\right)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\lambda^{k}k^{m}, \tag{1.2}$$

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where $m,n,p\in\mathbb{N}_0$ (*cf.* [40, Eq. (14)]).

Simsek [40] gave the following values for the numbers $y_6(m,n;\lambda,p)$ involving well-known finite sums:

When $p=1$, $\lambda=1$ and $m=0$ in (1.2), we have

$$y_6(0,n;1,1)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}=\frac{2^n}{n!}, \tag{1.3}$$

when $p=2$, $\lambda=1$ and $m=0$ in (1.2), we obtain

$$y_6(0,n;1,2)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^2=\frac{\binom{2n}{n}}{n!} \tag{1.4}$$

and putting $p=1$, $\lambda=-1$ and $m=0$ in (1.2), we have

$$y_6(0,n;-1,1)=\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0,$$

for detail, see [40]. The above sums have also been studied by many authors (*cf.* [1]-[46]).

Here we note that Simsek introduced many different combinatorial type numbers and polynomials. Thats why, he gave some notations to distinguish these numbers and polynomials from one another. So, the number 6 is just an index representation for the numbers $y_6(m,n;\lambda,p)$ (see, for detail, [37]-[42]).

Moreover, when $p=1$ in (1.2), the numbers $y_6(m,n;\lambda,p)$ are reduced to the following combinatorial numbers, which are called Simsek numbers by Goubi [10]:

$$y_6(m,n;\lambda,1)=y_1(m,n;\lambda) \tag{1.5}$$

which are defined by means of the following generating function:

$$\frac{(\lambda e^t+1)^v}{v!}=\sum_{m=0}^{\infty}y_1(m,v;\lambda)\frac{t^m}{m!}, \tag{1.6}$$

where $v\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$ and

$$y_1(m,v;\lambda)=\frac{1}{v!}\sum_{j=0}^{v}\binom{v}{j}j^m\lambda^j,$$

where $m\in\mathbb{N}_0$ (*cf.* [37, Eqs. (8) and (9)]).

The numbers $y_6(m,n;\lambda,p)$ are related to many different combinatorial type numbers and polynomials. Recently, many authors, such as Goubi [9, 10], Khan et al. [12], Kucukoglu [18], Kucukoglu and Simsek [19]-[21], Kilar [13] and Xu [46], have given results that include sums of powers of binomial coefficients and combinatorial type numbers. Besides, one method to approach the binomial coefficient is by means of the Stirlings approximation. The Stirling's approximation (or Stirling's formula), named after James Stirling, is a factorial approximation in mathematics. This formula has been studied by many authors and used in many fields, such as mathematics and physics. Moreover, this formula is a good and useful approximation that leads to accurate results even for small values of n (see, for detail, [4, 23, 26, 31, 33, 34, 40, 43, 45]). Therefore, the main motivation of this paper is to give some new relations and formulas including some special numbers and polynomials, with the help of an asymptotic expression of sums of powers of binomial coefficients, the Stirlings approximation formula, and generating functions with functional equations techniques. The numeric values of these results are displayed in tables. We also give some inequalities for these numbers.

We now briefly give some notations, definitions and generating functions of some special numbers and polynomials.

Let $\alpha \in \mathbb{C}$ and $r \in \mathbb{N}$,

$$\{\alpha\}_r = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-r+1) = \binom{\alpha}{r} r!$$

with $\{\alpha\}_0 = 1$ and

$$0^r = \begin{cases} 1 & r = 0 \\ 0 & r \in \mathbb{N} \end{cases}.$$

The Stirling numbers of the second kind are defined by

$$\frac{(e^t-1)^v}{v!} = \sum_{m=0}^{\infty} S_2(m,v) \frac{t^m}{m!}, \tag{1.7}$$

where $v \in \mathbb{N}_0$, and

$$x^m = \sum_{v=0}^{m} \{x\}_v S_2(m,v)$$

with $S_2(0,0) = 1$ and

$$\begin{aligned} S_2(m,v) &= 0, \quad (v > m) \\ S_2(m,0) &= 0, \ (m \in \mathbb{N}) \\ S_2(m,m) &= 1, \quad (m \in \mathbb{N}) \\ S_2(m,1) &= 1, \ (m \in \mathbb{N}) \end{aligned}$$

(*cf.* [1]-[46]).

By using (1.7), the numbers $S_2(m,v)$ are computed by

$$S_2(m,v) = \frac{1}{v!} \sum_{r=0}^{v} (-1)^r \binom{v}{r} (v-r)^m$$

(*cf.* [1]-[46]).

The Apostol-Bernoulli polynomials of order k are defined by

$$\left(\frac{t}{\lambda e^t - 1}\right)^k e^{tx} = \sum_{m=0}^{\infty} \mathcal{B}_m^{(k)}(x;\lambda) \frac{t^m}{m!}, \tag{1.8}$$

where $|t| < 2\pi$ when $\lambda = 1$; $|t| < |\ln \lambda|$ when $\lambda \neq 1$ and $k \in \mathbb{Z}$ (*cf.* [28, Eq. (9)], [44, 45]).

When $x = 0$ in (1.8), we have the Apostol-Bernoulli numbers of order k

$$\mathcal{B}_m^{(k)}(0;\lambda) = \mathcal{B}_m^{(k)}(\lambda)$$

(*cf.* [27, 44, 45]).

One can observe that

$$\mathcal{B}_m^{(1)}(x;\lambda) = \mathcal{B}_m(x;\lambda) \quad \text{and} \quad \mathcal{B}_m^{(1)}(\lambda) = \mathcal{B}_m(\lambda).$$

Taking $k = 0$ in (1.8), we have

$$\mathcal{B}_m^{(0)}(x;\lambda) = x^m$$

and also for $x = 0$ in the above equation, one has

$$\mathcal{B}_m^{(0)}(\lambda) = \begin{cases} 1, & m = 0 \\ 0, & m \in \mathbb{N} \end{cases}.$$

Substituting $p = 0$ into (1.2) and using (1.8) for $k = 1$, we get

$$(n-1)! y_6(m-1, n-1; \lambda, 0) = \sum_{k=0}^{n-1} \lambda^k k^{m-1} = \frac{\lambda^m \mathcal{B}_m(n; \lambda) - \mathcal{B}_m(\lambda)}{m}$$

(*cf.* [40]).

Substituting $\lambda = 1$ into (1.8), we have the Bernoulli polynomials of order k

$$\mathcal{B}_m^{(k)}(x; 1) = B_m^{(k)}(x)$$

which are defined by means of the following generating function:

$$\left(\frac{t}{e^t - 1}\right)^k e^{tx} = \sum_{m=0}^{\infty} B_m^{(k)}(x) \frac{t^m}{m!} \tag{1.9}$$

(*cf.* [27, 44, 45]).

When $x = 0$ in (1.9), we have

$$B_m^{(k)}(0) = B_m^{(k)}$$

which $B_m^{(k)}$ denote the Bernoulli numbers of order k (*cf.* [28, 44, 45]).

Using (1.7) and (1.9), we have

$$t^n e^{tv} = n! \sum_{m=0}^{\infty} S_2(m, n) \frac{t^m}{m!} \sum_{m=0}^{\infty} B_m^{(n)}(v) \frac{t^m}{m!}.$$

Hence

$$\sum_{m=0}^{\infty} \sum_{s=0}^{v} \binom{v}{s} \{m\}_{n+s} B_{m-n-s}^{(-s)} \frac{t^m}{m!} = n! \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} S_2(j, n) B_{m-j}^{(n)}(v) \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some calculations, we have the following presumably known formula:

$$\sum_{s=0}^{v} \binom{v}{s} \{m+n\}_{n+s} B_{m-s}^{(-s)} = n! \sum_{j=0}^{m+n} \binom{m+n}{j} S_2(j, n) B_{m+n-j}^{(n)}(v). \tag{1.10}$$

The Apostol-Euler polynomials of order k are defined by

$$\left(\frac{2}{\lambda e^t + 1}\right)^k e^{tx} = \sum_{m=0}^{\infty} \mathcal{E}_m^{(k)}(x; \lambda) \frac{t^m}{m!}, \tag{1.11}$$

where $|t| < \pi$ when $\lambda = 1$; $|t| < |\ln(-\lambda)|$ when $\lambda \neq 1$ and $k \in \mathbb{Z}$ (*cf.* [27, Eq. (1)], [44, 45]).

When $x = 0$ in (1.11), we get the Apostol-Euler numbers of order k

$$\mathcal{E}_m^{(k)}(0; \lambda) = \mathcal{E}_m^{(k)}(\lambda)$$

(*cf.* [44, 45]).

One can observe that

$$\mathcal{E}_m^{(1)}(x; \lambda) = \mathcal{E}_m(x; \lambda) \quad \text{and} \quad \mathcal{E}_m^{(1)}(\lambda) = \mathcal{E}_m(\lambda).$$

When $k=0$ in (1.11), we have

$$\mathcal{E}_m^{(0)}(x;\lambda)=x^m$$

and also for $x=0$ in the above equation, one can easily see that

$$\mathcal{E}_m^{(0)}(\lambda)=\left\{\begin{array}{ll}1, & m=0\\ 0, & m\in\mathbb{N}\end{array}\right. .$$

Substituting $\lambda=1$ into (1.11), we have the Euler polynomials of order k

$$\mathcal{E}_m^{(k)}(x;1)=E_m^{(k)}(x)$$

which are defined by means of the following generating function:

$$\left(\frac{2}{e^t+1}\right)^k e^{tx}=\sum_{m=0}^{\infty}E_m^{(k)}(x)\frac{t^m}{m!} \tag{1.12}$$

(*cf.* [44, 45]).

Setting $x=0$ in (1.12), we have

$$E_m^{(k)}(0)=E_m^{(k)}$$

which $E_m^{(k)}$ denote the Euler numbers of order k (*cf.* [28, 44, 45]).

Using (1.9) and (1.12), for $x=0$, we have

$$B_m^{(-r)}=2^{-m}\sum_{s=0}^{m}\binom{m}{s}B_s^{(-r)}E_{m-s}^{(-r)} \tag{1.13}$$

(*cf.* [17, Eq. (3.1)]).

The λ-Stirling numbers of the second kind are defined by

$$\frac{\left(\lambda e^t-1\right)^v}{v!}=\sum_{m=0}^{\infty}S_2(m,v;\lambda)\frac{t^m}{m!}, \tag{1.14}$$

where $v\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$ (*cf.* [29, 36, 44, 45]).

Taking $\lambda=1$ in (1.14), we have

$$S_2(m,v;1)=S_2(m,v).$$

The numbers $S_2(m,v;\lambda)$ are also defined by as follows:

$$\lambda^x x^m=\sum_{v=0}^{\infty}\binom{x}{v}v!S_2(m,v;\lambda), \tag{1.15}$$

(*cf.* [29, Eq. (98)]; see also [36, 44, 45]).

From (1.14), we have the following explicit formula for the numbers $S_2(m,v;\lambda)$:

$$S_2(m,v;\lambda)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^j\binom{v}{j}\lambda^{v-j}(v-j)^m,$$

where $v,m\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$ (*cf.* [29, 36, 44, 45]).

It also should be noted that

$$S_2(m,v;\lambda)=(-1)^v y_1(m,v;-\lambda) \tag{1.16}$$

(*cf.* [19]).

By using (1.8) and (1.14), we have

$$\mathcal{B}_m^{(-r)}(\lambda)=\binom{m+r}{r}^{-1}S_2(m+r,r;\lambda), \tag{1.17}$$

where $m, r \in \mathbb{N}$ (*cf.* [44, Eq. (7.16)]).

Putting $\lambda = 1$ in (1.17), we also have

$$B_m^{(-r)} = \binom{m+r}{r}^{-1} S_2(m+r, r) \tag{1.18}$$

(*cf.* [44, Eq. (7.17)]).

The Bell polynomials are defined by

$$e^{\left(e^t-1\right)x} = \sum_{m=0}^{\infty} Bl_m(x) \frac{t^m}{m!}, \tag{1.19}$$

(*cf.* [2, 3]).

Combining (1.19) with (1.7), we have the following formula:

$$Bl_m(x) = \sum_{j=0}^{m} S_2(m, j) x^j \tag{1.20}$$

(*cf.* [2, 3]).

The Legendre polynomials are defined by

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{m=0}^{\infty} P_m(x) t^m$$

(*cf.* [3]).

The polynomials $P_m(x)$ are computed by the following sum of power of binomial coefficients:

$$P_m(x) = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j}^2 (x-1)^{m-j} (x+1)^j \tag{1.21}$$

(*cf.* [3]).

The Fubini type polynomials of order v are defined by

$$\frac{2^v}{\left(2-e^t\right)^{2v}} e^{xt} = \sum_{m=0}^{\infty} a_m^{(v)}(x) \frac{t^m}{m!}, \tag{1.22}$$

where $v \in \mathbb{N}_0$ and $|t| < \ln 2$ (*cf.* [14]; see also [15, 16]).

When $x = 0$ in (1.22), we get the Fubini type numbers of order v

$$a_m^{(v)}(0) = a_m^{(v)}$$

(*cf.* [14]; see also [15, 16]).

The numbers $y_2(m, v; \lambda)$ are defined by

$$\frac{\left(\lambda e^t + \lambda^{-1} e^{-t} + 2\right)^v}{(2v)!} = \sum_{m=0}^{\infty} y_2(m, v; \lambda) \frac{t^m}{m!}, \tag{1.23}$$

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [37]).

From (1.23), we have

$$y_2(m, v; \lambda) = \frac{1}{(2v)!} \sum_{j=0}^{v} \binom{v}{j} 2^{v-j} \sum_{r=0}^{j} \binom{j}{r} (2r-j)^m \lambda^{2r-j},$$

where $m, v \in \mathbb{N}$ (*cf.* [37, Eq. (17)]).

The numbers $y_3(m,v;\lambda;a,b)$ are defined by

$$\frac{e^{vbt}}{v!}\left(\lambda e^{(a-b)t}+1\right)^v=\sum_{m=0}^{\infty} y_3(m,v;\lambda;a,b)\frac{t^m}{m!}, \tag{1.24}$$

where $a,b\in\mathbb{R}$, $v\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$ (or $\mathbb{R}$) (*cf.* [39, Eq. (1)]).

With the help of (1.24), we have

$$y_3(m,v;\lambda;a,b)=\frac{1}{v!}\sum_{j=0}^{v}\binom{v}{j}\lambda^j\left(bv+j(a-b)\right)^m$$

(*cf.* [37, Theorem 3.1]).

The combinatorial-type numbers $V_m(\lambda)$ are defined by

$$\frac{1-\lambda+\sqrt{(\lambda-1)^2+8\lambda^2 t}}{2\lambda^2 t}=\sum_{m=0}^{\infty}V_m(\lambda)t^m,$$

where $0<\left|\frac{\lambda^2 t}{(\lambda-1)^2}\right|\leq\frac{1}{8}$ (*cf.* [25]).

The rest of this paper is briefly organized as follows:

In Section 2, by using generating functions with functional equation methods, many identities containing the Apostol-Bernoulli and Apostol-Euler numbers of negative higher order, the Bernoulli and Euler numbers of negative higher order, the Bell polynomials, the Fubini type polynomials of higher order, the Stirling type numbers, and the combinatorial type numbers and polynomials are derived.

In Section 3, with the help of an asymptotic expression of sums of powers of binomial coefficients, some formulas for the numbers $y_6(m,n;\lambda,p)$ are given.

In Section 4, we give some results and remarks on the inequalities including binomial coefficients and the numbers $y_6(m,n;\lambda,p)$.

In Section 5, we give the conclusion section of this paper.

2. Relations for the numbers $y_6(m,n;\lambda,p)$ and certain special numbers and polynomials

In this section, we give some relations and identities involving the Apostol-Bernoulli and Apostol-Euler numbers of negative higher order, the Bernoulli and Euler numbers of negative higher order, the Bell polynomials, the Stirling numbers of the second kind, the λ-Stirling numbers, the Fubini type polynomials of higher order, the numbers $y_1(m,v;\lambda)$, the numbers $y_2(m,v;\lambda)$, the numbers $y_3(m,v;\lambda;a,b)$ and the numbers $y_6(m,n;\lambda,p)$. We also investigate some special cases of these results.

Theorem 2.1. *Let $m,n,p\in\mathbb{N}_0$ with $m\geq n$. Then we have*

$$y_6(m,n;\lambda,p)=\frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}^p\binom{k}{j}\{m\}_j\,\mathcal{B}_{m-j}^{(-j)}(\lambda). \tag{2.1}$$

Proof. By using (1.1), we have

$$\sum_{m=0}^{\infty}y_6(m,n;\lambda,p)\frac{t^m}{m!}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p\sum_{j=0}^{k}\binom{k}{j}\left(\lambda e^t-1\right)^j.$$

Combining the above equation with (1.8), we get

$$\sum_{m=0}^{\infty} y_6(m,n;\lambda,p)\frac{t^m}{m!}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p\sum_{j=0}^{k}\binom{k}{j}\sum_{m=0}^{\infty}\mathcal{B}_m^{(-j)}(\lambda)\frac{t^{m+j}}{m!}.$$

Therefore,

$$\sum_{m=0}^{\infty} y_6(m,n;\lambda,p)\frac{t^m}{m!}=\frac{1}{n!}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}^p\binom{k}{j}\{m\}_j\,\mathcal{B}_{m-j}^{(-j)}(\lambda)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Combining (2.1) with (1.17), we have the following corollary:

Corollary 2.2. *Let $m,n,p\in\mathbb{N}_0$. Then we have*

$$y_6(m,n;\lambda,p)=\frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}^p\{k\}_j\,S_2(m,j;\lambda).$$

Remark. *Substituting $x=k$, $k\in\mathbb{N}_0$ into* (1.15) *and combining the final equation with* (1.2)*, we also arrive at the Corollary* 2.2.

Theorem 2.3. *Let $m,n,p\in\mathbb{N}_0$ with $m\geq n$. Then we have*

$$y_6(m,n;\lambda,p)=\frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}^p\binom{k}{j}\lambda^k\{m\}_j\,B_{m-j}^{(-j)}. \tag{2.2}$$

Proof. Using (1.1), we have

$$\sum_{m=0}^{\infty} y_6(m,n;\lambda,p)\frac{t^m}{m!}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p\lambda^k\sum_{j=0}^{k}\binom{k}{j}\left(e^t-1\right)^j.$$

Combining the above equation with (1.9), we obtain

$$\sum_{m=0}^{\infty} y_6(m,n;\lambda,p)\frac{t^m}{m!}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p\lambda^k\sum_{j=0}^{k}\binom{k}{j}\sum_{m=0}^{\infty}\{m\}_j\,B_{m-j}^{(-j)}\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the equation (2.2). □

Here note that combining (1.10) with Theorem 18 in [40], we also arrive at the equation (2.2).

Remark. *Combining* (2.2) *with* (1.18)*, we have the following relation which was proved by Simsek* [40, Theorem 17]*:*

$$y_6(m,n;\lambda,p)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}^{p-1}\frac{\lambda^k S_2(m,j)}{(n-k)!(k-j)!}.$$

Combining (2.2) with (1.13), we obtain the following corollary:

Corollary 2.4.

$$y_6(m,n;\lambda,p)=\frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{s=0}^{m-j}\binom{n}{k}^p\binom{k}{j}\binom{m-j}{s}\frac{\lambda^k\{m\}_j}{2^{m-j}}B_s^{(-j)}E_{m-j-s}^{(-j)}.$$

Theorem 2.5. *Let $m, n, p \in \mathbb{N}_0$. Then we have*

$$y_6\left(m, n; \lambda, p\right) = \frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}^{p}\binom{k}{j}\left(-1\right)^{k-j}2^{j}\mathcal{E}_{m}^{(-j)}\left(\lambda\right).$$

Proof. By using (1.1), we have

$$\sum_{m=0}^{\infty} y_6\left(m, n; \lambda, p\right)\frac{t^m}{m!} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\sum_{j=0}^{k}\left(-1\right)^{k-j}\binom{k}{j}\left(\lambda e^t + 1\right)^{j}.$$

From the above equation and (1.11), we get

$$\sum_{m=0}^{\infty} y_6\left(m, n; \lambda, p\right)\frac{t^m}{m!} = \frac{1}{n!}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\left(-1\right)^{k-j}2^{j}\binom{n}{k}^{p}\binom{k}{j}\mathcal{E}_{m}^{(-j)}\left(\lambda\right)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.6. *Let $m, n, p \in \mathbb{N}_0$. Then we have*

$$y_6\left(m, n; \lambda, p\right) = \frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}\left(-1\right)^{k-j}\binom{n}{k}^{p}\binom{k}{j}\lambda^{k}2^{j}E_{m}^{(-j)}.$$

Proof. From (1.1) and (1.12), we obtain

$$\sum_{m=0}^{\infty} y_6\left(m, n; \lambda, p\right)\frac{t^m}{m!} = \frac{1}{n!}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\left(-1\right)^{k-j}\binom{k}{j}\binom{n}{k}^{p}\lambda^{k}2^{j}E_{m}^{(-j)}\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.7. *Let $m, n, p \in \mathbb{N}_0$. Then we have*

$$y_6\left(m, n; \lambda, p\right) = \frac{1}{n!}\sum_{k=0}^{n}\left(-\lambda\right)^{k}\binom{n}{k}^{p}\left(2k\right)!\sum_{j=0}^{m}\binom{m}{j}a_{j}^{(k)}\left(2k\right)y_2\left(m-j, k; -\frac{1}{2}\right).$$

Proof. Combining (1.1) with (1.22) and (1.23), and assuming $|t| < \ln 2$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} y_6\left(m, n; \lambda, p\right)\frac{t^m}{m!} &= \frac{1}{n!}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}^{p}\lambda^{k}\left(2k\right)!\sum_{m=0}^{\infty}y_2\left(m, k; -\frac{1}{2}\right)\frac{t^m}{m!}\\ &\quad\times\sum_{m=0}^{\infty}a_{m}^{(k)}\left(2k\right)\frac{t^m}{m!}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{m=0}^{\infty} y_6\left(m, n; \lambda, p\right)\frac{t^m}{m!} &= \frac{1}{n!}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}^{p}\lambda^{k}\left(2k\right)!\\ &\quad\times\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}a_{j}^{(k)}\left(2k\right)y_2\left(m-j, k; -\frac{1}{2}\right)\frac{t^m}{m!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we get the desired result. □

Theorem 2.8. *Let $m,n,p\in\mathbb{N}_0$. Then we have*

$$y_6\left(m,n;\lambda,p\right)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\left(2\lambda\right)^{k}\left(2k\right)!\sum_{j=0}^{m}\binom{m}{j}a_j^{(k)}\left(-k\right)y_3\left(m-j,2k;-\frac{1}{2};2,1\right).$$

Proof. Using (1.1), we get

$$\sum_{m=0}^{\infty}y_6\left(m,n;\lambda,p\right)\frac{t^m}{m!}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\left(2\lambda\right)^{k}\left(2k\right)!\frac{\left(-\frac{1}{2}e^t+1\right)^{2k}e^{2kt}}{\left(2k\right)!}\frac{2^ke^{-kt}}{\left(2-e^t\right)^{2k}}.$$

Combining the above equation with (1.22) and (1.24), we have

$$\begin{aligned}\sum_{m=0}^{\infty}y_6\left(m,n;\lambda,p\right)\frac{t^m}{m!}&=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\left(2\lambda\right)^{k}\left(2k\right)!\sum_{m=0}^{\infty}y_3\left(m,2k;-\frac{1}{2};2,1\right)\frac{t^m}{m!}\\&\quad\times\sum_{m=0}^{\infty}a_m^{(k)}\left(-k\right)\frac{t^m}{m!}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{m=0}^{\infty}y_6\left(m,n;\lambda,p\right)\frac{t^m}{m!}&=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\left(2\lambda\right)^{k}\left(2k\right)!\\&\quad\times\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}a_j^{(k)}\left(-k\right)y_3\left(m-j,2k;-\frac{1}{2};2,1\right)\frac{t^m}{m!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the desired result. $\square$

Theorem 2.9. *Let $r,n,p\in\mathbb{N}_0$. Then we have*

$$\sum_{m=0}^{r}y_6\left(m,n;\lambda,p\right)S_2\left(r,m\right)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\lambda^kBl_r\left(k\right).\tag{2.3}$$

Proof. Replacing t by e^t-1 in (1.1), we have

$$\sum_{m=0}^{\infty}y_6\left(m,n;\lambda,p\right)\frac{\left(e^t-1\right)^m}{m!}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\lambda^ke^{\left(e^t-1\right)k}.$$

Combining the above equation with (1.7) and (1.19), we obtain

$$\sum_{r=0}^{\infty}\sum_{m=0}^{r}y_6\left(m,n;\lambda,p\right)S_2\left(r,m\right)\frac{t^r}{r!}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}\lambda^k\sum_{r=0}^{\infty}Bl_r\left(k\right)\frac{t^r}{r!}.$$

Comparing the coefficients of $\frac{t^r}{r!}$ on both sides of the above equation, we arrive at the desired result. $\square$

Setting $p=1$ in (2.3) and combining the final equation with (1.5), we get the following corollary:

Corollary 2.10. *Let r and $n\in\mathbb{N}_0$. Then we have*

$$\sum_{m=0}^{r}y_1\left(m,n;\lambda\right)S_2\left(r,m\right)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda^kBl_r\left(k\right).\tag{2.4}$$

Combining (2.4) with (1.16), we obtain the following relation:

Corollary 2.11. *Let r and $n \in \mathbb{N}_0$. Then we have*

$$\sum_{m=0}^{r} S_2\left(m,n;-\lambda\right) S_2\left(r,m\right) = \frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda^k Bl_r\left(k\right).$$

Theorem 2.12. *Let $n \in \mathbb{N}_0$ and $\lambda \neq 1$. Then we have*

$$y_6\left(0,n;\lambda,2\right) = \frac{(\lambda-1)^n}{n!} P_n\left(\frac{\lambda+1}{\lambda-1}\right). \tag{2.5}$$

Proof. Substituting $m = 0$ and $p = 2$ into (1.2), we get the following result:

$$y_6\left(0,n;\lambda,2\right) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}^2 \lambda^k.$$

Combining the above equation with (1.21), we obtain

$$y_6\left(0,n;\lambda,2\right) = \frac{(\lambda-1)^n}{n!} P_n\left(\frac{\lambda+1}{\lambda-1}\right).$$

Thus, the proof of the theorem is completed. □

Notice that the different proof of the Theorem 2.12 was also given by Simsek, using relation between Michael Vowe polynomials and Legendre polynomials (see for detail, [40, Remark 10 and Remark 12]).

Remark. *When $\lambda = 3$ and $\lambda = -1$ in (2.5), respectively we have*

$$y_6\left(0,n;3,2\right) = \frac{2^n}{n!} P_n\left(2\right)$$

and

$$y_6\left(0,n;-1,2\right) = \frac{(-1)^n 2^n}{n!} P_n\left(0\right)$$

(cf. [40, p. 1337]).

3. Remarks on asymptotic expressions of the numbers $y_6\left(m,n;\lambda,p\right)$

In this section, we investigate asymptotic expressions of the numbers $y_6\left(m,n;\lambda,p\right)$. We present some relations related to these numbers.

For approximation we write $a_n \sim b_n$. This notation means that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \tag{3.1}$$

(*cf.* [33]).

By using the following Stirling's approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{3.2}$$

(*cf.* [5, 34, 45]), for $n \to \infty$ and $k \to \infty$, approximation of $\binom{n}{k}$ is given by as follows:

$$\binom{n}{k} \sim \frac{n^n}{k^k\left(n-k\right)^{n-k}} \sqrt{\frac{n}{2\pi k\left(n-k\right)}} \tag{3.3}$$

(*cf.* [43]).

By using (3.2), Polya and Szego [33, Problem 40, p. 55] gave an asymptotic expression of the finite sum of powers of binomial coefficients as follows:

$$\sum_{k=0}^{n}\binom{n}{k}^{p} \sim \frac{2^{pn}}{\sqrt{p}}\left(\frac{2}{\pi n}\right)^{\frac{p-1}{2}}. \tag{3.4}$$

It's time to give some relations involving the numbers $y_6(m,n;\lambda,p)$ with the help of the asymptotic formulas of the finite sum of powers of binomial coefficients given in the above equations.

Theorem 3.1. *Let $n,p\in\mathbb{N}$. Then we have*

$$y_6(0,n;1,p) \sim \frac{2^{pn-1}}{\sqrt{p}}\left(\frac{e}{n}\right)^{n}\left(\frac{2}{\pi n}\right)^{\frac{p}{2}}. \tag{3.5}$$

Proof. Substituting $\lambda=1$ and $m=0$ into (1.2), we get

$$y_6(0,n;1,p) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}. \tag{3.6}$$

By combining (3.6) with (3.4) and (3.2), after some elementary calculations, we obtain

$$\sqrt{p}\lim_{n\to\infty}\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} y_6(0,n;1,p)}{2^{pn}\left(\frac{2}{\pi n}\right)^{\frac{p-1}{2}}} = 1.$$

Assuming that $\frac{2^{pn}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\sqrt{p}}\left(\frac{2}{\pi n}\right)^{\frac{p-1}{2}} \neq 0$. Consequently, $y_6(0,n;1,p)$ is an asymptotically equal to $\frac{2^{pn}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\sqrt{p}}\left(\frac{2}{\pi n}\right)^{\frac{p-1}{2}}$. That is,

$$y_6(0,n;1,p) \sim \frac{2^{pn-1}}{\sqrt{p}}\left(\frac{e}{n}\right)^{n}\left(\frac{2}{\pi n}\right)^{\frac{p}{2}}.$$

Therefore, proof of theorem is completed. □

Remark. *Kucukoglu and Simsek, with the help of the Stirling's approximation, gave interesting and useful asymptotic formulas including combinatorial numbers (see, for detail,* [22, 23]*; and also* [26]*).*

We now give some special cases involving three equations for the numbers $y_6(m,n;\lambda,p)$. Putting $p=1$, $\lambda=1$ and $m=0$ in (1.2), and using (1.3), we have

$$y_6(0,n;1,1) = \frac{2^{n}}{n!}. \tag{3.7}$$

When $p=1$ in (3.5), we get

$$y_6(0,n;1,1) \sim \frac{2^{n-\frac{1}{2}}}{\sqrt{n\pi}}\left(\frac{e}{n}\right)^{n}. \tag{3.8}$$

From the equations (3.7) and (3.8), some values of the numbers $y_6(0,n;1,1)$ are given by Table 1:

n	$y_6(0,n;1,1)$	Approximate values of $y_6(0,n;1,1)$
1	2	2.16888
2	2	2.08441
3	1.33333	1.37075
4	0.666667	0.680672
5	0.266667	0.271142
6	0.0888889	0.0901309
600	$3.27877 \, . \, 10^{-1228}$	$3.27922 \, . \, 10^{-1228}$
6000	$5.63886 \, . \, 10^{-18260}$	$5.63894 \, . \, 10^{-18260}$
60000	$4.03148 \, . \, 10^{-242573}$	$4.03149 \, . \, 10^{-242573}$

TABLE 1. Some numeric and approximate values of the numbers $y_6(0,n;1,1)$.

When $p=2$, $\lambda=1$ and $m=0$ in (1.2), then using (1.4), we obtain

$$y_6(0,n;1,2)=\frac{(2n)!}{(n!)^3}. \tag{3.9}$$

Using (3.5), for $p=2$, we have

$$y_6(0,n;1,2)\sim\frac{e^n 2^{2n-\frac{1}{2}}}{\pi n^{n+1}}. \tag{3.10}$$

By using equations (3.9) and (3.10), some values of the numbers $y_6(0,n;1,2)$ are given by Table 2:

n	$y_6(0,n;1,2)$	Approximate values of $y_6(0,n;1,2)$
1	2	2.44731
2	3	3.32624
3	3.33333	3.57202
4	2.91667	3.07223
5	2.1	2.18921
6	1.28333	1.32863
600	$3.13305 \, . \, 10^{-1049}$	$3.13413 \, . \, 10^{-1049}$
6000	$6.21593 \, . \, 10^{-16456}$	$6.21614 \, . \, 10^{-16456}$
60000	$5.85535 \, . \, 10^{-224514}$	$5.85538 \, . \, 10^{-224514}$

TABLE 2. Some numeric and approximate values of the numbers $y_6(0,n;1,2)$.

It can be easily seen in Table 2 that when $n\to\infty$, ratio of the first column to the second column converges to 1 due to the asymptotic equality.

When $p=3$ in (3.5), we have the following asymptotic expression:

$$y_6(0,n;1,3)\sim\frac{2^{3n+1}}{\sqrt{6}}\left(\frac{e}{n}\right)^n\left(\frac{1}{\pi n}\right)^{\frac{3}{2}}.$$

Thus, some values of the numbers $y_6(0,n;1,3)$ are given by Table 3:

n	$y_6(0,n;1,3)$	Approximate values of $y_6(0,n;1,3)$
1	2	3.1887
2	5	6.12905
3	9.33333	10.7482
4	14.4167	16.0117
5	18.7667	20.4102
6	21.0889	22.6153
600	$3.45647 \,.\, 10^{-870}$	$3.45887 \,.\, 10^{-870}$
6000	$7.91195 \,.\, 10^{-14652}$	$7.91250 \,.\, 10^{-14652}$
60000	$9.81997 \,.\, 10^{-206455}$	$9.82004 \,.\, 10^{-206455}$

Table 3. Some numeric and approximate values of the numbers $y_6(0,n;1,3)$.

4. Further results and remarks on the inequalities involving the numbers $y_6(m,n;\lambda,p)$

In this section, we give both the upper bound and the lower bound for the numbers $y_6(m,n;\lambda,p)$ with the help of the binomial coefficients.

The binomial coefficient is related to the following inequalities

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!}, \tag{4.1}$$

where $1 \leq k \leq n$ (*cf.* [4, 43]).

By replacing n by $2n$ and k by n in (4.1), we have

$$2^n \leq \binom{2n}{n} \tag{4.2}$$

and

$$\frac{2^n n^n}{n!} \geq \binom{2n}{n}. \tag{4.3}$$

Combining (1.4) with (4.2) and (4.3), we have the following result including the lower and upper bound for the numbers $y_6(0,n;1,2)$:

Corollary 4.1. *Let $n \in \mathbb{N}$. Then we have*

$$\frac{2^n}{n!} \leq y_6(0,n;1,2) \leq \frac{2^n n^n}{(n!)^2}. \tag{4.4}$$

In [23, Eq. (6.7)], Kucukoglu and Simsek gave the following relation for the numbers $y_6(0,n;1,2)$:

$$V_n(\lambda) = \frac{(-1)^n 2^{n+1} \lambda^{2n} n!}{(n+1)(\lambda-1)^{2n+1}} y_6(0,n;1,2), \tag{4.5}$$

where $\lambda \neq 1$ and $n \in \mathbb{N}$. They also gave the lower and upper bound for the numbers $V_n(\lambda)$ as follows:

$$\frac{(-1)^n 2^{3n} \lambda^{2n}}{(n+1)\sqrt{n}(\lambda-1)^{2n+1}} \leq V_n(\lambda) \leq \frac{(-1)^n 2^{3n+1} \lambda^{2n}}{(n+1)(\lambda-1)^{2n+1}}, \tag{4.6}$$

where $\lambda \neq 1$ and $n \in \mathbb{N}$ (*cf.* [23, Eq. (6.35)]).

Combining (4.5) with (4.6), we also obtain the following inequalities for the numbers $y_6(0,n;1,2)$:

Corollary 4.2. *Let $n \in \mathbb{N}$. Then we have*

$$\frac{2^{2n-1}}{n!\sqrt{n}} \leq y_6(0,n;1,2) \leq \frac{2^{2n}}{n!}. \tag{4.7}$$

Remark. *Which of the inequalities given in* (4.4) *and* (4.7) *is more sharper can be investigated using different methods.*

Let $x \in [-1,1]$ and $n \in \mathbb{N}$. In [30], Martin gave the following inequality for the Legendre polynomials $P_n(x)$:

$$|P_n(x)| \leq \frac{1}{\left(1+n(n+1)\left(1-x^2\right)\right)^{\frac{1}{4}}}. \tag{4.8}$$

Using (4.8), we have

$$\left|P_n\left(\frac{\lambda+1}{\lambda-1}\right)\right| \leq \frac{1}{\left(1+n(n+1)\left(1-\left(\frac{\lambda+1}{\lambda-1}\right)^2\right)\right)^{\frac{1}{4}}},$$

where $\frac{\lambda+1}{\lambda-1} \in [-1,1]$ and $n \in \mathbb{N}$. Combining the above equation with (2.5), we obtain the following inequality for the numbers $y_6(0,n;\lambda,2)$:

Corollary 4.3. *Let $n \in \mathbb{N}$ and $\frac{\lambda+1}{\lambda-1} \in [-1,1]$. Then we have*

$$\left|\frac{n!y_6(0,n;\lambda,2)}{(\lambda-1)^n}\right| \leq \frac{1}{\left(1+n(n+1)\left(1-\left(\frac{\lambda+1}{\lambda-1}\right)^2\right)\right)^{\frac{1}{4}}}.$$

5. Conclusion

In this paper, some special numbers and polynomials, including finite sums of powers of binomial coefficients, were studied. Using generating functions for these numbers, some formulas involving the Apostol-Bernoulli and Apostol-Euler numbers of negative higher order, the Bernoulli and Euler numbers, the Stirling type numbers, the combinatorial numbers, the Bell polynomials, the Fubini type polynomials, and the Legendre polynomials were given. Further, some asymptotic expressions of the finite sums of powers of binomial coefficients and their numeric values for these numbers were presented. We also obtained some inequalities for these numbers. Consequently, the results of this paper have the potential to be considerable attention of many researchers such as mathematicians, physicists and engineers.

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