Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 14 Issue 1(2023), Pages 37-50
https://doi.org/10.54379/jiasf-2023-1-4

A SIMPLE AND EFFICIENT APPROACH BASED ON LAGUERRE POLYNOMIALS FOR SOLVING SCHLÖMILCH'S INTEGRAL EQUATION

AHMET ALTÜRK

Abstract. The Schlömilch's integral equation plays an important role in many ionospheric problems and is considered as an important and useful equation in atmospheric and terrestrial physics. The equation is classified as a Fredholm integral equation of the first kind. This classification enables us to use and employ the tools available for solving Fredholm equations of the first kind. The motivation behind this work is to develop an efficient method that based on the regularization method and Laguerre polynomials for solving various types of Schlömilch's integral equations.

1. Introduction

The subject of integral equations is one of the most important tools in mathematics. It has attracted the attention of many researchers due mainly to the fact that integral equations appear in many mathematical and physical models. Thus, constructing methods for obtaining analytical and numerical solutions of them have always been an active research field.

The Schlömilch's integral equation and it's solution have been used to obtain the electron density profile from the ionospheric ionograms for the case of quasi-transverse approximation. We will focus here more on mathematical aspect of the equation and its solutions. For more detailed explanation about physical aspect of the equation and its applications in engineering we refer the interested readers to [3, 4, 5, 6, 7, 8] and references therein.

We want to point out that the theoretical investigation of ionospheric problems, in particular the Schlömilch's integral equation, has been studied extensively in comparison to computational aspects of them, some of which have been mentioned below. To contribute to the latter, we aim to provide a simple and efficient algorithm that based on the regularization and Laguerre polynomials to solve different

1991 *Mathematics Subject Classification.* 45B05, 33C45, 47A52.
Key words and phrases. Integral equations; orthogonal polynomials; regularization.

Submitted December 7, 2022. Published February 24, 2023.
Communicated by Yilmaz Simsek.

types of Schlömilch's integral equation.

The main equation that we consider here, namely the standard *linear Schlömilch's integral equation*, has the following form:

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\sin\theta)\,d\theta, \quad -\pi \le x \le \pi, \tag{1.1}$$

where $y(x)$ is a continuous differential function on $[-\pi, \pi]$. A known solution to this equation reads

$$\phi(x) = y(0) + x\int_0^{\pi/2} y'(x\sin\theta)\,d\theta,$$

where the derivative y' is taken with respect to the argument $x\sin\theta$.

As it was emphasized in [9, 10], the Adomian decomposition method [1, 2] (ADM) cannot be applied directly to (1.1). To apply the ADM or any method that will be based on writing the solution in a series form should be adjusted properly so that the components of the solution completely determined recurrently or finding a relation that eventually produces the solution. Our motivation for this article is to introduce a method that based on a series and provides the exact solution. In what follows, we present some of the studies that make several noteworthy contributions to the literature on the Schlömilch's integral equation.

Bougoffa et al.[10] presented a new technique that extends application of the Adomian decomposition method in the sense that it can be applied to Fredholm integral equations of the first kind including Schlömilch's integral equations and a class of related integral equations of the first kind. The introduced method extends the applicability of the ADM. To develop the technique, the authors add the solution $\phi(x)$ to the both sides of the equation (1.1) and apply the ADM to the resulting equation. This new technique is easy to apply and yields very accurate results.

Wazwaz [11] used the regularization method combined with the ADM to solve various kinds of Schlömilch's integral equations. As we point out above, in order to use the ADM one needs to transform the equation (1.1) into a form that the unknown function appears outside of the integral as well. Wazwaz overcome this issue by applying the regularization method which will be discussed in the next section. In addition, he introduced and solved Schlömilch's-type integral equations.

Altürk [12] reveals the relation between solutions of the linear and nonlinear Schlömilch's integral equations and the well known gamma function. Altürk and Arabacıoğlu [13] intoduced a method based on the homotopy perturbation method to find solutions for various types of Schlömilch's integral equations. The method involves additional term to the standard convex homotopy so as to obtain an efficient algorithm.

Parand and Delkhosh [14] introduced a new numerical method based on the Chebyshev functions. To be more precise, they construct approximate solutions for linear and nonlinear Schlömilch's integral equations by using the generalized fractional order of the Chebyshev orthogonal functions.

Al-Jawary et al.[15] used the regularization method combined with the homotopy analysis method (HAM) and variational iteration method(VIM). To show the usefulness in finding the exact solutions, their implementation of aforementioned methods supported by several illustrative examples.

In addition to the standard Schlömilch's equation (1.1), there are two other types of Schlömilch's equations that will be considered in this work. One is the *generalized Schlömilch's integral equation*

$$y(x) = \frac{2}{\pi} \int_0^{\pi/2} \phi(x \sin^n \theta)\, d\theta, \quad n \geq 1.$$

and the other one is the *nonlinear Schlömilch's integral equation*

$$y(x) = \frac{2}{\pi} \int_0^{\pi/2} G\Big(\phi(x \sin \theta)\Big)\, d\theta,$$

where $G\big(\phi(x \sin \theta)\big)$ is a nonlinear function of ϕ and y is a continuous differential function on $-\pi \leq x \leq \pi$.

The rest of the article has been organized in the following way. The first section will examine the method of regularization briefly and state the main results that will be used in later sections. Then, basic properties of Laguerre polynomials are reviewed. The main section consists of method for solution of the Schlömilch's integral equations on a case by case basis. The detail is given in the first case which covers the basic linear Schlömilch's integral equation. For the rest, we will only emphasize the differences from the first case. Final section concludes the article.

2. The Regularization method

The regularization method was introduced by Philips [16], Lavrentiev[17], and Tikhonov [18, 19]. In this work, we will mainly focus on and employ the regularization method proposed by Lavrentiev. He introduced a regularization method to solve first kind integral equations with some mild restrictions on the kernel. Assume that a solution exists to the following ill-posed problem:

$$y(x) = \int_a^b k(x,t)\phi(t)\, dt, \quad a \leq x \leq b.$$

One can modify this equation by introducing a term that includes the regularization parameter γ as

$$y(x) = \int_a^b k(x,t)\phi(t)\, dt + \gamma\phi(x) \tag{2.1}$$

The equation (2.1) is a Fredholm integral equation of the second kind and if we denote its solution by $\phi_\gamma(x)$ and substitute into the equation (2.1) we get

$$\gamma\phi_\gamma(x) = y(x) - \int_a^b k(x,t)\phi_\gamma(t)\, dt,$$

Equivalently,

$$\phi_\gamma(x) = \frac{1}{\gamma} y(x) - \frac{1}{\gamma} \int_a^b k(x,t)\phi_\gamma(t)\, dt.$$

Now, it is an easy exercise to obtain $\phi_\gamma(x)$.

Under certain conditions, it is shown in [16, 18, 19] that

$$\phi(x) = \lim_{\gamma \to 0} \phi_\gamma(x)$$

3. Laguerre Polynomials

In this section, we give a brief review of the Laguerre polynomials and their properties that are needed for what follows. We note that there are different but equivalent ways to define Laguerre polynomials. Each has its own advantageous and disadvantageous for a practical application. Particular application will determine which approach is used as one's starting point. A few important properties which will be used throughout the article is listed below. For more information, we refer the reader to Arfgen and Weber [20].

A few Laguerre polynomial and the recursion relation is given below.

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= 1 - x, \\ L_2(x) &= 1 - 2x + \frac{x^2}{2}, \\ &\vdots \\ L_{n+1}(x) &= \frac{1 + 2n - x}{n+1} L_n(x) - \frac{n}{n+1} L_{n-1}(x), \quad n \geq 1. \end{aligned}$$

They form an orthonormal set of functions with weighting function e^{-x} in the interval $[0, \infty)$:

$$\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

4. Method for Solution

In order to find the solutions for the equations considered, we first apply the regularization method to bring them into such a form that there are more techniques to solve them, namely, bring them into the second kind integral equations. We then express the regularized solution function by the truncated Laguerre series. Finally, a matrix equation will be obtained and solution of that will lead us to get the desired solution.

Case 1: The Linear Schlömilch's Integral Equation

The standard *linear Schlömilch's integral equation* has the following form:

$$y(x) = \frac{2}{\pi} \int_0^{\pi/2} \phi(x \sin\theta)\, d\theta, \quad -\pi \leq x \leq \pi.$$

Applying the regularization method described above brings the equation into

$$\gamma \phi_\gamma(x) = y(x) - \frac{2}{\pi} \int_0^{\pi/2} \phi_\gamma(x \sin\theta)\, d\theta$$

or

$$\phi_\gamma(x) = \frac{1}{\gamma} y(x) - \frac{2}{\gamma\pi} \int_0^{\pi/2} \phi_\gamma(x \sin\theta)\, d\theta \tag{4.1}$$

Let $y(x)$ and $\phi_\gamma(x)$ be approximated by truncated Laguerre series as

$$\begin{aligned} y(x) &= \sum_{i=0}^{n} y_i L_i(x) \\ \phi_\gamma(x) &= \sum_{i=0}^{n} c_i L_i(x) \end{aligned} \tag{4.2}$$

Notation:

$$\mathbf{X} = [1,\, x,\, x^2 \,\ldots,\, x^n], \quad \mathbf{C} = [c_0,\, c_1,\, c_2 \,\ldots,\, c_n]^T, \quad \mathbf{Y} = [y_0,\, y_1,\, y_2 \,\ldots,\, y_n]^T \tag{4.3}$$

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & -1 & -2 & \ldots & -n \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ 0 & 0 & 0 & \ddots & \binom{n}{n-1}\frac{(-1)^{n-1}}{(n-1)!} \\ 0 & 0 & 0 & \ldots & \frac{(-1)^n}{n!} \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \sin(\theta) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \sin^n(\theta) \end{pmatrix}$$

With the help of this notation, the equation (4.1) can be expressed as

$$\mathbf{XQC} = \mathbf{XQ\tilde{Y}} - \mathbf{X\tilde{S}QC}, \tag{4.4}$$

where

$$\tilde{\mathbf{S}} = \frac{2}{\gamma\pi}\int_0^{\pi/2} \mathbf{S}\, d\theta \quad \text{and} \quad \tilde{\mathbf{Y}} = \frac{1}{\gamma}\mathbf{Y}$$

Using the fact that $\mathbf{Q}$ is invertible, rearranging the terms in (4.4) and then simplifying the resulting equation, we end up with

$$(\mathbf{I} + \mathbf{Q}^{-1}\tilde{\mathbf{S}}\mathbf{Q})\mathbf{C} = \tilde{\mathbf{Y}},$$

where $\mathbf{I}$ is the identity matrix.
If the matrix $\mathbf{I} + \mathbf{Q}^{-1}\tilde{\mathbf{S}}\mathbf{Q}$ is invertible, then

$$\mathbf{C} = (\mathbf{I} + \mathbf{Q}^{-1}\tilde{\mathbf{S}}\mathbf{Q})^{-1}\tilde{\mathbf{Y}}. \tag{4.5}$$

Substituting the coefficients c_i back into the equation (4.2), we get the solution $\phi_\gamma(x)$. The desired solution $\phi(x)$ is obtained by taking the limit of $\phi_\gamma(x)$ as $\gamma \to 0$.

We want to emphasize that in many applications and examples in the literature the data function y in (1.1) appears as a polynomial function. The following theorem will be useful for the examples considered here.

Theorem 4.1. [12] *The data function y in* (1.1) *is a polynomial function of degree n if and only if the solution function ϕ of* (1.1) *is a polynomial function of the same degree*

Before delving into the examples, we want to make a couple of remarks here:

Remark. *A well-known feature related to Fredholm integral equations of the first kind is that they are considered as "ill-posed" problems. This property indicates that a solution of the problem may not exist, or if it exists uniqueness and continuous dependence on the data are not guaranteed* [19]. *It is also this property that makes it difficult to find analytical solutions for the equation.*

Remark. *We don't claim for complete generality about the data function, but introduce a method that could be used to find the analytical solution under the mild conditions imposed on the data function and is comparable with existing techniques in the literature from which examples are mostly taken. In addition, a recent review paper* [21] *is recommended for those who are interested in an overview of numerical solutions for Fredholm integral equations of the first kind.*

Example 4.1. *Given the following Fredholm integral equation of the first kind* [11, 15]*:*

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\sin\theta)\,d\theta, \quad -\pi \le x \le \pi, \tag{4.6}$$

where $y(x) = 1 + \pi x^2$.

We first apply the regularization method to transform the equation into a second kind equation. Applying the regularization method, (4.6) becomes

$$\gamma\phi_\gamma(x) = 1 + \pi x^2 - \frac{2}{\pi}\int_0^{\pi/2} \phi_\gamma(x\sin\theta)\,d\theta$$

or

$$\phi_\gamma(x) = \frac{1}{\gamma}(1 + \pi x^2) - \frac{2}{\gamma\pi}\int_0^{\pi/2} \phi_\gamma(x\sin\theta)\,d\theta$$

Since we seek the solution $\phi_\gamma(x) = \sum_{i=0}^{n} c_i L_i(x)$ in a truncated Laguerre series form, we set $n = 2$. Following the steps explained above, we obtain

$$\begin{aligned}\mathbf{C} &= [c_0,\, c_1,\, c_2]^T\\ \mathbf{Y} &= [y_0,\, y_1,\, y_2]^T = [2\pi+1,\, -4\pi,\, 2\pi]^T\end{aligned}$$

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1\\ 0 & -1 & -2\\ 0 & 0 & \frac{1}{2}\end{pmatrix} \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 1 & 2\\ 0 & -1 & -4\\ 0 & 0 & 2\end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0\\ 0 & \sin(\theta) & 0\\ 0 & 0 & \sin^2(\theta)\end{pmatrix}$$

Substituting these into the equation (4.5), we get

$$\begin{aligned} c_0 &= \frac{2\alpha(2\pi+1) + 4\pi + 1}{2\alpha^2 + 3\alpha + 1},\\ c_1 &= \frac{-8\pi}{2\alpha+1},\\ c_2 &= \frac{4\pi}{2\alpha+1}.\end{aligned}$$

This in turn gives the regularized solution

$$\begin{aligned}\phi_\gamma(x) &= \frac{2\gamma(2\pi+1)+4\pi+1}{2\gamma^2+3\gamma+1}L_0(x) + \frac{-8\pi}{2\gamma+1}L_1(x) + \frac{4\pi}{2\gamma+1}L_2(x)\\ &= \frac{2\gamma(2\pi+1)+4\pi+1}{2\gamma^2+3\gamma+1} + \frac{-8\pi}{2\gamma+1}(1-x) + \frac{4\pi}{2\gamma+1}(1-2x+x^2/2)\\ &= \frac{2\gamma+1}{2\gamma^2+3\gamma+1} + \frac{2\pi}{2\gamma+1}x^2\end{aligned}$$

As the last step to obtain the desired solution $\phi(x)$, we take the limit as $\gamma \to 0$:

$$\begin{aligned}\phi(x) = \lim_{\gamma\to 0}\phi_\gamma(x) &= \lim_{\gamma\to 0}\Big(\frac{2\gamma+1}{2\gamma^2+3\gamma+1} + \frac{2\pi}{2\gamma+1}x^2\Big)\\ &= 1+2\pi x^2\end{aligned}$$

This is the exact solution which is also derived in [11, 15].

Remark. *We want to make a note here that sometimes the equation to be considered appeared in the form*

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2}\phi(x\sin(k\theta))\,d\theta, \quad -\pi \le x \le \pi,$$

where k is a constant.

The procedure to solve this types of problems is almost the same as that of Case 1. The only difference is that one needs to modify $\mathbf{S}$ in (4.3) as

$$\mathbf{S}_k = \begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ 0 & \sin(k\theta) & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \dots & \vdots\\ \vdots & \vdots & \ddots & \dots & \vdots\\ 0 & 0 & 0 & 0 & \sin^n(k\theta)\end{pmatrix}$$

and replace $\mathbf{S}$ with $\mathbf{S}_k$ wherever it appears. In other words, with the help of this notation, the equation (4.4) can be expressed as

$$\mathbf{XQC} = \mathbf{XQ\tilde{Y}} - \mathbf{X\tilde{S}_kQC}, \tag{4.7}$$

where

$$\mathbf{\tilde{S}_k} = \frac{2}{\gamma\pi}\int_0^{\pi/2}\mathbf{S}_k\,d\theta \quad \text{and} \quad \mathbf{\tilde{Y}} = \frac{1}{\gamma}\mathbf{Y}$$

Using the fact that $\mathbf{Q}$ is invertible, rearranging the terms in (4.7) and then simplifying the resulting equation, we end up with

$$(\mathbf{I} + \mathbf{Q}^{-1}\mathbf{\tilde{S}_kQ})\mathbf{C} = \mathbf{\tilde{Y}},$$

where $\mathbf{I}$ is the identity matrix.
If the matrix $\mathbf{I} + \mathbf{Q}^{-1}\mathbf{\tilde{S}_kQ}$ is invertible, then

$$\mathbf{C} = (\mathbf{I} + \mathbf{Q}^{-1}\mathbf{\tilde{S}_kQ})^{-1}\mathbf{\tilde{Y}}. \tag{4.8}$$

To illustrate this case, we consider the following example.

Example 4.2. *Given the following Fredholm integral equation of the first kind* [11]*:*

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\sin 3\theta)\, d\theta, \quad -\pi \le x \le \pi, \tag{4.9}$$

where $y(x) = 2x$.

We first apply the regularization method to transform the equation into a second kind equation. Applying the regularization method, (4.9) becomes

$$\gamma\phi_\gamma(x) = 2x - \frac{2}{\pi}\int_0^{\pi/2} \phi_\gamma(x\sin 3\theta)\, d\theta$$

or

$$\phi_\gamma(x) = \frac{1}{\gamma}(2x) - \frac{2}{\gamma\pi}\int_0^{\pi/2} u_\gamma(x\sin t)\, dt$$

Since we seek the solution $\phi_\gamma(x) = \sum_{i=0}^{n} c_i L_i(x)$ in a truncated Laguerre series form, we set $n = 1$. This implies

$$\mathbf{C} = [c_0,\, c_1]^T$$
$$\mathbf{Y} = [y_0,\, y_1]^T = [2,\, -2]^T$$

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$
$$\mathbf{S_3} = \begin{pmatrix} 1 & 0 \\ 0 & \sin(3\theta) \end{pmatrix}$$

Substituting these into the equation (4.8), we get

$$c_0 = \frac{6\pi}{3\alpha\pi + 2},$$
$$c_1 = \frac{-6\pi}{3\alpha\pi + 2}.$$

This in turn gives the solution

$$\begin{aligned}\phi_\gamma(x) &= \frac{6\pi}{3\gamma\pi + 2}L_0(x) - \frac{6\pi}{3\gamma\pi + 2}L_1(x) \\ &= \frac{6\pi}{3\gamma\pi + 2} - \frac{6\pi}{3\gamma\pi + 2}(1 - x) \\ &= \frac{6\pi}{3\gamma\pi + 2}x\end{aligned}$$

As the last step to obtain the desired solution $\phi(x)$, we take the limit as $\gamma \to 0$:

$$\begin{aligned}\phi(x) = \lim_{\gamma\to 0}\phi_\gamma(x) &= \lim_{\gamma\to 0}\big(\frac{6\pi}{3\gamma\pi + 2}x\big) \\ &= 3\pi x\end{aligned}$$

This is the exact solution which is also derived in [11].

Case 2: The Generalised Schlömilch's Integral Equation

The *generalized Schlömilch's integral equation* admits the following form:

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\sin^k\theta)\,d\theta, \quad x \in \Omega, \tag{4.10}$$

where Ω is a closed and bounded domain of x. The regularization method is employed to the equation (4.10) to get

$$\gamma\phi_\gamma(x) = y(x) - \frac{2}{\pi}\int_0^{\pi/2} u_\gamma(x\sin^k\theta)\,d\theta$$

or

$$\phi_\gamma(x) = \frac{1}{\gamma}y(x) - \frac{2}{\gamma\pi}\int_0^{\pi/2} u_\gamma(x\sin^k\theta)\,d\theta$$

Let $y(x)$ and $\phi_\gamma(x)$ be approximated by truncated Laguerre series as before

$$y(x) = \sum_{i=0}^{n} y_i L_i(x)$$

$$\phi_\gamma(x) = \sum_{i=0}^{n} c_i L_i(x)$$

The solution follows from the steps explained in Case 1 with a small modification in $\mathbf{S}$.

$$\mathbf{S}_k = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \sin^k(\theta) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \sin^{nk}(\theta) \end{pmatrix}$$

That is, $\mathbf{S}$ will be replaced with $\mathbf{S}_k$ wherever it appears.

Example 4.3. *Given the following Fredholm integral equation of the first kind* [11, 15]*:*

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\sin^2\theta)\,d\theta, \quad -\pi \le x \le \pi, \tag{4.11}$$

where $y(x) = x + 3x^2$.

We first apply the regularization method to transform the equation into a second kind equation. Applying the regularization method, (4.11) becomes

$$\gamma y_\gamma(x) = x + 3x^2 - \frac{2}{\pi}\int_0^{\pi/2} \phi_\gamma(x\sin^2\theta)\,d\theta$$

or

$$\phi_\gamma(x) = \frac{1}{\gamma}(x+3x^2) - \frac{2}{\gamma\pi}\int_0^{\pi/2} \phi_\gamma(x\sin^2\theta)\,d\theta$$

Since we seek the solution $\phi_\gamma(x) = \sum_{i=0}^{n} c_i L_i(x)$ in a truncated Laguerre series form, $k = 2$ and we set $n = 2$. This implies

$$\mathbf{C} = [c_0,\, c_1,\, c_2]^T$$
$$\mathbf{Y} = [y_0,\, y_1,\, y_2]^T = [7,\, -13,\, 6]^T$$

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 2 \end{pmatrix}$$
$$\mathbf{S_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2(\theta) & 0 \\ 0 & 0 & \sin^4(\theta) \end{pmatrix}$$

Substituting these into the equation (4.8) with $\mathbf{S}$ replaced with $\mathbf{S_2}$ so that means $\tilde{\mathbf{S}}$ is replaced with $\tilde{\mathbf{S}}_\mathbf{2}$, we get

$$c_0 = \frac{2(56\alpha + 27)}{16\alpha^2 + 14\alpha + 3},$$
$$c_1 = \frac{-2(104\alpha + 51)}{16\alpha^2 + 14\alpha + 3},$$
$$c_2 = \frac{48}{8\alpha + 3}.$$

This in turn gives the regularized solution

$$\begin{aligned} \phi_\gamma(x) &= \frac{2(56\gamma + 27)}{16\gamma^2 + 14\gamma + 3} L_0(x) + \frac{-2(104\gamma + 51)}{16\gamma^2 + 14\gamma + 3} L_1(x) + \frac{48}{8\gamma + 3} L_2(x) \\ &= \frac{2(56\gamma + 27)}{16\gamma^2 + 14\gamma + 3} + \frac{-2(104\gamma + 51)}{16\gamma^2 + 14\gamma + 3}(1 - x) + \frac{48}{8\gamma + 3}(1 - 2x + x^2/2) \\ &= \frac{16\gamma + 6}{16\gamma^2 + 14\gamma + 3} x + \frac{24}{8\gamma + 3} x^2 \end{aligned}$$

As the last step to obtain the desired solution $u(x)$, we take the limit as $\gamma \to 0$:

$$\begin{aligned} \phi(x) = \lim_{\gamma\to 0} \phi_\gamma(x) &= \lim_{\gamma\to 0} \big(\frac{16\gamma + 6}{16\gamma^2 + 14\gamma + 3} x + \frac{24}{8\gamma + 3} x^2\big) \\ &= 2x + 8x^2 \end{aligned}$$

Case 3: The Nonlinear Schlömilch's Integral Equation

We consider *the nonlinear Schlömilch's integral equation* which has the following form:

$$y(x) = \frac{2}{\pi} \int_0^{\pi/2} G(\phi(x \sin\theta))\, d\theta, \quad -\pi \le x \le \pi, \tag{4.12}$$

where $G(\phi(x \sin\theta))$ is a nonlinear function of $\phi(x \sin\theta)$.

We assume that G is invertible so that letting that $G(\phi(x \sin\theta)) = z(x \sin\theta)$ will imply that

$$\phi(x \sin\theta) = G^{-1}(z(x \sin\theta)).$$

Thus, with this transformation, (4.12) becomes

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2} z(x\sin\theta)\,d\theta,$$

which is equivalent to (1.1). We solve this equation for $z(x)$ and then use the inverse transform G^{-1} to get $\phi(x)$.

Example 4.4. *Given the following Fredholm integral equation of the first kind* [11, 15]*:*

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi^2(x\sin\theta)\,d\theta,$$

where $y(x) = 5x^6$.

We first apply the transformation $z = \phi^2$ to convert the equation into

$$y(x) = \frac{2}{\pi}\int_0^{\pi/2} z(x\sin\theta)\,d\theta. \tag{4.13}$$

Employing the regularization method, (4.13) becomes

$$\gamma z_\gamma(x) = 5x^6 - \frac{2}{\pi}\int_0^{\pi/2} z_\gamma(x\sin\theta)\,d\theta$$

or

$$z_\gamma(x) = \frac{1}{\gamma}(5x^6) - \frac{2}{\gamma\pi}\int_0^{\pi/2} z_\gamma(x\sin\theta)\,d\theta$$

Since we seek the solution $z_\gamma(x) = \sum_{i=0}^{n} c_i L_i(x)$ in a truncated Laguerre series form, we set $n = 6$. This implies

$\mathbf{C} = [c_0,\, c_1,\, c_2,\, c_3,\, c_4,\, c_5,\, c_6]^T$

$\mathbf{Y} = [y_0,\, y_1,\, y_2,\, y_3,\, y_4,\, y_5,\, y_6]^T = [3600,\, -21600,\, 54000,\, -72000,\, 54000,\, -21600,\, 3600]^T$

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 & -5 & -6 \\ 0 & 0 & 1/2 & 3/2 & 3 & 5 & 15/2 \\ 0 & 0 & 0 & -1/6 & -2/3 & -5/3 & -10/3 \\ 0 & 0 & 0 & 0 & 1/24 & 5/24 & 5/8 \\ 0 & 0 & 0 & 0 & 0 & -1/120 & -1/20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/720 \end{pmatrix}$$

$$\mathbf{Q}^{-1} = \begin{pmatrix} 1 & 1 & 2 & 6 & 24 & 120 & 720 \\ 0 & -1 & -4 & -18 & -96 & -600 & -4320 \\ 0 & 0 & 2 & 18 & 144 & 1200 & 10800 \\ 0 & 0 & 0 & -6 & -96 & -1200 & -14400 \\ 0 & 0 & 0 & 0 & 24 & 600 & 10800 \\ 0 & 0 & 0 & 0 & 0 & -120 & -4320 \\ 0 & 0 & 0 & 0 & 0 & 0 & 720 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin^2\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin^3\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin^4\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin^5\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin^6\theta \end{pmatrix}$$

Substituting these into the equation (4.5), we get

$$\begin{aligned} c_0 = c_6 &= \frac{57600}{(16\gamma+5)}, \\ c_1 = c_5 &= -\frac{345600}{(16\gamma+5)}, \\ c_2 = c_4 &= \frac{864000}{(16\gamma+5)}, \\ c_3 &= -\frac{1152000}{(16\gamma+5)}. \end{aligned}$$

This in turn gives the regularized solution

$$\begin{aligned} z_\gamma(x) &= \frac{57600}{(16\gamma+5)}L_0(x) - \frac{345600}{(16\gamma+5)}L_1(x) + \frac{864000}{(16\gamma+5)}L_2(x) \\ &\quad - \frac{1152000}{(16\gamma+5)}L_3(x) + \frac{864000}{(16\gamma+5)}L_4(x) - \frac{345600}{(16\gamma+5)}L_5(x) + \frac{57600}{(16\gamma+5)}L_6(x) \\ &= \frac{57600}{720(16\gamma+5)}x^6 \end{aligned}$$

As the last step to obtain the desired solution $\phi(x)$, we take the limit as $\gamma \to 0$:

$$z(x) = \lim_{\gamma\to 0} z_\gamma(x) = \lim_{\gamma\to 0} \frac{57600}{720(16\gamma+5)} x^6 = 16x^6$$

Since $z = \phi^2$, then $\phi(x) = \pm 4x^3$.

5. Conclusion

In this work, we proposed a new method which is based on a combination of the regularization method and Laguerre polynomials. It is an efficient way for deriving exact solutions of Schlömilch's integral equations of different kinds. The proposed algorithm is easy to apply and removes all unnecessary mathematical calculations that appear in application of other methods. In addition, examples which are taken from the literature for comparison are given to show applicability of the method for each type of equation.

5.1. **Acknowledgement.** Some part of this paper has been presented at "The 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022) held in Antalya, Turkey on October 27-30, 2022."

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Ahmet Altürk
Dept. of Mathematics, Amasya University, Ipekköy, Amasya, Turkey
E-mail address: ahmet.alturk@amasya.edu.tr