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A SIMPLE PROOF OF IVÁDY DOUBLE INEQUALITY

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Abstract. In this paper, we give a new proof of Ivády double inequality: for $x > 0$, we have

$$\sqrt{1 - exp\left(-\frac{x^2}{(1+x^2)^{\frac{1}{2}}}\right)} < \tanh x < \sqrt[3]{1 - exp\left(-\frac{x^3}{(1+x^3)^{\frac{1}{2}}}\right)}.$$

1. Introduction

In [2] Problem 51, Ivády proposed that the double inequality

$$\sqrt{1 - exp\left(-\frac{x^2}{(1+x^2)^{\frac{1}{2}}}\right)} < \tanh x < \sqrt[3]{1 - exp\left(-\frac{x^3}{(1+x^3)^{\frac{1}{2}}}\right)} \tag{1.1}$$

holds for $x > 0$. In [3], Ivády's own proof of the left side of the double inequality (1.1) is correct, but the proof of the right side is incorrect. In [5], Zhang and Chen showed the correct proof for the right-hand side of the double inequality (1.1). Recently, Araki et al. [1] proved the following Ivády's type double inequality: for $x > 0$, we have

$$\sqrt{1 - exp\left(-\frac{x^2}{(1+x^2)^{\alpha}}\right)} < \tanh x < \sqrt{1 - exp\left(-\frac{x^2}{(1+x^2)^{\beta}}\right)}, \tag{1.2}$$

where the constants $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{6}$ are the best possible. The double inequality (1.1) is an interesting inequality that evaluates $\tanh x$ from both the left and right sides. The known proofs of the Ivády inequality so far are proofs using inverse function of $\tanh x$. In this paper, we will give a simple and new Ivády double inequality that is differently proven.

Theorem 1.1. *The double inequality (1.1) holds for $x > 0$.*

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2. Two Lemmas to prove Theorem 1.1

Lemma 2.1. *For $x > 0$, we have*

$$\tanh x > \sqrt{1 - exp\left(-\frac{x^2}{(1+x^2)^{\frac{1}{2}}}\right)}.$$

Proof. We consider the equation

$$\tanh x = \sqrt{1 - e^{a(x)}}$$

for $a(x) < 0$, then we have

$$1 - \frac{2}{1+e^{2x}} = \sqrt{1 - e^{a(x)}},$$
$$a(x) = \ln \frac{4e^{2x}}{(1+e^{2x})^2}.$$

It suffices to show that

$$f_1(x) = a(x) + \frac{x^2}{\sqrt{1+x^2}} = \ln \frac{4e^{2x}}{(1+e^{2x})^2} + \frac{x^2}{\sqrt{1+x^2}} < 0$$

for $x > 0$. The derivative of $f_1(x)$ is

$$f_1'(x) = \frac{x\left(2+x^2\right)}{(1+x^2)^{\frac{3}{2}}} - \frac{2\left(e^x - 1\right)\left(1+e^x\right)}{1+e^{2x}}.$$

We set

$$f_2(x) = \left(\frac{x\left(2+x^2\right)}{(1+x^2)^{\frac{3}{2}}}\right)^2 - \left(\frac{2\left(e^x - 1\right)\left(1+e^x\right)}{1+e^{2x}}\right)^2$$

for $x > 0$ and

$$f_3(y) = \frac{2(y-1)(1+y)}{1+y^2}$$

for $y > 1$. Since the derivative of $f_3(y)$ is

$$f_3'(y) = \frac{8y}{(1+y^2)^2} > 0$$

for $y > 1$, $f_3(y)$ is strictly increasing for $y > 1$. By $e^x > 1 + x + \frac{x^2}{2}$ for $x > 0$, we have

$$\begin{aligned} f_2(x) &= \left(\frac{x\left(2+x^2\right)}{(1+x^2)^{\frac{3}{2}}}\right)^2 - \left(f_3(e^x)\right)^2 < \left(\frac{x\left(2+x^2\right)}{(1+x^2)^{\frac{3}{2}}}\right)^2 - \left(f_3\left(1+x+\frac{x^2}{2}\right)\right)^2 \\ &= \left(\frac{x\left(2+x^2\right)}{(1+x^2)^{\frac{3}{2}}}\right)^2 - \left(\frac{2\left(\left(1+x+\frac{x^2}{2}\right)-1\right)\left(1+\left(1+x+\frac{x^2}{2}\right)\right)}{1+\left(1+x+\frac{x^2}{2}\right)^2}\right)^2 \\ &= \frac{-x^4 f_4(x)}{(1+x^2)^3\left(8+8x+8x^2+4x^3+x^4\right)^2} \end{aligned}$$

for $x > 0$, where $f_4(x) = 256 + 576x + 1024x^2 + 1312x^3 + 1284x^4 + 1024x^5 + 632x^6 + 304x^7 + 104x^8 + 24x^9 + 3x^{10} > 0$. Therefore, we obtain $f_2(x) < 0$ and $f_1'(x) < 0$

for $x > 0$. Since $f_1(x)$ is strictly decreasing for $x > 0$ and $\lim_{x\to 0} f_1(x) = 0$, we can get $f_1(x) < 0$ for $x > 0$. □

Lemma 2.2. *For $x > 0$, we have*

$$\tanh x < \sqrt[3]{1 - exp\left(-\frac{x^3}{(1+x^3)^{\frac{1}{2}}}\right)}.$$

Proof. We consider the equation

$$\tanh x = \sqrt[3]{1 - e^{b(x)}}$$

for $b(x) < 0$, then we have

$$1 - \frac{2}{1+e^{2x}} = \sqrt[3]{1-e^{b(x)}}\,,$$
$$b(x) = \ln \frac{2\left(1+3e^{4x}\right)}{\left(1+e^{2x}\right)^3}\,.$$

It suffices to show that

$$g_1(x) = b(x) + \frac{x^3}{\sqrt{1+x^3}} = \ln \frac{2\left(1+3e^{4x}\right)}{\left(1+e^{2x}\right)^3} + \frac{x^3}{\sqrt{1+x^3}} > 0$$

for $x > 0$. The derivative of $g_1(x)$ is

$$g_1'(x) = \frac{3x^2\left(2+x^3\right)}{2\left(1+x^3\right)^{\frac{3}{2}}} - \frac{6e^{2x}\left(1-e^x\right)^2\left(1+e^x\right)^2}{\left(1+e^{2x}\right)\left(1+3e^{4x}\right)}\,.$$

We set

$$g_2(x) = \left(\frac{3x^2\left(2+x^3\right)}{2\left(1+x^3\right)^{\frac{3}{2}}}\right)^2 - \left(\frac{6e^{2x}\left(1-e^x\right)^2\left(1+e^x\right)^2}{\left(1+e^{2x}\right)\left(1+3e^{4x}\right)}\right)^2$$

for $x > 0$ and

$$g_3(y) = \frac{6y^2(1-y)^2(1+y)^2}{\left(1+y^2\right)\left(1+3y^4\right)}$$

for $y > 1$. Since the derivative of $g_3(y)$ is

$$g_3'(y) = \frac{12(1-y)y(1+y)\left(1+y-y^2+3y^3\right)\left(1-y-y^2-3y^2\right)}{\left(1+y^2\right)^2\left(1+3y^4\right)^2} > 0$$

for $y > 1$, $g_3(y)$ is strictly increasing for $y > 1$. From $\lim_{y\to\infty} g_3(y) = 2$, we have

$$0 < \frac{6e^{2x}\left(1-e^x\right)^2\left(1+e^x\right)^2}{\left(1+e^{2x}\right)\left(1+3e^{4x}\right)} < 2$$

for $x > 0$. Therefore, we have

$$g_2(x) = \left(\frac{3x^2\left(2+x^3\right)}{2\left(1+x^3\right)^{\frac{3}{2}}}\right)^2 - \left(g_3(e^x)\right)^2 > \left(\frac{3x^2\left(2+x^3\right)}{2\left(1+x^3\right)^{\frac{3}{2}}}\right)^2 - 2^2$$
$$= \frac{g_4(x)}{4(1+x)^3(1-x+x^2)^3}\,,$$

where

$$\begin{aligned} g_4(x) &= -16 - 48x^3 + 36x^4 - 48x^6 + 36x^7 - 16x^9 + 9x^{10} \\ &\geq -16 - 48x^3 + 36x^3 \cdot 2 - 48x^6 + 36x^6 \cdot 2 - 16x^9 + 9x^9 \cdot 2 \\ &= -16 + 24x^3 + 24x^6 + 2x^9 > -16 + 24 \cdot 2^3 = 176 > 0 \end{aligned}$$

for $x \geq 2$. Thus, we have $g_2(x) > 0$ for $x \geq 2$. Since the inequality $e^x < \frac{2+x}{2-x}$ holds for $0 < x < 2$ (see [4] in pp 269), we have

$$\begin{aligned} g_2(x) &= \left(\frac{3x^2\left(2+x^3\right)}{2\left(1+x^3\right)^{\frac{3}{2}}}\right)^2 - \left(g_3(e^x)\right)^2 > \left(\frac{3x^2\left(2+x^3\right)}{2\left(1+x^3\right)^{\frac{3}{2}}}\right)^2 - \left(g_3\left(\frac{2+x}{2-x}\right)\right)^2 \\ &= \left(\frac{3x^2\left(2+x^3\right)}{2\left(1+x^3\right)^{\frac{3}{2}}}\right)^2 - \left(\frac{6\left(\frac{2+x}{2-x}\right)^2\left(1-\frac{2+x}{2-x}\right)^2\left(1+\frac{2+x}{2-x}\right)^2}{\left(\left(1+\frac{2+x}{2-x}\right)^2\right)\left(1+3\left(\frac{2+x}{2-x}\right)^4\right)}\right)^2 \\ &= \frac{9x^6 g_5(x)}{4(1+x)^3(4+x^2)^2(1-x+x^2)^3(16+16x+24x^2+4x^3+x^4)^2} \end{aligned}$$

for $0 < x < 2$, where $g_5(x) = 49152 + 32768x + 14336x^2 + 45056x^3 + 35840x^4 - 1024x^5 - 5184x^6 + 10368x^7 - 4064x^8 - 8224x^9 + 900x^{10} + 2080x^{11} + 1296x^{12} + 292x^{13} + 72x^{14} + 8x^{15} + x^{16}$. Here, we consider the sign of $g_5(x)$ for $0 < x < 2$. For $0 < x \leq \frac{3}{2}$, we have

$$\begin{aligned} \frac{g_5(x)}{x^3} &> 45056 + 35840x - 1024x^2 - 5184x^3 + 10368x^4 - 4064x^5 - 8224x^6 \\ &\geq 45056 + 35840x - 1024x\left(\frac{3}{2}\right) - 5184x\left(\frac{3}{2}\right)^2 + 10368x^4 \\ &\quad - 4064x^4\left(\frac{3}{2}\right) - 8224x^4\left(\frac{3}{2}\right)^2 \\ &= 45056 + 22640x - 14232x^4 \geq 45056 + 22640x - 14232x\left(\frac{3}{2}\right)^3 \\ &= 45056 - 25393x \geq 45056 - 25393\left(\frac{3}{2}\right) = \frac{13933}{2} > 0\,. \end{aligned}$$

For $\frac{3}{2} < x < 2$, we have

$$\begin{aligned} \frac{g_5(x)}{x^5} &> -1024 - 5184x + 10368x^2 - 4064x^3 - 8224x^4 + 900x^5 + 2080x^6 + 1296x^7 \\ &> -1024 - 5184 \cdot 2 + 10368\left(\frac{3}{2}\right)^2 - 4064x^3 - 8224x^4 + 900x^4\left(\frac{3}{2}\right) \\ &\quad + 2080x^4\left(\frac{3}{2}\right)^2 + 1296x^4\left(\frac{3}{2}\right)^3 \\ &= 11936 - 4064x^3 + 2180x^4 > 11936 - 4064x^3 + 2180x^3\left(\frac{3}{2}\right) \\ &= 11936 - 794x^3 = 11936 - 794 \cdot 2^3 > 5584 > 0\,. \end{aligned}$$

Hence, we obtain $g_2(x) > 0$ for $0 < x < 2$ and $g_1'(x) > 0$ for $x > 0$. Since $g_1(x)$ is strictly increasing for $x > 0$ and $\lim_{x\to 0} g_1(x) = 0$, we can get $g_1(x) > 0$ for $x > 0$. □

Proof of Theorem 1.1. Theorem 1.1 is immediately proven from Lemmas 2.1 and 2.2. □

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