Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 14 Issue 1(2023), Pages 1-15
https://doi.org/10.54379/jiasf-2023-1-1

SOME PROBABILITY THEORY-BASED INEQUALITIES FOR THE INCOMPLETE GAMMA FUNCTION

STEVEN G. FROM

Abstract. In this paper, we present new inequalities and bounds involving the incomplete gamma function and related functions. Many of these new inequalities and bounds are based upon results, some well known and some not as well known, from probability theory and reliability theory. When these results are combined with other mathematical techniques, some very good upper and lower bounds are obtained. In particular, improvements of previously discussed bounds are presented.

1. Introduction

The gamma function is defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\,\mathrm{d}t,\ a > 0. \tag{1.1}$$

The incomplete gamma functions are given by

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t}\,\mathrm{d}t \tag{1.2}$$

and

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\,\mathrm{d}t. \tag{1.3}$$

Clearly, $\Gamma(a) = \gamma(a,x) + \Gamma(a,x)$ for $a > 0$, $x > 0$. Also,

$$\gamma(a,x) = a^{-1} x^a g(x), \tag{1.4}$$

where

$$g(x) = \int_0^1 e^{-tx} h(t)\,\mathrm{d}t \tag{1.5}$$

with

$$h(t) = a t^{a-1},\ 0 < t < 1. \tag{1.6}$$

1991 *Mathematics Subject Classification.* 26D07, 26D15.
Key words and phrases. gamma function; incomplete gamma function; Jensens inequality; moment-generating function; reliability theory.

Submitted August 13, 2022. Published December 16, 2022.
Communicated by Alberto Fiorenza.

Some related integrals of interest are, for $p > 0$,

$$\int_x^\infty e^{-t^p}\,\mathrm{d}t = \frac{1}{p}\Gamma\left(\frac{1}{p}, x^p\right) \tag{1.7}$$

and

$$\int_0^x e^{-t^p}\,\mathrm{d}t = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) - \frac{1}{p}\Gamma\left(\frac{1}{p}, x^p\right). \tag{1.8}$$

The aim of this paper is to present new bounds and inequalities involving the incomplete gamma function $\gamma(a, x)$ given in (1.2). This will also allow us to give new bounds for the related integral in (1.8).

Before presenting the new results, we first discuss some previously proposed bounds and inequalities.

Theorem 1.1 (**Theorem 4.1 of Neuman** [13]).

$$\exp\left(\frac{-ax}{a+1}\right) \le \frac{a}{x^a}\gamma(a,x) = g(x) \le {}_1F_1(a; a+1; -x) \le \frac{1+ae^{-x}}{a+1}, \tag{1.9}$$

where ${}_1F_1$ is Kummer's confluent hypergeometric function.

Theorem 1.2 (**Theorem 1 of Alzer** [1]). *Let $p \neq 1$ be a positive real number. If $0 < p < 1$, let $\alpha = 1$ and $\beta = [\Gamma(1 + \frac{1}{p})]^{-p}$. If $p > 1$, let $\alpha = [\Gamma(1 + \frac{1}{p})]^{-p}$ and $\beta = 1$. Then for $x > 0$,*

$$[1 - e^{-\beta x^p}]^{1/p} < \frac{1}{\Gamma(1+\frac{1}{p})}\int_0^x e^{-t^p}\,\mathrm{d}t < [1 - e^{\alpha x^p}]^{\frac{1}{p}}. \tag{1.10}$$

Theorem 1.3 (**Theorem 2 of Luo et al.** [11]). *Let $a, b, p > 0$. Then*

$$1 - \frac{1 - e^{-\alpha x^p}}{a(p+1)} < \frac{1}{x}\int_0^x e^{-t^p}\,\mathrm{d}t < 1 - \frac{1 - e^{-bx^p}}{b(p+1)} \tag{1.11}$$

holds for all $x > 0$ iff $0 < a \le a_0 = \frac{1}{p+1}$ and $b \ge a_0^* = \frac{p+1}{2p+1}$. In particular,

$$\exp\left(\frac{-x^p}{p+1}\right) < \frac{1}{x}\int_0^x e^{-t^p}\,\mathrm{d}t < \left(\frac{p}{p+1}\right)^2 + \frac{2p+1}{(p+1)^2}\exp\left(-\left(\frac{p+1}{2p+1}\right)x^p\right) \tag{1.12}$$

holds for $x > 0$ with the best constants $a_0 = \frac{1}{p+1}$ and $a_0^* = \frac{p+1}{2p+1}$.

After presenting the new bounds in Section 3, we shall compare them to the bounds given in Theorems 1.1–1.3.

For many other bounds and other inequalities involving the functions in (1.1)–(1.8), see the related works [6, 7, 8, 9, 12, 14, 15, 16, 17] and the references contained in these works.

2. Some Preliminary Results

To derive the new bounds and other inequalities, we shall need various definitions and results from probability theory and reliability theory as well as a few other results. These are presented next. See Barlow and Proschan [2, 3] for information on the reliability theory results used in this paper.

Definition 2.1. *Let Y denote a nonnegative continuous random variable with cumulative distribution function (cdf) $H(y) = P(Y \le y)$, and suppose that Y has a probability density function (pdf) $h(y) = H'(y)$. Assuming that all the integrals given below are convergent, define the following:*

(a) *The survivor function is* $S(y) = P(Y > y) = 1 - H(y)$.
(b) *The mean (expected) value of* Y *is* $E(Y) = \int_0^\infty yh(y)\,dy = \int_0^\infty S(y)\,dy$.
(c) *The second moment of* Y *is*

$$E(Y^2) = \int_0^\infty y^2 h(y)\,dy = \int_0^\infty 2yS(y)\,dy.$$

(d) *The moment-generating function (mgf) of* Y *is*

$$M_Y(s) = E(e^{sY}) = \int_0^\infty e^{sy}h(y)\,dy.$$

(e) *The failure rate function is* $r(y) = \frac{h(y)}{S(y)}$ *if* $S(y) > 0$.
(f) *The mean residual life function of* Y *is the conditional expected value*

$$L(y) = E(Y - y) \mid Y > y) = \frac{\int_y^\infty S(u)\,du}{S(y)} = \frac{\int_y^\infty (u-y)\,dH(u)}{S(y)}$$

if $S(y) > 0$.

The functions in parts (d), (e), and (f) of Definition 2.1 play prominent roles in probability theory, statistics, and reliability theory. We shall see that this is the case when we derive new bounds and inequalities in Section 3. In part (d), if Y is a bounded random variable. (which will be the case later), then $M_Y(s)$ will exist for all real s.

Definition 2.2. *Let* X *be a nonnegative continuous random variable with cdf* $H(x) = P(X \le x)$ *and survivor function* $S(x) = 1 - H(x)$.

(a) H *is a decreasing failure rate (DFR) distribution if* $\frac{S(x+t)}{S(t)}$ *is nondecreasing in* $t \ge 0$ *for every* $x \ge 0$. *If* X *has a pdf* $h(x) = H'(x)$, *then this is equivalent to the function* $r(x) = \frac{h(x)}{S(x)}$ *being nonincreasing in* x.
(b) H *is a decreasing failure rate on average (DFRA) distribution if* $\frac{-Ln(S(x))}{x}$ *is decreasing in* $x \ge 0$ *(equivalently, if* $[S(x)]^{1/x}$ *is increasing in* $x \ge 0$*).*
(c) H *is a new is worse than used (NWU) distribution if* $S(x+y) \ge S(x)S(y)$, $x \ge 0,\ y \ge 0$.

It is well known that if H is DFR, then H is DFRA. Also, if H is DFRA, then H is NWU. (See, for example, Barlow and Proschan [3].) Sometimes we shall say that the random variable X is DFR, DFRA, or NWU, meaning that H, the cdf of X, is a member of that class.

Lemma 2.1. *Suppose that* X *is DFR, and let*

$$\mu_X = E(X) = \int_0^\infty S(x)\,\mathrm{d}x,\ 0 < \mu_X < \infty.$$

Then

$$S(x) \le e^{-x/\mu_X} \text{ if } 0 \le x \le \mu_X \tag{2.1}$$

and

$$S(x) \le \frac{e^{-1}\mu_X}{x} \text{ if } x \ge \mu_X. \tag{2.2}$$

Proof. See either Theorem 6.10 of Barlow and Proschan [3] or Barlow and Proschan [2], pp. 31–32. ∎

Definition 2.3. *Let I be an indexing set, and suppose that for every $t \in I$, F_t is a cdf. Let $\{F_t : t \in I\}$ be the set of all these cdfs, and let P be a probability measure on I. Then $F = \int_I F_t \, dP(t)$ is called the mixture of the distributions in $\{F_t : t \in I\}$ with respect to the mixing distribution P. Written in terms of survivor functions, it is easily seen that $S = \int_I S_t \, dP(t)$, where $S_t = 1 - F_t$, $t \in I$, and $S = 1 - F$.*

In Section 3, we shall make use of Definition 2.3 in the following way: Let $G(t)$ be the cdf corresponding to the probability measure P, and let T denote an random variable with cdf $G(t)$. Now given that $T = t$, let X denote an random variable having cdf $F_t(x)$. Then by the theorem of total probability and assuming that $P(X \geq 0) = 1$ and $I \subseteq [0, \infty)$, X has cdf

$$F(x) = P(X \leq x) = \int_0^\infty F_t(x) \, \mathrm{d}G(t)$$

and X has survivor function

$$S(x) = P(X > x) = \int_0^\infty S_t(x) \, \mathrm{d}G(t),$$

where $S_t = 1 - F_t$.

Lemma 2.2. *Let X be a nonnegative continuous random variable having survivor function $S(x)$ of the form given by*

$$S(x) = \int_I S_t(x) \, \mathrm{d}G(t). \tag{2.3}$$

Then the following hold:

(a) *If $S_t(x)$ is DFR for all $t \in I$, then $S(x)$ is DFR.*
(b) *If $S_t(x)$ is DFRA for all $t \in I$, then $S(x)$ is DFRA.*

Proof. See Theorem 4.7 on p. 103 of Barlow and Proschan [3]. □

Lemma 2.3. *Let T be a nonnegative random variable with mean*

$$\mu_T = E(T) = \int_0^\infty S(t) \, \mathrm{d}t$$

and second moment

$$\mu_T^{(2)} = E(T^2) = \int_0^\infty 2tS(t) \, \mathrm{d}t < \infty.$$

Then the following hold:

(a) *Suppose T is a bounded random variable, that is, there is a positive real number M with $P(0 \leq T \leq M) = 1$. Then for all real s, the moment-generating function of T has upper bound*

$$M_T(s) = E(e^{sT}) \leq 1 - \frac{\mu_T}{M} + \frac{\mu_T}{M} \exp(sM). \tag{2.4}$$

(b) *Let $\alpha = \mu_T$ and $\beta = \mu_T^{(2)}$, and suppose that $s \leq 0$. Then*

$$M_T(s) = E(e^{sT}) \leq 1 - \frac{\alpha^2}{\beta} + \frac{\alpha^2}{\beta} \exp\left(\frac{\beta s}{\alpha}\right). \tag{2.5}$$

Proof. See Brook [4], pp. 171–173. □

Definition 2.4. *Let A be an interval subset of the real numbers, and let $\mathcal{D} = \{P_a(x)\colon a \in A\}$ be a family of pdfs in the real variable x. Then $\mathcal{D}$ is a monotone likelihood ratio family of pdfs if whenever $a_1, a_2 \in A$ with $a_1 < a_2$, $\frac{P_{a_2}(x)}{P_{a_1}(x)}$ is a nondecreasing function of $w(x)$ for some real-valued function $w(x)$ of x.*

Lemma 2.4. *Suppose $\mathcal{D} = \{h_a(x)\colon a \in A\}$ is a family of pdfs with monotone likelihood ratio, with $w(x) = x$, let $\psi(x)$ be any nondecreasing function of x, let X denote an random variable with pdf $h_a(x)$, and let $E_a[\psi(X)] = \int_A \psi(x)\,\mathrm{d}H(x)$. Then $E_a[\psi(X)]$ is a nondecreasing function of $a \in A$.*

Proof. See Lehmann and Romano [10], pp. 70–71. □

Lemma 2.5. *Let T be a continuous random variable with support in a compact interval $[a, b]$, and suppose that T has cdf $H(t)$ satisfying $H(a) = 0$, $H(b) = 1$, and $0 < H(t) < 1$, $a < t < b$. In addition, let $f(t)$ have second derivative $f''(t)$ continuous on $[a, b]$, and let*

$$q_1(t) = \inf\{f''(u)\colon t \le u \le t + L(t)\}, \tag{2.6}$$

$$q_2(t) = \sup\{f''(u)\colon t \le u \le t + L(t)\}, \tag{2.7}$$

$$L_1 = \frac{1}{2}\int_a^b q_1(t)(L(t))^2\,\mathrm{d}H(t), \tag{2.8}$$

and

$$U_1 = \frac{1}{2}\int_a^b q_2(t)(L(t))^2\,\mathrm{d}H(t). \tag{2.9}$$

Then

$$L_1 \le \int_a^b f(t)\,\mathrm{d}H(t) - f\left(\int_a^b t\,\mathrm{d}H(t)\right) \le U_1. \tag{2.10}$$

Proof. See From [5], pp. 14–19. □

Note that (2.10) is a generalization of Jensen's inequality to possibly nonconvex functions $f(t)$ but also improves on Jensen's inequality for nonlinear convex functions.

Lemma 2.6. *Let $a > 0$, and let T be a continuous random variable with pdf $h_a(t) = at^{a-1}$ for $0 < t < 1$, and $h_a(t) = 0$ otherwise. Then the following hold:*

(a)
$$E(T) = \frac{a}{a+1} \text{ and } E(T^2) = \frac{a}{a+2}. \tag{2.11}$$

(b) *T has mean residual life function*
$$L(t) = \begin{cases} \frac{a}{a+1}\left(\frac{1-t^{a+1}}{1-t^a}\right) - t, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases} \tag{2.12}$$

(c) *Let $L_a(t) = L(t)$, emphasizing the dependence of $L(t)$ on a. Then $L_a(t)$ is increasing in a for $a > 0$, $0 \le t < 1$.*

(d) *If $0 < a \le 1$, then*
$$0 \le \frac{1-t}{-\mathrm{Ln}(t)} - t \le L_a(t) \le \frac{1-t}{2},\ 0 < t < 1. \tag{2.13}$$

(e) *If $a \geq 1$, then*

$$\frac{1-t}{2} \leq L(t) \leq \frac{a}{a+1}(1+t^a) - t \leq \frac{a-t}{a+1}, \ 0 \leq t < 1. \tag{2.14}$$

Proof. The proofs of (a) and (b) are straightforward and omitted.

To prove (c), apply Lemma 2.4 with $A = (0, \infty)$, fix t with $0 \leq t < 1$, and define a family $\mathcal{D} = \{P_a(x) \colon a \in A\}$ of probability density functions with $A = (0, \infty)$ by

$$P_a(x) = \begin{cases} \frac{h_a(x)}{1-t^a}, & t \leq x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

In addition, let X be an random variable with pdf $P_a(x)$, given t, and suppose that $a_1, a_2 \in A$ with $a_1 < a_2$. Then

$$\frac{P_{a_2}(x)}{P_{a_1}(x)} = \frac{h_{a_2}(x)}{h_{a_1}(x)} = \frac{a_2}{a_1} x^{a_2 - a_1}$$

is nondecreasing in $w(x) = x$. Thus $\mathcal{D}$ is a monotone likelihood ratio family of pdfs in x. Let $\psi(x) = x$. Clearly, $\psi(x)$ is nondecreasing in x. By Lemma 2.4, $E_a[\psi(X)]$ is nondecreasing in $a > 0$ but $E_a[\psi(X)] = L_a(t)$. Thus $L_a(t)$ is nondecreasing in a for $a > 0$.

To prove (d), let $m(a) = E_a[\psi(X)] = L_a(t)$, emphasizing the dependence of $L(t)$ on a. By part (c),

$$L(0^+) \leq L_a(t) \leq m(1) = \frac{1-t}{2}.$$

By l'Hôpital's Rule and part (b),

$$L(0^+) = \lim_{a \to 0^+} L_a(t) = \frac{1-t}{-\mathrm{Ln}(t) - t}$$

The proof of part (e) is similar to that of part (d) and follows from parts (b) and (c) of the Lemma, hence it is omitted. $\square$

3. New Bounds and Inequalities

In this section, we present new bounds for various functions and integrals discussed in Section 1.

First, let's consider bounds for $g(x) = \frac{a}{x^a}\gamma(a, x)$. Before presenting new bounds, we show that the upper bound $g(x) \leq \frac{1+ae^{-x}}{a+1}$ of Neuman [13] in Theorem 1.1 can be derived immediately from Lemma 2.3(a). Since

$$g(x) = \int_0^1 e^{-xt} a t^{a-1} \, \mathrm{d}t, \ a > 0, \tag{3.1}$$

letting T be an random variable with pdf $h(t) = at^{a-1}, \ \ 0 < t < 1$, we obtain $\mu_T = E(T) = \frac{a}{a+1}$. With $M = 1$ in Lemma 2.3(a) and letting $s = -x$, we obtain

$$g(x) = M_T(-x) \leq 1 - \frac{\mu_T}{M} + \frac{\mu_T}{M}\exp(-xM) = \frac{1+ae^{-x}}{a+1}.$$

The lower bound of Neuman [13] in Theorem 1.1 also follows immediately from Jensen's inequality using $\mu_T = \frac{a}{a+1}$ and the fact that for any fixed $x \geq 0, \ f(x) = e^{-tx}$ is a convex function of t for t with $0 \leq t \leq 1$.

Now let's apply Lemma 2.3(b) to obtain a new upper bound for $g(x)$ which improves on the upper bound of Neuman [13].

Theorem 3.1. *For all $a > 0$ and $x \geq 0$, $g(x)$ satisfies the upper bound inequality*

$$g(x) \leq \frac{1}{(a+1)^2}\left[1 + a(a+2)e^{-\left(\frac{a+1}{a+2}\right)x}\right] \leq \frac{1+ae^{-x}}{a+1} \tag{3.2}$$

Equality holds in the last half iff $x = 0$.

Proof. Apply Lemmas 2.3(b) and 2.6(a) with $\alpha = \mu_T = \frac{a}{a+1}$, $\beta = \mu_T^{(2)} = \frac{a}{a+2}$, and $s = -x$, to get

$$M_T(-x) = g(x) \leq 1 - \frac{\alpha^2}{\beta} + \frac{\alpha^2}{\beta}\exp\left(-\frac{\beta}{\alpha}x\right) = \frac{1}{(a+1)^2}\left[1 + a(a+2)e^{-\left(\frac{a+1}{a+2}\right)x}\right].$$

This proves the first half of (3.2). To prove the second half, we use the fact that

$$\frac{1 + a(a+2)e^{-\left(\frac{a+1}{a+2}\right)x}}{(a+1)^2} \leq \frac{1+ae^{-x}}{a+1} \text{ iff } R(x) \leq 0,$$

where

$$R(x) = \frac{1}{(a+1)^2}\left(1 + a(a+2)e^{-\left(\frac{a+1}{a+2}\right)x}\right) - \left(\frac{1+ae^{-x}}{a+1}\right).$$

Then differentiating with respect to x, some algebra gives

$$R'(x) \leq 0 \text{ iff } \frac{a}{a+1}\left(e^{-x} - e^{-\left(\frac{a+1}{a+2}\right)x}\right) \leq 0,$$

which is obviously true, with strict inequality unless $x = 0$. Since $R(0) = 0$, the Mean Value Theorem gives $R(x) \leq 0$, with $R(x) < 0$ unless $x = 0$. □

Later, we shall present several more improvements of Theorem 1.1 bounds.

Theorem 3.2. *Suppose $a > 1$. Then for $x \geq 0$,*

$$g(x) \leq \exp\left(-\left(\frac{a-1}{a}\right)x\right),\ 0 \leq x \leq \frac{a}{a-1} \tag{3.3}$$

and

$$g(x) \leq \frac{\left(\frac{a}{a-1}\right)e^{-1}}{x},\ x > \frac{a}{a-1}. \tag{3.4}$$

Proof.

$$g(x) = \int_0^1 e^{-xt} \cdot at^{a-1}\,\mathrm{d}t = \int_0^1 S_t(x)\,\mathrm{d}G(t),$$

where $S_t(x) = e^{-xt}$ and

$$G(t) = \begin{cases} 0, & t \leq 0 \\ t^a, & 0 < t < 1 \\ 1, & t \geq 1. \end{cases}$$

Clearly, $S_t(x)$ is DFR for all $t \in I = [0,1]$ and all $x \geq 0$.

By Lemma 2.2(a), $g(x)$ is a DFR distribution in $x \geq 0$. Thus $g(x)$ is a DFR survivor function that corresponds to a nonnegative random variable X. Letting $S(x) = g(x)$ in Lemma 2.1, and noting that $\mu_X = E(X) < \infty$ iff $a > 1$, we get

$$S(x) \leq e^{-x/\mu_X},\ 0 \leq x \leq \mu_X \tag{3.5}$$

and

$$S(x) \leq \frac{e^{-1}\mu_X}{x},\ x > \mu_X. \tag{3.6}$$

Upon applying Fubini's Theorem, we get that for $a > 1$,

$$\mu_X = \int_0^\infty S(x)\,\mathrm{d}x = \int_0^\infty \int_0^1 e^{-xt} a t^{a-1}\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^1 a t^{a-1} \int_0^\infty e^{-tx}\,\mathrm{d}x\,\mathrm{d}t \; = \frac{a}{a-1}.$$

Substituting $\frac{a}{a-1}$ for μ_X in (3.5) and (3.6), we obtain the Theorem. $\square$

Theorem 3.3. *For all $a > 0$ and $x \geq 0$,*

$$g(x) \leq e^{-x}\left[1 + \frac{a}{(a+1)(a+2)}x + \frac{2}{(a+1)(a+2)}(e^x - 1)\right], \tag{3.7}$$

which is strictly less than the Theorem 1.1 upper bound of Neuman [13] *for $x > 0$.*

Proof. Let $w(x) = e^x g(x)$. Then a simple computation gives

$$\begin{aligned} w'(x) &= e^x \int_0^1 e^{-tx}[a t^{a-1}(1-t)]\,\mathrm{d}t \\ &= \frac{e^x}{a+1}\int_0^1 e^{-tx}[a(a+1)t^{a-1}(1-t)]\,\mathrm{d}t. \end{aligned} \tag{3.8}$$

Now the expression in brackets in the integrand in (3.8) is a pdf. Applying Lemma 2.3(a) and proceeding as in the proof of Theorem 3.1, we obtain (using $w(0) = 1$)

$$w'(x) \leq \frac{e^x}{a+1}\left(1 - \frac{a}{a+2} + \frac{a}{a+2}e^{-x}\right)$$

Thus

$$\begin{aligned} w(x) &= w(0) + \int_0^x w'(t)\,\mathrm{d}t \\ &\leq \left[1 + \frac{a}{(a+1)(a+2)}x + \frac{2}{(a+1)(a+2)}(e^x - 1)\right] \end{aligned}$$

Multiplication by e^{-x} produces (3.7), and this proves the first half of the Theorem.

To show the improvement of the new upper bound over the Theorem 1.1 upper bound, note that the improvement occurs iff

$$\frac{2}{a+2} + \left(\frac{a(a+1)}{a+2} + \frac{a}{a+2}\right)e^{-x} < 1 + ae^{-x}, \tag{3.9}$$

which holds iff $(x-1)e^{-x} < 1$ for $x > 0$. Since $(x-1)e^{-x} \leq e^{-2} < 1$ for $x > 0$, (3.9) holds, and this proves the second half of the Theorem. $\square$

Next, we show one of several ways to improve on the lower bound of Theorem 1.1 given in Neuman [13].

Theorem 3.4. *For all $a > 0$ and $x \geq 0$,*

$$g(x) \geq \frac{a}{2a+2}e^{-x} + \frac{a+2}{2a+2}e^{-\left(\frac{a}{a+2}\right)x}, \tag{3.10}$$

which is strictly greater than the lower bound $e^{-\left(\frac{a}{a+1}\right)x}$ in Theorem 1.1 for $x > 0$.

Proof. Proceeding as in the proof of Theorem 3.3,

$$w'(x) = \frac{e^x}{a+1} \int_0^1 e^{-tx}[a(a+1)t^{a+1}(1-t)]\,\mathrm{d}t$$

Applying Jensen's inequality to the integral, we obtain $w'(x) \geq \frac{1}{a+1} e^{\left(\frac{2}{a+2}\right)x}$. Thus

$$\begin{aligned} g(x) &\geq \left(1 + \int_0^x \frac{1}{a+1} e^{\left(\frac{2}{a+2}\right)t}\,\mathrm{d}t\right) e^{-x} \\ &= \frac{a}{2a+2} e^{-x} + \frac{a+2}{2a+2} e^{-\left(\frac{a}{a+2}\right)x}. \end{aligned}$$

This completes the proof of the first part of the Theorem.

To show the improvement of the new lower bound over the Theorem 1.1 lower bound, note that

$$\frac{a}{2a+2} e^{-x} + \frac{a+2}{2a+2} e^{-\left(\frac{a}{a+2}\right)x}$$

is a weighted arithmetic mean of e^{-x} and $e^{-\left(\frac{a}{a+2}\right)x}$, which (for $x > 0$) is strictly greater than the corresponding weighted geometric mean,

$$\left(e^{-x}\right)^{\frac{a}{2a+2}} \left(e^{-\left(\frac{a}{a+2}\right)}x\right)^{\frac{a+2}{2a+2}} = e^{-\left(\frac{a}{a+1}\right)x},$$

which is the lower bound of Theorem 1.1. □

Theorem 3.5. *For all $a > 0$ and $x \geq 0$,*

$$g(x) \leq e^{-x}\left[1 + \frac{1}{(a+1)^2(a+2)}\left(2e^x - 2 + \frac{1}{2}a(a+3)^2\left(\exp\left(\frac{2x}{a+3}\right) - 1\right)\right)\right]. \tag{3.11}$$

Proof. Proceeding as in the proofs of Theorems 3.3 and 3.4, let $w(x) = e^x g(x)$. Then applying Lemma 2.3(b) and Lemma 2.6(a), we obtain

$$w'(x) = \frac{e^x}{a+1} \int_0^1 e^{-tx}[a(a+1)t^{a+1}(1-t)]\,\mathrm{d}t$$

with $\frac{\alpha^2}{\beta} = \frac{a(a+3)}{(a+1)(a+2)}$ and $\frac{\beta}{\alpha} = \frac{a}{a+3}$, hence

$$\begin{aligned} w'(x) &\leq \frac{e^x}{a+1}\left[1 - \frac{a(a+3)}{(a+1)(a+2)} + \frac{a(a+3)}{(a+1)(a+2)} \exp\left(-\left(\frac{a+1}{a+3}\right)x\right)\right] \\ &= \frac{1}{(a+1)^2(a+2)}\left[2e^x + a(a+3)\exp\left(\left(\frac{2}{a+3}\right)x\right)\right]. \end{aligned}$$

Thus

$$\begin{aligned} w(x) &= 1 + \int_0^x w'(t)\,\mathrm{d}t \\ &\leq \left[1 + \frac{1}{(a+1)^2(a+2)}\left(2e^x - 2 + \frac{1}{2}a(a+3)^2\left(\exp\left(\frac{2x}{a+3}\right) - 1\right)\right)\right]. \end{aligned}$$

Multiplying by e^{-x}, we obtain (3.11). □

Theorem 3.6. *For all $a \ge 1$ and $x \ge 0$,*

$$g(x) \ge e^{-\left(\frac{a}{a+1}\right)x} + \frac{x^2 \exp\left(-\left(\frac{a(2a+3)}{(a+1)(a+3)}x\right)\right)}{4(a+1)(a+2)}, \tag{3.12}$$

which is clearly greater than the Theorem 1.1 lower bound of $e^{-\left(\frac{a}{a+1}\right)x}$ for $x > 0$.

Proof. By Lemma 2.6(e),

$$\frac{1-t}{2} \le L(t) \le \frac{a-t}{a+1},\ 0 \le t \le 1. \tag{3.13}$$

Let T be a random variable with pdf $h(t) = h_a(t) = at^{a-1},\ 0 < t < 1$. Applying Lemma 2.5 with $f(t) = e^{-tx}$, we get

$$q_1(t) = x^2 e^{-(t+L(t))x} \ge x^2 e^{-\left(\frac{a}{a+1}\right)(1+t)x}$$

by (3.13). Thus we obtain

$$\begin{aligned} g(x) &\ge e^{-\left(\frac{a}{a+1}\right)x} + \frac{1}{2}x^2 \int_0^1 e^{-\left(\frac{a}{a+1}\right)(1+t)x} \left(\frac{1-t}{2}\right)^2 at^{a-1}\,\mathrm{d}t \\ &= e^{-\left(\frac{a}{a-1}\right)x} \\ &\quad + \frac{x^2 \exp\left(-\left(\frac{a}{a+1}\right)x\right)}{4(a+1)(a+2)} \int_0^1 e^{-\left(\frac{a}{a+1}\right)xt} \left[at^{a-1}(1-t)^2 \frac{(a+1)(a+2)}{2}\right] \mathrm{d}t. \end{aligned} \tag{3.14}$$

The expression in brackets in the integrand in (3.14) is a pdf with support in $(0,1)$. Since $e^{-\left(\frac{a}{a+1}\right)xt}$ is convex in t for all $x \ge 0$, applying Jensen's inequality to this integrand and using

$$\int_0^1 \left[at^a(1-t)^2 \frac{(a+1)(a+2)}{2}\right] \mathrm{d}t = \frac{a}{a+3},$$

we obtain the following after simplification:

$$g(x) \ge e^{-\left(\frac{a}{a+1}\right)x} + \frac{x^2 \exp\left(-\frac{a(2a+3)}{(a+1)(a+3)}x\right)}{4(a+1)(a+2)}.$$

$\square$

Theorem 3.7. *For all a with $0 < a \le 1$ and all $x \ge 0$,*

$$\begin{aligned} g(x) \le e^{-\left(\frac{a}{a+1}\right)x} + \frac{x^2}{4(a+1)(a+2)} \Bigg[1 - \frac{a(a+4)}{(a+1)(a+3)} \Bigg(1 \\ - \exp\left(-\left(\frac{a+1}{a+4}\right)x\right)\Bigg)\Bigg]. \end{aligned} \tag{3.15}$$

Proof. We will proceed as in the proof of Theorem 3.6. Lemma 2.6(d) gives

$$0 \le L(t) \le \frac{1-t}{2}. \tag{3.16}$$

Then applying Lemma 2.5 with $f(t) = e^{-tx}$, we get $q_2(t) = x^2 e^{-tx}$. Thus we obtain

$$\begin{aligned} g(x) &\le e^{-\left(\frac{a}{a+1}\right)x} + \frac{x^2}{2}\int_0^1 e^{-tx}\left(\frac{1-t}{2}\right)^2 at^{a-1}\,\mathrm{d}t \\ &= e^{-\left(\frac{a}{a+1}\right)x} + \frac{x^2}{4(a+1)(a+2)}\int_0^1 e^{-tx}\left[\frac{(a+1)(a+2)}{2}(1-t)^2 at^{a-1}\right]\mathrm{d}t. \end{aligned} \tag{3.17}$$

The expression in brackets in the integrand in (3.17) is a pdf with mean $\alpha = \frac{a}{a+3}$ and second moment $\beta = \frac{a(a+1)}{(a+3)(a+4)}$.

Applying Lemma 2.3(b) with the random variable T having the same pdf, we obtain

$$\begin{aligned} g(x) &\le e^{-\left(\frac{a}{a+1}\right)x} + \frac{x^2}{4(a+1)(a+2)}\left[1 - \frac{a(a+4)}{(a+1)(a+3)}\right. \\ &\qquad \left. + \frac{a(a+4)}{(a+1)(a+3)}\exp\left(-\left(\frac{a+1}{a+4}\right)x\right)\right] \\ &= e^{-\left(\frac{a}{a+1}\right)x} + \frac{x^2}{4(a+1)(a+2)}\left[1 - \frac{a(a+4)}{(a+1)(a+3)}\left(1\right.\right. \\ &\qquad \left.\left. - \exp\left(-\left(\frac{a+1}{a+4}\right)x\right)\right)\right]. \end{aligned}$$

□

Next, we give new bounds for the function $\frac{1}{x}\int_0^x e^{-t^p}\,\mathrm{d}t$ for $p > 0$ and $x > 0$. We shall see that we can recover the upper bounds of Luo et al. [11] given in Theorem 1.3 and even improve on them.

Theorem 3.8. *Let $p > 0$, and let $a = \frac{1}{p}$. Then for $x > 0$,*

$$\begin{aligned} \frac{1}{x}\left[\frac{a}{2a+2}e^{-x^p} + \frac{a+2}{2a+2}e^{-\left(\frac{a}{a+2}\right)x^p}\right] &\le \frac{1}{x}\int_0^x e^{-t^p}\,\mathrm{d}t \\ &\le e^{-x^p}\left[1 + \frac{1}{(a+1)^2(a+2)}\left(2e^{x^p} - 2\right.\right. \\ &\qquad \left.\left. + \frac{1}{2}a(a+3)^2\left(\exp\left(\left(\frac{2}{a+3}\right)x^p\right) - 1\right)\right)\right]. \end{aligned} \tag{3.18}$$

Proof. Note that $\frac{1}{x}\int_0^x e^{-t^p}\,\mathrm{d}t = \frac{1}{x}g(x^p)$, since a change of variables gives

$$\begin{aligned} \frac{1}{x}\int_0^x e^{-t^p}\,\mathrm{d}t &= \int_0^{x^p} e^{-w}\left[\frac{1}{x}\left(\frac{1}{p}\right)w^{\frac{1}{p}-1}\right]\mathrm{d}w = \frac{1}{x}g(x^p) \\ &= \int_0^{x^p} e^{-w}h_a(w)\,\mathrm{d}w, \end{aligned} \tag{3.19}$$

where $a = \frac{1}{p}$. Now apply Theorem 3.5 to get an upper bound on $g(x^p)$. This completes the proof of the upper bound in (3.18).

The proof of the lower bound in (3.18) follows immediately from Theorem 3.4 after applying (3.19). □

Remark 3.1. *From* (3.19)*, it can be shown that if we apply Lemma 2.3(a) to the integral relation*

$$\frac{1}{x}\int_0^x e^{-t^p}\,dt = \int_0^{x^p} e^{-w}\left[\frac{1}{x}\left(\frac{1}{p}\right)w^{\frac{1}{p}-1}\right]dw,$$

this results in $\frac{1}{x}\int_0^x e^{-t^p}\,dt < 1-\frac{1-e^{-x^p}}{(p+1)}$*, which is the upper bound of Luo et al.* [11] *given in* (3.19) *for* $b=1$*. If we apply Lemma 2.3(b) instead, we obtain*

$$\frac{1}{x}\int_0^x e^{-t^p} < 1-\frac{(1+2p)}{(p+1)^2}+\frac{1+2p}{(p+1)^2}\exp\left(-\left(\frac{p+1}{2p+1}\right)x^p\right),$$

which is the upper bound of Luo et al. [11] *given in* (1.12).

Theorem 3.9. *For all* $a>0$ *and* $x\geq 0$,

$$g(x)\geq \frac{1}{2}e^{-x}\left[(1+x-a)+\sqrt{(1+x-a)^2+4a}\right]. \tag{3.20}$$

Proof. Since $g(x)$ is DFR, the failure rate function is $r(x)=-\frac{g'(x)}{g(x)}$, which is decreasing in $x\geq 0$. Thus $r(x)$ is log convex, with $g(x)g''(x)\geq (g'(x))^2$. By Leibniz's Rule for differentiating an integral and integration by parts,

$$-g'(x)=\int_0^1 e^{-tx}at^a\,\mathrm{d}t=-\frac{a}{x}e^{-x}+\frac{a}{x}g(x) \tag{3.21}$$

and

$$g''(x)=-\frac{a}{x}e^{-x}-\frac{a}{x^2}e^{-x}-\frac{a^2}{x^2}e^{-x}+\left(\frac{a^2+a}{x^2}\right)g(x) \tag{3.22}$$

By (3.21) and (3.22), we obtain the following after simplification:

$$\begin{aligned} &g(x)g''(x)-(g'(x))^2\geq 0\\ \text{iff }&(g(x))^2+(a-x-1)e^{-x}g(x)-ae^{-2x}\geq 0.\end{aligned} \tag{3.23}$$

Solving the quadratic (3.23) for $g(x)$ and throwing out the negative root, we obtain the following after simplification:

$$g(x)\geq \frac{1}{2}e^{-x}\left[(1+x-a)+\sqrt{(1+x-a)^2+4a}\right].$$

□

Among the bounds presented thus far, extensive comparisons show that Theorems 3.4, 3.5, and 3.8 provide the best lower and upper bounds overall and are easily computed. All the previously discussed bounds, both old and new, can be improved upon, and this will be discussed later. We considered several different methods to obtain bounds, since each method can be applied to special functions besides the ones considered in this work. In most cases, no one bound is uniformly superior to all other bounds (except for a few bounds discussed earlier), so for this reason a good number of bounds was presented. In any case, several new bounds—utilizing several different methods—that improve on previously discussed bounds given in the literature have been presented.

Next, we present some interesting new inequalities by applying various reliability theory concepts.

Theorem 3.10. *For all* $a>0$, $g(x)=\int_0^1 e^{-tx}at^{a-1}\,\mathrm{d}t$ *satisfies the following inequalities:*

(a)

$$(g(y))^{1/y} \geq (g(x))^{1/x},\ 0 \leq x \leq y < \infty \tag{3.24}$$

(b)

$$g(x+y) \geq g(x)g(y),\ x \geq 0,\ y \geq 0.$$

Proof. By Lemma 2.2(a), $g(x)$ is DFR for $x \geq 0$. By part (b) of that lemma, $g(x)$ is DFRA. By Definition 2.2(b), $(g(x))^{1/x}$ is increasing in x. This proves part (a).

Part (b) of the Theorem follows immediately from Definition 2.2(c) and the fact that DFRA distributions are NWU. ☐

Theorem 3.11. *Let $g(x) = g_a(x) = \int_0^1 e^{-tx} a t^{a-1}\,\mathrm{d}t$ to emphasize the dependence of g on $a > 0$. Suppose m is a positive integer such that $m \leq a \leq m+1$. Then*

$$\frac{(m+1)!}{x^{m+1}}\left[1-\sum_{j=0}^{m}\frac{x^j e^{-x}}{j!}\right] \leq g_a(x) \leq \frac{m!}{x^m}\left[1-\sum_{j=0}^{m-1}\frac{x^j e^{-x}}{j!}\right]. \tag{3.25}$$

Proof. Let $p_a(t) = at^{a-1},\quad a \geq 1$. Then $\mathcal{D} = \{p_a(t)\colon a \geq 1\}$ is a family of pdfs with monotone likelihood ratio in t, since $1 \leq a_1 \leq a_2 < \infty$ implies that $\frac{p_{a_2}(t)}{p_{a_1}(t)} = \frac{a_2}{a_1}t^{a_2-a_1}$ is increasing in t.

Since $\psi(t) = e^{-tx}$ is decreasing in t for $x \geq 0$, Lemma 2.4 gives that $g_a(x)$ is decreasing in a for all $x \geq 0$. Thus $g_{m+1}(x) \leq g_a(x) \leq g_m(x)$ for all $x \geq 0$. Also, $g_m(x) = \frac{m}{x^m}(m-1)!\left[1-\sum_{j=0}^{m-1}\frac{x^j e^{-x}}{j!}\right]$, by repeated integration by parts. Substituting $m+1$ for m completes the proof. ☐

The next theorem does not give bounds on $g_a(x)$, but it is an interesting inequality of a new type.

Theorem 3.12. *Let $g_a(x) = \int_0^1 e^{-tx} a t^{a-t}\,\mathrm{d}t,\ a > 0,\ x \geq 0$. Then for x_1, x_2 with $0 \leq x_1 \leq x_2 < \infty$,*

$$g_{a+1}(x_1)g_a(x_2) \geq g_a(x_1)g_{a+1}(x_2). \tag{3.26}$$

Proof. In the proof of Theorem 3.2, it was established that $g_a(x)$ is DFR in $x \geq 0$ for all $a > 0$. By Definition 2.1, $r(x) = \frac{-\frac{\mathrm{d}}{\mathrm{d}x}(g_a(x))}{g_a(x)}$ is decreasing in $x \geq 0$ for $a > 0$. However, $-\frac{\mathrm{d}}{\mathrm{d}x}(g_a(x)) = \left(\frac{a}{a+1}\right)g_{a+1}(x)$. Thus $r(x) = \frac{\left(\frac{a}{a+1}\right)g_{a+1}(x)}{g_a(x)}$ is decreasing in x, from which (3.26) follows. ☐

Many of the bounds in this paper can be improved by using repeated integration by parts or by differentiation of the function for which bounds or inequalities are desired with respect to x or a. We give one such example showing how to greatly improve on the upper bound on $g(x) = g_a(x) = \int_0^1 e^{-tx} a t^{a-1}\,\mathrm{d}t$. Integration by parts of this integral produces the recurrence relation

$$g_a(x) = e^{-x} + \frac{x}{a+1}g_{a+1}(x). \tag{3.27}$$

Iteration of (3.27) and a simple induction argument using repeated integration by parts gives

$$
\begin{aligned}
g_a(x) = e^{-x}&\left[1+\sum_{j=1}^{k}\left(\prod_{i=1}^{j}\frac{1}{a+i}\right)x^j\right] \\
&\qquad + \left(\frac{x^{k+1}}{\prod_{i=1}^{k+1}(a+i)}\right)g_{a+k+1}(x),\ k=1,2,3,\ldots
\end{aligned}
\tag{3.28}
$$

Clearly, any upper bound on $g_{a+k+1}(x)$ gives an upper bound on $g_a(x)$. This method usually improves on an upper bound on $g_a(x)$ if the same formula for an upper bound is used. A similar statement holds for lower bounds. In addition, previous bounds which required either the condition $0 < a < 1$ or the condition $a \geq 1$ will extend to the other condition to give bounds valid for all $a > 0$. In particular, this is the case if (3.28) is applied to Theorem 3.2 bounds (3.3) and (3.4). For $k = 1$, we obtain the following for all $a > 0$:

$$g(x) = g_a(x) \leq e^{-x} + \frac{x}{a+1}\exp\left(-\left(\frac{a}{a+1}\right)x\right),\ 0 \leq x \leq \frac{a+1}{a}$$

and

$$g(x) = g_a(x) \leq e^{-x} + \frac{x}{a+1}\left(\frac{\left(\frac{a+1}{a}\right)e^{-1}}{x}\right) = e^{-x} + \frac{e^{-1}}{a},\ x > \frac{a+1}{a}.$$

For x with $0 \leq x \leq \frac{a+1}{a}$, this bound is inferior to the upper (Theorem 1.1) bound of Neuman [13], but as $x \to \infty$ it becomes significantly better. Even greater improvement of bounds results if the best bounds (overall) given in Theorems 3.4, 3.11, and 3.8 are used to get bounds on $g_{a+k+1}(x)$ and are then substituted into (3.28) to get superior bounds on $g_a(x)$ or on $\int_0^x e^{-t^p}\,\mathrm{d}t$ for $p > 0$. The bounds given in Theorems 3.6 and 3.7 are not as good as these bounds, but the methods used to obtain them can be used to obtain bounds on other special functions, such as the beta function and other special functions related to the gamma function. Hopefully, we will report on this in future works.

4. Concluding Remarks

In this paper, many new bounds and inequalities for the incomplete gamma function have been discussed. Many of these were derived using results from reliability theory and probability theory results on bounding the moment-generating function of a random variable, in conjunction with other methods. The new bounds are often a significant improvement on previously proposed bounds. Moreover, the methods used can be used for other special functions and will be discussed in future works.

References

[1] Alzer, H. *On some inequalities for the incomplete gamma function* Math. Comp. 66(218), 771–778 (1997).

[2] Barlow, R.E., Proschan, F. *Mathematical Theory of Reliability* Wiley (1965).

[3] Barlow, R.E., Proschan, F. *Statistical Theory of Reliability and Life Testing Probability Models* To Begin With (1981).

[4] Brook, D. *Bounds for moment generating functions and for extinction probabilities* J. Appl. Prob. 3(1), 171–178 (1966).

[5] From, S. *Some new generalizations of Jensen's inequality with related results and applications* Aust. J. Math. Anal. Appl. 13(1), 1–29 (2016).
[6] Gautschi, W. *Some elementary inequalities relating to the gamma and incomplete gamma functions* J. Math. Phys. 38, 77–81 (1959).
[7] Giordano, C., Laforgia, A. *Inequalities and monotonicity properties for the gamma function* J. Comp. Appl. Math. 133(1–2), 387–396 (2001).
[8] Laforgia, A., Natalini, P. *On some inequalities for the gamma function* Adv. Dynam. Sys. and Appl. 8(2), 261–267 (2013).
[9] Laforgia, A., Natalini, P. *Exponential, gamma, and polygamma functions: Simple proofs of classical and new inequalities* J. Math. Anal. Appl. 407(2), 495–504 (2013).
[10] Lehmann, E.L., Romano, J.P. *Testing Statistical Hypotheses, 3rd Ed* Springer (2005).
[11] Luo, T., Lv, H., Yang, Z., Zheng, S. *New sharp approximations involving incomplete gamma functions* Results Math. 72, 1007–1020 (2017).
[12] Neuman, E. *Some inequalities for the gamma function* Appl. Math. Comput. 218, 4349–4352 (2011).
[13] Neuman, E. *Inequalities and bounds for the incomplete gamma function* Results Math. 63, 1209–1214 (2013).
[14] Qi, F. *Monotonicity results and inequalities for the gamma and incomplete gamma functions* Math. Inequal. Appl. 5, 61–67 (2002).
[15] Qi, F., Mei, J. *Some inequalities for the incomplete gamma and related functions* Z. Anal. Anwendungen 18, 793–799 (1999).
[16] Yang, Z. Zhang, W., Chu, Y. *Monotonicity of the incomplete gamma function with applications* J. Inequal. Appl. 215 (2016). doi: 10.1186/s13660-016-1197-7.
[17] Zhu, L. *New bounds for the function involving incomplete gamma function* RACSAM, 113, 901-908 (2019).

Steven G. From
Department of Mathematics, University of Nebraska at Omaha, Omaha, Nebraska 68182, USA
E-mail address: sfrom@unomaha.edu