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A NEW VERSION OF HILBERT INTEGRAL INEQUALITY OF THREE VARIABLES WITH HYPERBOLIC SINE FUNCTION

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Abstract. In this paper we give a new form for the Hilbert integral inequality for three variables with hyperbolic sine and cosine functions. The reverse form and equivalent forms are also obtained. The constant which we obtained is the best constant.

1. Introduction

If we assume that $\int\limits_0^\infty f^2(x)dx \in (0,\infty)$ and $\int\limits_0^\infty g^2(x)dx \in (0,\infty)$, then the following well-known Hilbert integral inequality holds

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left(\int\limits_0^\infty f^2(x)dx \int\limits_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant π is the best possible (see [6]).

Quite a number of authors established various formulations of Hilbert's inequalities in discrete, integral and half discrete forms of multiple variables. For example Batbold Tserendorj and Laith Emil Azar, presented discrete [16], half discrete, and integral forms [15] given by

$$\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda}dxdydz < C \left(\int\limits_0^\infty \int\limits_0^\infty \frac{f^p(x,y)}{(x+y)^{\lambda-2p+p\gamma+2}}dxdy \right)^{\frac{1}{p}} \left(\int\limits_0^\infty \frac{g^q(z)}{z^{\lambda-q-q\xi+1}}dz \right)^{\frac{1}{q}} \tag{1.2}$$

where $\lambda > 0$, $\xi \in (\frac{-\lambda}{p}, \frac{\lambda}{q})$, $f(x,y)$ is a non-negative function defined on $(0,\infty)\times(0,\infty)$, and $g(z) > 0$ on $(0,\infty)$. Further, the constant $C = B(\frac{\lambda}{p}+\xi, \frac{\lambda}{q}-\xi)$ is the best possible.

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In recent years, authors have continued to establish new integral forms of the Hilbert–inequality type with different new kernel functions (see [12,13]). For further results on the latter types, refer to Refs. [5],[6],[8], [9] and [11].

2. Preliminaries and Lemmas

In this work, we apply the following special functions:

$$\Gamma(\varkappa) = \int_0^\infty t^{\varkappa-1} e^{-t} dt, \quad \varkappa > 0, \tag{2.1}$$

$$B(\delta, \sigma) = \int_0^\infty \frac{t^{\delta-1}}{(t+1)^{\delta+\sigma}} dt, \ \ \delta, \sigma > 0. \tag{2.2}$$

Further, we make use of other useful representations for Eqs.(2.1) and (2.2) as follows:

$$\frac{1}{\omega^{\varkappa}} = \frac{1}{\Gamma(\varkappa)} \int_0^\infty t^{\varkappa-1} e^{-\omega t} dt. \tag{2.3}$$

$$B(\delta, \sigma) = \frac{\Gamma(\delta)\Gamma(\sigma)}{\Gamma(\delta+\sigma)} \tag{2.4}$$

$$B(\delta, \sigma) = B(\delta, \sigma). \tag{2.5}$$

Also, we will use the following inequality in several locations in this paper:

$$\cosh x \geq 1, \ \ \forall x \in (-\infty, \infty). \tag{2.6}$$

Next, the following lemmas are proven:

Lemma 2.1. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $g(z) > 0$ *on* $(0, \infty)$. *Then for* $t > 0$ *and* $\vartheta < \frac{1}{p}$, *we have*

$$\int_0^\infty e^{-\sinh z t} g(z) dz \leq t^{\vartheta - \frac{1}{p}} \Gamma^{\frac{1}{p}}(1 - p\vartheta) \left(\int_0^\infty (\sinh z)^{q\vartheta} e^{-\sinh z t} g^q(z) dz \right)^{\frac{1}{q}}.$$

Proof. Using Hölder inequlity, using the substitution $\sinh z = \omega$,we get

$$\begin{aligned}
\int_0^\infty e^{-\sinh z t} g(z)dz &= \int_0^\infty \left\{ (\sinh z)^{-\vartheta} e^{-\frac{\sinh z}{p} t} \right\} \left\{ (\sinh z)^{\vartheta} e^{-\frac{\sinh z}{q} t} g(z) \right\} dz \\
&\le \left(\int_0^\infty (\sinh z)^{-p\vartheta} e^{-\sinh z t} dz \right)^{\frac{1}{p}} \left(\int_0^\infty (\sinh z)^{q\vartheta} e^{-\sinh z t} g^q(z) dz \right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \omega^{-p\vartheta} e^{-\omega t} \frac{d\omega}{\cosh z} \right)^{\frac{1}{p}} \left(\int_0^\infty (\sinh z)^{q\vartheta} e^{-\sinh z t} g^q(z) dz \right)^{\frac{1}{q}} \\
&\le t^{\vartheta - \frac{1}{p}} \Gamma^{\frac{1}{p}} (1 - p\vartheta) \left(\int_0^\infty (\sinh z)^{q\vartheta} e^{-\sinh z t} g^q(z) dz \right)^{\frac{1}{q}}.
\end{aligned}$$

□

Lemma 2.2. *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t > 0$, $\tau < \frac{2}{q}$ *and* $f(x, y) > 0$ *is a non-negative function defined and integrable on* $(0, \infty) \times (0, \infty)$*. As a result, we get*

$$\left(\int_0^\infty \int_0^\infty f(x, y) e^{-[\sinh x + \sinh y]t} dxdy \right)^p \le t^{\tau q - 2} \Gamma(2 - \tau q) \int_0^\infty \int_0^\infty \frac{e^{-[\sinh x + \sinh y]t}}{[\sinh x + \sinh y]^{-\tau p}} f^p(x, y) dxdy.$$

Proof. Using Hölder's inequality, and making the substitutions $\sinh y = u \sinh x$ and $\sinh x = v$, $v \ge 0$, (to evaluate the first double integral on the right-hand-side of the inequality below), this yields

$$\begin{aligned}
&\left(\int_0^\infty \int_0^\infty f(x, y) e^{-[\sinh x + \sinh y]t} dxdy \right)^p \\
&= \left(\int_0^\infty \int_0^\infty \left\{ \frac{e^{-\frac{[\sinh x + \sinh y]}{q} t}}{[\sinh x + \sinh y]^{\tau}} dxdy \right\} \left\{ \frac{e^{-\frac{[\sinh x + \sinh y]}{p} t}}{[\sinh x + \sinh y]^{-\tau}} f(x, y) \right\} dxdy \right)^p \\
&\le \left(\left\{ \int_0^\infty \int_0^\infty \frac{e^{-\sinh x [1+u] t}}{(\sinh x)^{\tau q} [1+u]^{\tau q}} \left(\frac{\sinh x dv du}{(1+u) \cosh x \cosh y} \right) \right\}^{\frac{1}{q}} \left\{ \int_0^\infty \int_0^\infty \frac{e^{-[\sinh x + \sinh y]t} f^p(x, y)}{[\sinh x + \sinh y]^{-\tau p}} dxdy \right\}^{\frac{1}{p}} \right)^p = \\
&\left(\left\{ \int_0^\infty \int_0^\infty \frac{(\sinh x)^{1 - \tau q} e^{-\sinh x [1+u] t}}{[1+u]^{\tau q + 1}} \left(\frac{1}{\cosh x} \right) \left(\frac{1}{\cosh y} \right) dv du \right\}^{\frac{1}{q}} \left\{ \int_0^\infty \int_0^\infty \frac{e^{-[\sinh x + \sinh y]t} f^p(x, y)}{[\sinh x + \sinh y]^{-\tau p}} dxdy \right\}^{\frac{1}{p}} \right)^p
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_0^\infty\int_0^\infty \frac{v^{1-\tau q}e^{-v[1+u]t}}{[1+u]^{\tau q+1}}\times 1\times 1\times dvdu\right)^{\frac{p}{q}}\int_0^\infty\int_0^\infty \frac{e^{-[\sinh x+\sinh y]t}}{[\sinh x+\sinh y]^{-\tau p}}f^p(x,y)dxdy \\
&\leq \left(\int_0^\infty\int_0^\infty \frac{ve^{-vt}}{(1+u)^2}dudv\right)^{\frac{p}{q}}\int_0^\infty\int_0^\infty \frac{e^{-[\sinh x+\sinh y]t}}{[\sinh x+\sinh y]^{-\tau p}}f^p(x,y)dxdy \\
&= \left(\int_0^\infty \frac{1}{(1+u)^2}du\int_0^\infty v^{1-\tau q}e^{-vt}dv\right)^{\frac{p}{q}}\int_0^\infty\int_0^\infty \frac{e^{-[\sinh x+\sinh y]t}}{[\sinh x+\sinh y]^{-\tau p}}f^p(x,y)dxdy \\
&= t^{\tau p-2\frac{p}{q}}\Gamma^{\frac{p}{q}}(2-\tau q)\int_0^\infty\int_0^\infty \frac{e^{-[\sinh x+\sinh y]t}}{[\sinh x+\sinh y]^{-\tau p}}f^p(x,y)dxdy
\end{aligned}
$$

□

Remark : Note that if $0<p<1$ and $\frac{1}{p}+\frac{1}{q}=1$, then by the reverse form of Hölder's inequality and the same previous method we can prove the reverse forms of the inequalities in Lemma 2.1 and Lemma 2.2 which are needed in Theorem 3.2.

3. Main Result

Theorem 3.1. *Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>0$, $-\frac{\lambda}{p}<\xi<\frac{\lambda}{q}$, $f(x,y)$ a non- negative function defined on $(0,\infty)\times(0,\infty)$, and $g(z)$ a positive function on $(0,\infty)$. If $\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy<\infty$ and $\int_0^\infty \frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz<\infty$, then*

$$
\begin{aligned}
\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdydz \quad &\leq \quad C\left(\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}} \\
&\times\left(\int_0^\infty \frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}}, \qquad (3.1)
\end{aligned}
$$

where $C=B(\frac{\lambda}{p}+\xi,\frac{\lambda}{q}-\xi)$ is the best possible. In particular:

***a)** If we take $\lambda=\frac{3}{2}$, $\xi=\frac{1}{4}$, and $p=q=2$, then (3.1) leads*

$$
\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^{\frac{3}{2}}}dxdydz\leq 2\left(\int_0^\infty\int_0^\infty f^2(x,y)dxdy\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(z)dz\right)^{\frac{1}{2}}
$$

***b)** If we take $\lambda=2$, $\xi=0$, $p=\frac{4}{3}$, $q=4$, then (3.1) becomes*

$$
\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^2}dxdydz\leq \frac{\pi}{2}\left(\int_0^\infty\int_0^\infty \frac{f^{\frac{4}{3}}(x,y)}{[\sinh x+\sinh y]^{\frac{4}{3}}}dxdy\right)^{\frac{3}{4}}\left(\int_0^\infty \frac{g^4(z)}{(\sinh z)^{-1}}dz\right)^{\frac{1}{4}}
$$

Proof. Let

$$\begin{aligned}
I &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdydz \\
&= \frac{1}{\Gamma(\lambda)}\int_0^\infty\int_0^\infty\int_0^\infty f(x,y)g(z)\left(\int_0^\infty t^{\lambda-1}e^{-[\sinh x+\sinh y+\sinh z]t}dt\right)dxdydz. \\
&= \frac{1}{\Gamma(\lambda)}\int_0^\infty\left(t^{\frac{\lambda-1}{p}+\xi}\int_0^\infty\int_0^\infty f(x,y)e^{-[\sinh x+\sinh y]t}dxdy\right)\left(t^{\frac{\lambda-1}{q}-\xi}\int_0^\infty e^{-\sinh zt}g(z)dz\right)dt \\
&\le \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1+\xi p}\left(\int_0^\infty\int_0^\infty f(x,y)e^{-[\sinh x+\sinh y]t}dxdy\right)^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^\infty t^{\lambda-1-\xi q}\left(\int_0^\infty e^{-\sinh zt}g(z)dz\right)^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

From Lemma 2.1 and Lemma 2.2, we obtain respectively

$$\begin{aligned}
I &\le \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1+\xi p}t^{\tau p-2\frac{p}{q}}\Gamma^{\frac{p}{q}}(2-\tau q)\int_0^\infty\int_0^\infty \frac{e^{-[\sinh x+\sinh y]t}}{[\sinh x+\sinh y]^{-\tau p}}f^p(x,y)dxdydt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^\infty t^{\lambda-1-\xi q}\left(t^{\vartheta-\frac{1}{p}}\Gamma^{\frac{1}{p}}(1-p\vartheta)\left(\int_0^\infty(\sinh z)^{q\vartheta}e^{-\sinh zt}g^q(z)dz\right)^{\frac{1}{q}}\right)^q dt\right)^{\frac{1}{q}}. \\
&\le \frac{\Gamma^{\frac{1}{q}}(2-\tau q)\Gamma^{\frac{1}{p}}(1-p\vartheta)}{\Gamma(\lambda)}\left(\int_0^\infty\int_0^\infty\frac{f^p(x,y)}{[\sinh x+\sinh y]^{-\tau p}}\left(\int_0^\infty t^{\lambda-1+p\xi+\tau p-2\frac{p}{q}}e^{-[\sinh x+\sinh y]t}dt\right)dxdy\right)^{\frac{1}{p}} \\
&\times\left(\int_0^\infty(\sinh z)^{q\vartheta}g^q(z)\left(\int_0^\infty t^{\lambda-1-\xi q+q\vartheta-\frac{q}{p}}e^{-\sinh zt}dt\right)dz\right)^{\frac{1}{q}} \\
&= \frac{\Gamma^{\frac{1}{q}}(2-\tau q)\Gamma^{\frac{1}{p}}(1-p\vartheta)\Gamma^{\frac{1}{q}}(\lambda-\xi q+q\vartheta-\frac{q}{p})\Gamma^{\frac{1}{p}}(\lambda+\xi p+\tau p-2\frac{p}{q})}{\Gamma(\lambda)} \\
&\times\left(\int_0^\infty\int_0^\infty\frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty\frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}} \\
&= C_{\tau,\vartheta}\left(\int_0^\infty\int_0^\infty\frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty\frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}}
\end{aligned}$$

If we set $\tau=\frac{2-\lambda-\xi p+2\frac{p}{q}}{pq}$, and $\vartheta=\frac{1-\lambda+\xi q+\frac{q}{P}}{pq}$, we obtain the proof of our main result with $C_{\frac{2-\lambda-\xi p+2\frac{p}{q}}{pq},\frac{1-\lambda+\xi q+\frac{q}{P}}{pq}}=C_{\tau,\vartheta}=C$.

The next step in this paper is to show that the constant C given in (3.1) is the best possible. Let ε be a very small real number, and let us define the functions $f_\epsilon(x,y)=0$, on $(0,a)\times(0,a)$ and $f_\epsilon(x,y)=(\sinh x+\sinh y)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}$ on $[a,\infty)\times[a,\infty)$, and $g_\epsilon(z)=0$,on $(0,a)$ and $g_\epsilon(z)=(\sinh z)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}$ on $[a,\infty)$, where $a=ln(1+\sqrt{2})$.

Suppose that C is not the best possible, then there exists a constant K where $0<K<C$, and again as in Lemma 2.1, Lemma 2.2, use the substitutions $\sinh y=u\sinh x$, $\sinh x=v$, $\sinh z=\omega$, and $h=\frac{1}{\sinh x}\leq 1,\forall x\geq a$, such that:

$$\begin{aligned}
I &\leq K\left(\int_0^\infty\int_0^\infty\frac{f^p(x,y)}{(\sinh x+\sinh y)^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty\frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}}\\
&=K\left(\int_a^\infty\int_a^\infty\frac{\left[(\sinh x+\sinh y)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}\right]^P}{(\sinh x+\sinh y)^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}}\left(\int_a^\infty\frac{(\sinh z)^{\lambda-\xi q-\frac{q}{p}-\epsilon-1}}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}}\\
&=K\left(\int_a^\infty\int_a^\infty(\sinh x+\sinh y)^{-\epsilon-2}\,dxdy\right)^{\frac{1}{p}}\left(\int_a^\infty(\sinh z)^{-\epsilon-1}\,dz\right)^{\frac{1}{q}}\\
&=K\left(\int_{u=h}^\infty\int_{v=1}^\infty\left(\frac{(\sinh x)^{-\epsilon-1}\,dv}{\cosh x}\right)\left(\frac{(1+u)^{-\epsilon-2}}{\cosh y}du\right)\right)^{\frac{1}{p}}\left(\int_{\omega=1}^\infty\frac{\omega^{-\epsilon-1}d\omega}{\cosh z}\right)^{\frac{1}{q}}\\
&<K\left(\int_h^\infty\int_1^\infty v^{-\epsilon-1}\,(1+u)^{-\epsilon-2}\,dv\frac{du}{\cosh x\cosh y}\right)^{\frac{1}{p}}\left(\int_1^\infty\omega^{-\epsilon-1}d\omega\right)^{\frac{1}{q}}\\
&<K\left(\int_1^\infty v^{-\epsilon-1}\left(\int_h^\infty u^{-\epsilon-2}\frac{du}{\cosh x\cosh y}\right)dv\right)^{\frac{1}{p}}\left(\int_1^\infty\omega^{-\epsilon-1}d\omega\right)^{\frac{1}{q}}=\frac{K}{(\epsilon)^{\frac{1}{p}}(\epsilon)^{\frac{1}{q}}}=\frac{K}{\epsilon}.
\end{aligned}$$

On the other hand, to estimate the left-hand-side of (3.1), using $(\cosh c)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}\varpi=\frac{\sinh z}{\sinh x+\sinh y}$, $c>x,c>y,c>z$, $\cosh c>2^{\epsilon+1}$, choose $m>5$ such that $\frac{(\cosh c)^{m-3}(\cosh c)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}}{\cosh z}>1$,and let $b=\frac{(\cosh c)^{\frac{m}{\frac{\epsilon-\lambda}{q}+\xi+1}}}{\sinh x+\sinh y}$, we obtain:

$$I=\int_0^\infty\int_0^\infty\int_0^\infty\frac{f_\epsilon(x,y)g_\epsilon(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdydz$$

$$= \int_a^\infty\int_a^\infty\int_a^\infty \frac{(\sinh x+\sinh y)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}(\sinh z)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdydz$$

$$= \int_a^\infty\int_a^\infty\int_a^\infty \frac{(\sinh x+\sinh y)^{\frac{\lambda}{P}+\xi-\frac{\epsilon}{p}-2}(\sinh z)^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(\sinh x+\sinh y)^\lambda(1+\frac{\sinh z}{\sinh x+\sinh y})^\lambda}dxdydz$$

$$= \int_a^\infty\int_a^\infty (\sinh x+\sinh y)^{\frac{\lambda}{P}+\xi-\frac{\epsilon}{p}-2-\lambda}\left(\int_a^\infty \frac{(\sinh z)^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+\frac{\sinh z}{\sinh x+\sinh y})^\lambda}dz\right)dxdy$$

$$= \int_a^\infty\int_a^\infty (\sinh x+\sinh y)^{\frac{\lambda}{P}+\xi-\frac{\epsilon}{p}-2-\lambda+\frac{\lambda-\epsilon}{q}-\xi-1}$$

$$\times\left(\int_b^\infty \frac{(\cosh c)^m\varpi^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+(\cosh c)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}\varpi)^\lambda}\left(\frac{(\sinh x+\sinh y)(\cosh c)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}}{\cosh z}d\varpi\right)\right)dxdy$$

$$= \int_h^\infty\int_1^\infty v^{-\epsilon-2}(1+u)^{-\epsilon-2}\cosh c\left(\frac{\cosh c dv}{\cosh x}\right)\left(\frac{\sinh x\cosh c}{\cosh y}du\right)$$

$$\times\left(\int_0^\infty \frac{\varpi^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+\varpi)^\lambda}d\varpi-\int_0^b \frac{\varpi^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+\varpi)^\lambda}d\varpi\right)$$

$$\geq \left(\int_1^\infty v^{-\epsilon-1}\left(\cosh c\int_1^\infty (1+u)^{-\epsilon-2}\frac{\cosh^2 c du}{\cosh x\cosh y}\right)dv\right)\left(B(\frac{\lambda-\epsilon}{q}-\xi,\frac{\lambda}{p}+\frac{\epsilon}{q}+\xi)-\int_0^b \varpi^{\frac{\lambda-\epsilon}{q}-\xi-1}d\varpi\right)$$

$$\geq \frac{B(\frac{\lambda-\epsilon}{q}-\xi,\frac{\lambda}{p}+\frac{\epsilon}{q}+\xi)}{\epsilon(\epsilon+1)}-O(1).$$

It is clear that when $\epsilon\to 0^+$, then from the above two estimates we lead to a contradiction, therefore, we have completely proven Theorem 3.1. □

Next, the reverse form of the inequality given in Theorem 3.1 is introduced.

Theorem 3.2. *Let* $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>0$, $-\frac{\lambda}{p}<\xi<\frac{\lambda}{q}$, $f(x,y)$ *a non- negative function defined on* $(0,\infty)\times(0,\infty)$, *and* $g(z)$ *a positive function on* $(0,\infty)$. *If* $\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy<\infty$ *and* $\int_0^\infty \frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz<\infty$, *then*

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdydz \geq C\left(\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}}\times\left(\int_0^\infty \frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}} \quad (3.2)$$

where $C=B(\frac{\lambda}{p}+\xi,\frac{\lambda}{q}-\xi)$ *is the best possible.*

Proof. Using the reverse form of Hölder's inequality, and the same procedure as in Theorem 3.1, we obtain the proof of Theorem 3.2. □

4. Equivalent Forms

In this part, we give two equivalent forms of each of our main theorems, all of them are with the best constant.

Theorem 4.1. *Under the conditions of theorem 1, we give the following two inequalities:*

$$\int_0^\infty (\sinh z)^{(p-1)\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdy\right)^p dz$$
$$\leq C^p\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy \tag{4.1}$$

and

$$\int_0^\infty\int_0^\infty (\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2}\left(\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dz\right)^q dxdy$$
$$\leq C^q\int_0^\infty (\sinh z)^{\xi q+\frac{q}{p}-\lambda}\, g^q(z)dz. \tag{4.2}$$

Inequalities (4.1) and (4.2) are equivalent to (3.1), also here the constants C^p and C^q are the best possible.

Proof. To prove (4.2), we set

$$g(z)=(\sinh z)^{(p-1)\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdy\right)^{p-1}$$

so that from (3.1) one gets

$$\int_0^\infty (\sinh z)^{(p-1)\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdy\right)^p dz$$
$$=\int_0^\infty (\sinh z)^{(p-1)\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdy\right)^{p-1}$$

$$\times\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdy\right)dz$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdydz$$

$$\leq C\left(\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}}$$

$$=C\left(\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty(\sinh z)^{(p-1)\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdy\right)^p dz\right)^{\frac{1}{q}}.$$

By dividing the last inequality by $\left(\int_0^\infty(\sinh z)^{(p-1)\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdy\right)^p dz\right)^{\frac{1}{q}}$, we obtain (4.1).

To show the equivalence between (3.1) and (4.2), let

$$f(x,y)=(\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2}\left(\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dz\right)^{q-1}.$$

Then using (3.1), this yields

$$\int_0^\infty\int_0^\infty(\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2}\left(\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dz\right)^{q}dxdy$$

$$=\int_0^\infty\int_0^\infty(\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2}\left(\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dz\right)^{q-1}$$

$$\times\left(\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dz\right)dxdy$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdydz$$

$$\leq C\left(\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}}$$

$$= C\left(\int_0^\infty\int_0^\infty (\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2}\left(\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^{\lambda}}dz\right)^q dxdy\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty \frac{g^q(z)}{(\sinh z)^{\lambda-\xi q-\frac{q}{p}}}dz\right)^{\frac{1}{q}}$$

Divide the two sides of the last inequlity by

$$\left(\int_0^\infty\int_0^\infty (\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2}\left(\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^{\lambda}}dz\right)^q dxdy\right)^{\frac{1}{p}},$$

this results in eq.(4.2).

Moreover, from Hölder's inequality and (4.2), one arrives at

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^{\lambda}}dxdydz = \int_0^\infty\int_0^\infty \left((\sinh x+\sinh y)^{\frac{-[q\xi+(q-1)\lambda-2]}{q}} f(x,y)\right)$$

$$\times\left((\sinh x+\sinh y)^{\frac{[q\xi+(q-1)\lambda-2]}{q}}\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^{\lambda}}dz\right)dxdy$$

$$\leq\left(\int_0^\infty\int_0^\infty (\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2} f^p(x,y)dxdy\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty\int_0^\infty (\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2}\left(\int_0^\infty \frac{g(z)}{(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^{\lambda}}dz\right)^q dxdy\right)^{\frac{1}{q}}$$

$$\leq\left(\int_0^\infty\int_0^\infty (\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2} f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(C^q\int_0^\infty (\sinh z)^{-\lambda+\xi q+\frac{q}{p}} q(z)dz\right)^{\frac{1}{q}}.$$

By this, the equivalence between (4.2) and (3.1) has been proven. The constants are the best possible as in (3.1). The theorem is proved. □

Theorem 4.2. *Under the same conditions as in Theorem 3.2, we obtain:*

$$\int_0^\infty \sinh z^{p\xi+(p-1)\lambda-+1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{(\sinh x+\sinh y+\sinh z)^{\lambda}}dxdy\right)^p dz$$

$$\geq C^p\int_0^\infty\int_0^\infty \frac{f^p(x,y)}{[\sinh x+\sinh y]^{\lambda+p\xi-2\frac{p}{q}}}dxdy, \tag{4.3}$$

and

$$\int_0^\infty\int_0^\infty (\sinh x+\sinh y)^{q\xi+(q-1)\lambda-2}\left(\int_0^\infty \frac{g(z)}{(\sinh x+\sinh y+\sinh z)^{\lambda}}dz\right)^q dxdy$$

$$\geq \quad C^q\int_0^\infty (\sinh z)^{\xi q+\frac{q}{p}-\lambda}\, g^q(z)dz, \tag{4.4}$$

where the constants here $C^p = B^p(\frac{\lambda}{p}+\xi, \frac{\lambda}{q}-\xi)$ *and* $C^q = B^q(\frac{\lambda}{p}+\xi, \frac{\lambda}{q}-\xi)$ *are the best possible..*

Proof. Since the method of proof of the above inequalities is the same as that of the method in Theorem 4.1, we leave it for the reader. □

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