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ON THE DENSITY OF LAGUERRE FUNCTIONS IN SOME BANACH FUNCTION SPACES

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Abstract. Let $\lambda > 0$ and $\Phi_\lambda := \{\varphi_{1,\lambda}, \varphi_{2,\lambda}, \dots\}$ be the system of dilated Laguerre functions. We show that if $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ is embedded into a separable Banach function space $X(\mathbb{R}^+)$, then the linear span of Φ_λ is dense in $X(\mathbb{R}^+)$. This implies that the linear span of Φ_λ is dense in every separable rearrangement-invariant space $X(\mathbb{R}^+)$ and in every separable variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^+)$.

1. Introduction

For $n \in \mathbb{N} \cup \{0\}$, the n-th Laguerre polynomial is defined by

$$L_n(x) := \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k, \quad x \in \mathbb{R} \tag{1.1}$$

(see, e.g., [18, Section 5.1]). It is well known that the system of Laguerre functions $\Phi := \{\varphi_1, \varphi_2, \dots\}$ defined by

$$\varphi_n(x) := L_{n-1}(x) e^{-x/2}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{1.2}$$

is an orthonormal system in $L^2(\mathbb{R}^+)$, that is,

$$\int_0^\infty \varphi_n(x)\varphi_m(x)\, dx = \begin{cases} 1, & n = m, \\ 0, & n \neq m \end{cases}$$

(see, e.g., [11, Section 4.8.2]). Moreover, it is complete in $L^2(\mathbb{R}^+)$, that is, if $g \in L^2(\mathbb{R}^+)$ and

$$\int_0^\infty \varphi_n(x) g(x)\, dx = 0 \quad \text{for all} \quad n \in \mathbb{N},$$

then $g(x) = 0$ for almost every $x \in \mathbb{R}^+$ (see, e.g., [11, Section 4.8.3] or [13, Ch. VIII, §4.3]).

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The system of Laguerre functions or its modifications arise in many areas of mathematics (see, e.g., [8], [9, Ch. 4], [14, Ch. 4], [17], [19], [20, Ch. 8], [21, Ch. VIII, § 5], to mention just a few works, where Laguerre functions play an important role). Our motivations come from the theory of Wiener-Hopf operators

$$(Wf)(x) := \int_0^\infty k(x-y)f(y)\,dy$$

on Lebesgue spaces $L^p(\mathbb{R}^+)$, $1 < p < \infty$, where the system of dilated Laguerre functions $\Phi_2 := \{\varphi_{1,2}, \varphi_{2,2}, \dots\}$ with $\varphi_{n,2}(x) = \varphi_n(2x)$ for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, arises naturally (see, e.g., [7, Ch. I, Section 3], [10, Ch. I, §8.3], [15, Sections 4.2–4.3]). In particular, the density of the linear span of Φ_2 in $L^p(\mathbb{R}^+)$ for $1 < p < \infty$ plays a crucial role in the proof of the fact that the Banach algebra $\operatorname{alg}(W(C(\dot{\mathbb{R}}))$ generated by Wiener-Hopf operators with continuous symbols contains all compact operators on $L^p(\mathbb{R}^+)$ (see, [4, Section 9.9] and also [12, Lemmas 5.2–5.3]).

The aim of this paper is to prove that for every $\lambda > 0$ the linear span of the system of dilated Laguerre functions

$$\Phi_\lambda := \{\varphi_1(\lambda x), \varphi_2(\lambda x), \dots\}$$

is dense in a separable Banach function space $X(\mathbb{R}^+)$ under a natural additional assumption that $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ is embedded into $X(\mathbb{R}^+)$. We postpone the technical definition of a Banach function space to Section 2.1 (see also [2, Ch. 1] for a complete account on the theory of Banach function spaces). Here we only mention that the class of Banach function spaces is very large, it contains all Lebesgue spaces $L^p(\mathbb{R}^+)$, all Orlicz spaces $L^\Phi(\mathbb{R}^+)$, and all Lorentz spaces $L^{p,q}(\mathbb{R}^+)$; which are rearrangement-invariant (see Section 2.2 and [2, Ch. 2]); as well as, all variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^+)$ (see Section 2.3 and [5, 6]); which are not rearrangement-invariant.

Theorem 1.1 (Main result). *Let $\lambda > 0$. If $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ is embedded into a separable Banach function space $X(\mathbb{R}^+)$, then the linear span of Φ_λ is dense in $X(\mathbb{R}^+)$.*

The paper is organized as follows. In Section 2, we recall definitions of the class of Banach function spaces and their associate spaces, of its subclass of rearrangement-invariant Banach function spaces, as well as, of variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^+)$, which constitute a distinguished example of non-rearrangement-invariant Banach function spaces. We pay special attention to the mutually associate rearrangement-invariant Banach function spaces $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ and $L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)$. Further, we state Lerch's theorem and recall a suitable form of Stirling's formula.

In Section 3, we show that the system of dilated Laguerre functions Φ_λ is contained in $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. Further, following the scheme of the proof of [11, Section 4.8.3], we show that Φ_λ is complete in $L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)$. Finally, we prove Theorem 1.1 and state its corollary for separable rearrangement-invariant Banach function spaces and separable variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^+)$.

We plan to employ the results obtained in this work to the study of Banach algebras generated by Wiener-Hopf operators on variable Lebesgue spaces in a forthcoming publication.

2. Preliminaries

2.1. **Banach function spaces.** The set of all Lebesgue measurable extended complex-valued functions on $\mathbb{R}^+$ is denoted by $\mathcal{M}(\mathbb{R}^+)$. Let $\mathcal{M}^+(\mathbb{R}^+)$ be the subset of functions in $\mathcal{M}(\mathbb{R}^+)$ whose values lie in $[0,\infty]$. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^+$ is denoted by $|E|$ and its characteristic function is denoted by χ_E. Following [2, Ch. 1, Definition 1.1], a mapping $\rho : \mathcal{M}^+(\mathbb{R}^+) \to [0,\infty]$ is called a Banach function norm if, for all functions f, g, f_n ($n \in \mathbb{N}$) in $\mathcal{M}^+(\mathbb{R}^+)$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{R}^+$, the following properties hold:

$$\text{(A1)} \qquad \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \quad \rho(af) = a\rho(f), \quad \rho(f+g) \leq \rho(f) + \rho(g),$$

$$\text{(A2)} \qquad 0 \leq g \leq f \text{ a.e.} \;\Rightarrow\; \rho(g) \leq \rho(f) \quad \text{(the lattice property)},$$

$$\text{(A3)} \qquad 0 \leq f_n \uparrow f \text{ a.e.} \;\Rightarrow\; \rho(f_n) \uparrow \rho(f) \quad \text{(the Fatou property)},$$

$$\text{(A4)} \qquad |E| < \infty \Rightarrow \rho(\chi_E) < \infty,$$

$$\text{(A5)} \qquad |E| < \infty \Rightarrow \int_E f(x)\,dx \leq C_E \rho(f)$$

with $C_E \in (0,\infty)$ which may depend on E and ρ but is independent of f. When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R}^+)$ of all functions $f \in \mathcal{M}(\mathbb{R}^+)$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathbb{R}^+)$, the norm of f is defined by

$$\|f\|_{X(\mathbb{R}^+)} := \rho(|f|).$$

Under the natural linear space operations and under this norm, the set $X(\mathbb{R}^+)$ becomes a Banach space (see [2, Ch. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathcal{M}^+(\mathbb{R}^+)$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{R}^+} f(x)g(x)\,dx \;:\; f \in \mathcal{M}^+(\mathbb{R}^+),\; \rho(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+(\mathbb{R}^+),$$

which is a Banach function norm itself [2, Ch. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{R}^+)$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{R}^+)$. The associate space $X'(\mathbb{R}^+)$ is naturally identified with a subspace of the Banach dual space $X^*(\mathbb{R}^+)$ (see [2, pp. 19–20]).

2.2. **Rearrangement-invariant Banach function spaces.** Let $\mathcal{M}_0(\mathbb{R}^+)$ and $\mathcal{M}_0^+(\mathbb{R}^+)$ be the classes of a.e. finite functions in $\mathcal{M}(\mathbb{R}^+)$ and $\mathcal{M}^+(\mathbb{R}^+)$, respectively. The distribution function m_f of $f \in \mathcal{M}_0(\mathbb{R}^+)$ is given by

$$m_f(\lambda) := |\{x \in \mathbb{R}^+ : |f(x)| > \lambda\}|, \quad \lambda \geq 0.$$

Two functions $f, g \in \mathcal{M}_0(\mathbb{R}^+)$ are said to be equimeasurable if $m_f(\lambda) = m_g(\lambda)$ for all $\lambda \geq 0$.

A Banach function norm $\rho : \mathcal{M}^+(\mathbb{R}^+) \to [0,\infty]$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+(\mathbb{R}^+)$, the equality $\rho(f) = \rho(g)$ holds. In that case, the Banach function space $X(\mathbb{R}^+)$ generated by ρ is said to be a rearrangement-invariant Banach function space (or simply a rearrangement-invariant space). Lebesgue spaces $L^p(\mathbb{R}^+)$, $1 \leq p \leq \infty$, Orlicz spaces $L^\Phi(\mathbb{R}^+)$, and Lorentz spaces $L^{p,q}(\mathbb{R}^+)$ are classical examples of rearrangement-invariant Banach function spaces (see, e.g., [2] and the references therein). By [2, Ch. 2, Proposition 4.2], if a Banach function space $X(\mathbb{R}^+)$ is rearrangement-invariant, then its associate space $X'(\mathbb{R}^+)$ is also rearrangement-invariant.

2.3. **Variable Lebesgue spaces.** Let $p(\cdot):\mathbb{R}^+\to[1,\infty]$ be a measurable function called variable exponent. For a measurable function $f:\mathbb{R}^+\to\mathbb{C}$, consider the functional:

$$\varrho_{p(\cdot)}(f):=\int_{\mathbb{R}^+\setminus\{y\in\mathbb{R}^+\,:\,p(y)=\infty\}}|f(x)|^{p(x)}dx+\operatorname*{ess\,sup}_{x\in\{y\in\mathbb{R}^+\,:\,p(y)=\infty\}}|f(x)|.$$

The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^+)$ consists of all measurable functions $f:\mathbb{R}^+\to\mathbb{C}$ such that $\varrho_{p(\cdot)}(f/\lambda)<\infty$ for some $\lambda=\lambda(f)>0$. It is well known that $L^{p(\cdot)}(\mathbb{R}^+)$ is a Banach function space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^+)}:=\inf\{\lambda>0\ :\ \varrho_{p(\cdot)}(f/\lambda)\le 1\}$$

(see, e.g., [5, Theorem 2.71 and Section 2.10.3] and also [6, Theorem 3.2.13] for an equivalent norm). Moreover, $L^{p(\cdot)}(\mathbb{R}^+)$ is separable if and only if

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^+}p(x)<\infty$$

(see, e.g., [5, Theorem 2.78]). Variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^+)$ are not rearrangement-invariant (see, e.g., [5, Example 3.14] for a counter-example).

2.4. **Spaces $L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$ and $L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)$.** For a function f in the intersection $L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$, let

$$\|f\|_{L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)}:=\max\left\{\|f\|_{L^1(\mathbb{R}^+)},\|f\|_{L^\infty(\mathbb{R}^+)}\right\}.$$

Following [2, Ch. 2, Definition 6.1], the space $L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)$ consists of all functions $f\in\mathcal{M}_0(\mathbb{R}^+)$ that are representable as a sum $f=g+h$ of functions $g\in L^1(\mathbb{R}^+)$ and $h\in L^\infty(\mathbb{R}^+)$. For each $f\in L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)$, let

$$\|f\|_{L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)}:=\inf\left\{\|g\|_{L^1(\mathbb{R}^+)}+\|h\|_{L^\infty(\mathbb{R}^+)}\right\},$$

where the infimum is taken over all representations $f=g+h$ of the kind described above. In view of [2, Ch. 2, Theorem 6.4], the spaces $L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$ and $L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)$ are rearrangement-invariant Banach function spaces and they are mutually associate to each other. Therefore, the following version of Hölder's inequality is an immediate consequence of [2, Ch. 1, Theorem 2.4].

Lemma 2.1. *Suppose $f\in L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$ and $g\in L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)$. Then $fg\in L^1(\mathbb{R}^+)$ and*

$$\|fg\|_{L^1(\mathbb{R}^+)}\le\|f\|_{L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)}\|g\|_{L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)}.$$

The spaces $L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$ and $L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)$ are the smallest and the largest spaces, respectively, among all rearrangement-invariant Banach function spaces. A similar property is also true for variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^+)$ with $p(\cdot):\mathbb{R}^+\to[1,\infty]$.

Theorem 2.2. *If $X(\mathbb{R}^+)$ is a rearrangement-invariant Banach function space or a variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^+)$ with $p(\cdot):\mathbb{R}^+\to[1,\infty]$, then*

$$L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)\hookrightarrow X(\mathbb{R}^+)\hookrightarrow L^1(\mathbb{R}^+)+L^\infty(\mathbb{R}^+)$$

(here $\hookrightarrow$ denotes the continuous embedding of corresponding Banach spaces).

For rearrangement-invariant Banach function spaces, the proof of the above theorem is contained in [2, Ch. 2, Theorem 6.6], while for variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^+)$ with $p(\cdot):\mathbb{R}^+\to[1,\infty]$, its proof is given in [6, Theorem 3.3.11] (see also [5, Theorem 2.51]).

2.5. **Lerch's theorem.** The proof of the following result, usually attributed to Lerch, can be found, e.g., in [22, Section 6.5, Corollary 5.1] or [11, Section 3.5.8].

Theorem 2.3. *If* $\varphi \in L^1(0,1)$ *and*

$$\int_0^1 \varphi(t)t^n\,dt = 0 \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\},$$

then $\varphi = 0$.

2.6. **Stirling's formula.** We will use Stirling's formula in the following form.

Lemma 2.4 ([16])**.** *For all* $n \in \mathbb{N}$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n+1}\right) < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n}\right).$$

3. Proof of the main result

3.1. **The system Φ_λ is contained in $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$.**

Lemma 3.1. *Let* $\lambda > 0$ *and*

$$g_{n,\lambda}(x) := (\lambda x)^n e^{-\lambda x/2}, \quad x \in \mathbb{R}^+, \quad n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

Then

$$\|g_{n,\lambda}\|_{L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)} \le C_\lambda 2^{n+1} n!\,, \tag{3.2}$$

where

$$C_\lambda := \max\left\{\frac{1}{\lambda}, \frac{1}{2\sqrt{2\pi}}\right\}.$$

Consequently, $\Phi_\lambda \subset L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$.

Proof. Since

$$g'_{n,\lambda}(x) = \lambda^n x^{n-1} e^{-\lambda x/2}\left(n - \frac{\lambda x}{2}\right),$$

we have

$$\|g_{n,\lambda}\|_{L^\infty(\mathbb{R}^+)} = \sup_{x\in\mathbb{R}^+} g_{n,\lambda}(x) = g\left(\frac{2n}{\lambda}\right) = 2^n \left(\frac{n}{e}\right)^n. \tag{3.3}$$

On the other hand,

$$\begin{aligned}\|g_{n,\lambda}\|_{L^1(\mathbb{R}^+)} &= \int_0^\infty (\lambda x)^n e^{-\lambda x/2} dx = \frac{2^{n+1}}{\lambda}\int_0^\infty t^n e^{-t} dt \\ &= \frac{2^{n+1}}{\lambda}\Gamma(n+1) = \frac{2^{n+1} n!}{\lambda}.\end{aligned} \tag{3.4}$$

Taking into account (3.3)–(3.4), with the aid of Lemma 2.4, we get

$$\begin{aligned}\|g_{n,\lambda}\|_{L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)} &= \max\left\{\|g_{n,\lambda}\|_{L^1(\mathbb{R}^+)}, \|g_{n,\lambda}\|_{L^\infty(\mathbb{R}^+)}\right\} \\ &= \max\left\{\frac{2^{n+1}n!}{\lambda}, 2^n\left(\frac{n}{e}\right)^n\right\} \\ &\le \max\left\{\frac{2^{n+1}n!}{\lambda}, \frac{1}{2\sqrt{2\pi n}}\exp\left(-\frac{1}{12n+1}\right)2^{n+1}n!\right\} \\ &\le \max\left\{\frac{1}{\lambda}, \frac{1}{2\sqrt{2\pi}}\right\}2^{n+1}n!,\end{aligned}$$

which completes the proof of (3.2). Finally, it follows from (1.1)–(1.2) that each dilated Laguerre function $\varphi_n(\lambda x)$, $n \in \mathbb{N}$, is a linear combination of functions $g_{k,\lambda}(x)$, $k \in \{0, \dots, n\}$. Thus $\Phi_\lambda \subset L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. □

3.2. **Completeness of Φ_λ in $L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)$.** The following theorem extends the result on the completeness of Φ_1 in $L^2(\mathbb{R}^+)$ (see, e.g., [11, Section 4.8.3]).

Theorem 3.2. *Let $\lambda > 0$. If $f \in L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)$ and*

$$\int_0^\infty f(x)\varphi_n(\lambda x)\,dx = 0 \quad \text{for all} \quad n \in \mathbb{N}, \tag{3.5}$$

then $f = 0$.

Proof. The proof is analogous to that one given in [11, Section 4.8.3, pp. 165-166]. It follows from (1.1) that for every $n \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}^+$,

$$[L_0(x)\ L_1(x)\ \dots\ L_n(x)]^T = A[x^0\ x^1\ \dots\ x^n]^T$$

where B^T denotes the transpose of a matrix B and

$$A := \begin{bmatrix} \binom{0}{0}\frac{1}{0!} & 0 & 0 & \ddots & 0 & 0 \\ \binom{1}{0}\frac{1}{0!} & \binom{1}{1}\frac{(-1)}{1!} & 0 & \ddots & 0 & 0 \\ \binom{2}{0}\frac{1}{0!} & \binom{2}{1}\frac{(-1)}{1!} & \binom{2}{2}\frac{1}{2!} & \ddots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \ddots & \ddots \\ \binom{n-1}{0}\frac{1}{0!} & \binom{n-1}{1}\frac{(-1)}{1!} & \binom{n-1}{2}\frac{1}{2!} & \dots & \binom{n-1}{n-1}\frac{(-1)^{n-1}}{(n-1)!} & 0 \\ \binom{n}{0}\frac{1}{0!} & \binom{n}{1}\frac{(-1)}{1!} & \binom{n}{2}\frac{1}{2!} & \dots & \binom{n}{n-1}\frac{(-1)^{n-1}}{(n-1)!} & \binom{n}{n}\frac{(-1)^n}{n!} \end{bmatrix}.$$

Since

$$\det A = \prod_{k=0}^{n} \frac{(-1)^k}{k!} \neq 0,$$

we see that A is invertible and

$$[x^0\ x^1\ \dots\ x^n]^T = A^{-1}[L_0(x)\ L_1(x)\ \dots\ L_n(x)]^T,$$

whence x^n can be expressed as a linear combination of $L_0(x), L_1(x), \dots, L_n(x)$. Therefore, (3.5) implies that

$$\int_0^\infty f(x)g_{n,\lambda}(x)\,dx = 0 \quad \text{for all} \quad n \in \mathbb{N}, \tag{3.6}$$

where the functions $g_{n,\lambda}$, $n \in \mathbb{N} \cup \{0\}$ are given by (3.1). Consider

$$h_\lambda(x) := f(x)e^{-\lambda x/2} = f(x)g_{0,\lambda}(x), \quad x \in \mathbb{R}^+.$$

Then (3.6) yields

$$\int_0^\infty h_\lambda(x)x^n\,dx = 0 \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{3.7}$$

It follows from Lemmas 2.1 and 3.1 that $h_\lambda \in L^1(\mathbb{R}^+)$. Then its Laplace transform

$$H_\lambda(z) := \int_0^\infty h_\lambda(x)e^{-zx}dx$$

exists for $\operatorname{Re} z \geq 0$ and is analytic in the domain $\operatorname{Re} z > 0$ (see, e.g., [1, Theorem 12.8]).

Let $y \geq 0$. Expanding e^{-yx} in the Maclaurin series, we get

$$H_\lambda(y) = \int_0^\infty h_\lambda(x) \left(\sum_{n=0}^\infty \frac{(-yx)^n}{n!} \right) dx.$$

We are going to justify the interchange of order of integration and summation in the above integral. We will show that the series $\sum_{n=0}^\infty a_{n,\lambda} y^n$ converges absolutely in $(-R_\lambda, R_\lambda)$ for some $R_\lambda \in (0, +\infty)$, where

$$a_{n,\lambda} := \frac{1}{n!} \int_0^\infty |h_\lambda(x)| x^n \, dx = \frac{1}{\lambda^n n!} \int_0^\infty |f(x)| g_{n,\lambda}(x) \, dx, \quad n \in \mathbb{N} \cup \{0\}.$$

Indeed, it follows from Lemmas 2.1 and 3.1 that for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} a_{n,\lambda} &\leq \frac{1}{\lambda^n n!} \|f\|_{L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)} \|g_{n,\lambda}\|_{L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)} \\ &\leq C_\lambda \|f\|_{L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)} \frac{2^{n+1}}{\lambda^n} =: b_{n,\lambda}. \end{aligned}$$

Hence the radius of convergence R_λ of the series $\sum_{n=0}^\infty a_{n,\lambda} y^n$ is not less than the radius of convergence of the series $\sum_{n=0}^\infty b_{n,\lambda} y^n$, which is equal to $\lambda/2$. Hence, for $0 \leq y < \lambda/2$,

$$\sum_{n=0}^\infty \frac{1}{n!} \left(\int_0^\infty |h_\lambda(x)| x^n \, dx \right) y^n < \infty.$$

In this case, the Tonelli and Fubini theorems (see, e.g., [3, Ch. 4, Theorems 3.1–3.2]) imply that for $0 \leq y < \lambda/2$, one has

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(\int_0^\infty h_\lambda(x) x^n \, dx \right) y^n = \int_0^\infty h_\lambda(x) \left(\sum_{n=0}^\infty \frac{(-yx)^n}{n!} \right) dx = H_\lambda(y). \quad (3.8)$$

It follows from (3.7) and (3.8) that $H_\lambda(y) = 0$ for $y \in [0, \lambda/2)$. Since $H_\lambda(z)$ is analytic for $\operatorname{Re} z > 0$, by the identity theorem for analytic functions (see, e.g., [23, Theorem 8.12]), we conclude that $H_\lambda(y) = 0$ for all $y \in [0, \infty)$, that is,

$$\int_0^\infty h_\lambda(x) e^{-yx} dx = 0, \quad y \geq 0.$$

By employing the substitution $x = -\ln t$, we can rewrite this as

$$\int_0^1 h_\lambda(-\ln t) t^{y-1} dt = 0, \quad y \geq 0.$$

In particular,

$$\int_0^1 h_\lambda(-\ln t) t^{n-1} dt = 0, \quad n \in \mathbb{N}.$$

By Lerch's theorem (see, Theorem 2.3), $h_\lambda(-\ln t) = 0$ for a.e. $t \in (0,1)$, that is, $h_\lambda(x) = 0$ for a.e. $x \in \mathbb{R}^+$. Finally, this implies that $f = 0$. □

3.3. **Proof of Theorem 1.1.** It follows from Lemma 3.1 and the hypotheses of the theorem that

$$\Phi_\lambda \subset L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \subset X(\mathbb{R}^+), \tag{3.9}$$

where $\Phi_\lambda = \{\varphi_{1,\lambda}, \varphi_{2,\lambda}, \dots\}$ and

$$\varphi_{n,\lambda}(x) := \varphi_n(\lambda x), \quad x \in \mathbb{R}^+, \quad n \in \mathbb{N},$$

are dilated Laguerre functions. Then [2, Ch. 1, Theorem 1.8] yields

$$L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \hookrightarrow X(\mathbb{R}^+). \tag{3.10}$$

Since $L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)$ is the associate space of $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, the continuous embedding in (3.10) and [2, Ch. 1, Proposition 2.10] imply that

$$X'(\mathbb{R}^+) \hookrightarrow L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+). \tag{3.11}$$

Let $G \in X^*(\mathbb{R}^+)$ be such that

$$G\varphi_{n,\lambda} = 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.12}$$

Since $X(\mathbb{R}^+)$ is separable, it follows from [2, Ch. 1, Corollaries 4.3 and 5.6] that the Banach dual $X^*(\mathbb{R}^+)$ of $X(\mathbb{R}^+)$ is canonically isometrically isomorphic to the associate space $X'(\mathbb{R}^+)$ of $X(\mathbb{R}^+)$. Therefore, there is a unique function $g \in X'(\mathbb{R}^+)$ such that

$$Gf = \int_0^\infty f(x)g(x)\,dx \quad \text{for all} \quad f \in X(\mathbb{R}^+). \tag{3.13}$$

It follows from (3.9) and (3.12)–(3.13) that

$$\int_0^\infty g(x)\varphi_n(\lambda x)\,dx = 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.14}$$

Since $g \in L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+)$, Theorem 3.2 and (3.14) imply that $g = 0$. Therefore, it follows from (3.13) that $G = 0$.

So, we have proved that if $G \in X^*(\mathbb{R}^+)$ satisfies (3.12), then $G = 0$. By a corollary of the Hahn-Banach theorem (see, e.g., [3, Ch. 7, Theorem 4.2]), the above fact is equivalent to the density of the linear span of Φ_λ in $X(\mathbb{R}^+)$. □

3.4. **Corollary of the main result for rearrangement-invariant Banach function spaces and variable Lebesgue spaces.**

Corollary 3.3. *Let $\lambda > 0$. If $X(\mathbb{R}^+)$ is either a separable rearrangement-invariant Banach function space or a separable variable Lebesgue space, then the linear span of Φ_λ is dense in $X(\mathbb{R}^+)$.*

This result follows immediately form Theorems 1.1 and 2.2.

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