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REDUCED (p, q)-DIFFERENTIAL TRANSFORM METHOD AND APPLICATIONS

PANKAJ JAIN, CHANDRANI BASU AND VIVEK PANWAR

Abstract. In this paper, Reduced Differential Transform method in the framework of (p, q)-calculus, denoted by $\mathrm{R}_{p,q}\mathrm{DT}$, has been introduced and applied in solving a variety of differential equations such as diffusion equation, 2D-wave equation, K-dV equation, Burgers equations and Ito system. While the diffusion equation has been studied for the special case $p = 1$, i.e., in the framework of q-calculus, the other equations have not been studied even in q-calculus.

1. Introduction

Differential equations (ordinary as well as partial) are important ingredients in all branches of science and engineering since most of the phenomenon occurring in nature can be modelled in terms of differential equations. A lot of new and efficient methods keep arising to solve variety of differential equations, both analytically as well as numerically. In the present paper, we are concerned with the Differential Transform Method (DTM) which is a semi-analytical method proposed by Zhou [22] to solve linear and non-linear initial value problems in electric circuit analysis. Since its introduction, DTM has been applied successfully in a variety of linear and non-linear differential equations.

As a generalization of DTM, Keskin and Oturance [14, 15, 16, 17] proposed the so called Reduced Differential Transform Method (RDTM) and immediately it got much attention and applied effectively to number of differential equations, initial value problems and boundary value problems.

During the last two decades, tremendous work has been done using quantum calculus or q-calculus which sometimes is called calculus without limits. It has been applied to several areas including differential equations, operator theory, inequalities, integral transforms, linear algebra etc. Although the idea of q-calculus was initiated by Jackson [6] but its much usage is no more than a couple of decades old. For some basics of q-calculus and certain applications, one may refer to the books

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Key words and phrases. Reduced Differential Transform Method; diffusion equation; 2D-wave equation; Burgers equation; K-dV equation; Ito system; (p, q)-calculus.

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[4, 10]. The q-calculus has been applied for RTDM to solve certain differential equations [7].

The quantum calculus has been generalized to post-quantum calculus, commonly known as (p,q)-calculus, [2, 3, 19]. Very recently, in [8, 9], the present authors have also contributed in the study of integral transforms in the (p,q)-framework.

The aim of the present paper is to develop RTDM in the framework of (p,q)-calculus and apply it to solve partial differential equations and initial value problems. In particular, we consider the following:

(i) Diffusion equation
(ii) 2D-Wave equation
(iii) K-dV equation
(iv) Burgers equation
(v) Ito system.

We point out that in the framework of q-calculus, diffusion equation was considered in [20]. To the best of our knowledge, the other equations and Ito system mentioned above have not been considered even in q-calculus.

The paper is organized as follows: Section 2 is preliminaries where we collect some basic notions of q and (p,q) calculus. Moreover, q-analogue of the reduced differential transform method has also been discussed here. In Section 3, we introduce and study the (p,q)-analogue of the reduced differential transform method. Applications of the results proved in Section 3 in solving diffusion equation, 2D-wave equation, K-dV equation and Burgers equation have been given in Section 4 whereas application for solving Ito system has been provided in Section 5. Finally, in Section 6, some concluding remarks have been provided.

2. Preliminaries

2.1. **q-Calculus.** Throughout this paper, we shall take $q \in (0,1)$. Here we shall give some basic notions and notations used in q-calculus. Let $x \in \mathbb{C}$, and $n \in \mathbb{N}$. The q-analogue of x and q-factorial of n are defined, respectively, by

$$[x]_q = \frac{1-q^x}{1-q},$$

and

$$[n]_q! = [1]_q[2]_q \dots [n]_q, \qquad [0]_q! = 1.$$

For $n, k \in \mathbb{N}$, we shall be using the notation

$$\binom{n}{k}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

The q-derivatives $D_q f$ of a function at $x \neq 0$ is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

For two functions f and g, the q-analogue of Leibnitz formula [10] is given by

$$D_q^n\Big(f(t)g(t)\Big) = \sum_{k=0}^{n} \binom{n}{k}_q D_q^k f(tq^{n-k}) D_q^{n-k} g(t).$$

If f is differentiable, then $D_q f(x)$ becomes the classical derivative $f^{'}(x)$ as $q \to 1$.

The q-Taylor formula [10] of f in some neighbourhood of a point a is given by

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-a)_q^k D_q^k f(a)}{[k]_q!},$$

where

$$(x-a)_q^k=(x-a)(x-qa)(x-q^2x)\dots(x-q^{k-1}a).$$

The q-analogue of exponential function [10] has two forms, given respectively by

$$e_q^x=\sum_{k=0}^{\infty}\frac{x^k}{[k]_q!}\qquad\text{and}\qquad E_q^x=\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}x^k}{[k]_q!}.$$

The partial q-derivatives of a multi-variable function $f(\vec{x})$, $\vec{x}=(x_1,x_2\dots x_n)$ with respect to a variable x_i is defined by

$$D_{q,x_i}f(\vec{x})=\frac{f(x_1,x_2,\dots,qx_i,\dots,x_n)-f(x_1,x_2,\dots,x_i,\dots,x_n)}{(q-1)x_i}.$$

2.2. (p,q)**-Calculus.** In this section, we give a brief introduction of (p,q)-calculus. Throughout this paper we shall take $0<q<p\le 1$. Let $x\in\mathbb{C}$, and $n\in\mathbb{N}$. The (p,q)-analogue of x and (p,q)-factorial of n are defined, respectively, by

$$[x]_{p,q}=\frac{p^x-q^x}{p-q},$$

and

$$[n]_{p,q}!=\prod_{k=1}^{n}[k]_{p,q},\qquad n\ge 1,\qquad [0]_{p,q}!=1.$$

For $n,k\in\mathbb{N}$, we shall be using the notation

$$\begin{bmatrix} n\\ k\end{bmatrix}=\frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}.$$

The (p,q)-derivative of a function f is defined by

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x}.$$

For two functions f and g, the (p,q)-analogue of Leibnitz formula [2] is given by

$$D_{p,q}^n\Big(f(x)g(x)\Big)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}D_{p,q}^k f(xp^{n-k})D_{p,q}^{n-k}g(xq^k).$$

The (p,q)-Taylor formula [19] of degree N in some neighbourhood of a point a is given by

$$f(x)=\sum_{k=0}^{N}p^{-\binom{k}{2}}\frac{D_{p,q}^k f(ap^{-k})}{[k]_{p,q}!}(x-a)_{p,q}^k,$$

where

$$(x-a)_{p,q}^k=(x-a)(px-qa)\dots(p^{k-1}x-q^{k-1}a).$$

The (p,q)-analogue of exponential function has three forms as given below:

$$e_{p,q}^x=\sum_{k=0}^{\infty}\frac{p^{\binom{k}{2}}x^k}{[k]_{p,q}!},\ E_{p,q}^x=\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}x^k}{[k]_{p,q}!}\ \text{and}\ \tilde{e}_{p,q}^x=\tilde{e}_{p,q}(x)=\sum_{k=0}^{\infty}\frac{x^k}{[k]_{p,q}!}.$$

For various properties and relationships among the three exponential forms, one may refer to [3].

The (p,q)-analogue of $\sin x$ and $\cos x$ [2] is given by:

$$\widetilde{\sin}_{p,q}(x) = \frac{\tilde{e}_{p,q}(ix) - \tilde{e}_{p,q}(ix)}{2i} \qquad \text{and} \qquad \widetilde{\cos}_{p,q}(x) = \frac{\tilde{e}_{p,q}(ix) + \tilde{e}_{p,q}(ix)}{2}. \tag{2.1}$$

The following theorem was proved in [2].

Theorem A. *Let f be (p,q)-differentiable of order n. Then*

$$f(p^n x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k p^{\binom{k}{2}} (p-q)^k D^k_{p,q} f(x). \tag{2.2}$$

The partial (p,q)-derivatives of a multi-variable function $f(x_1, x_2 \ldots x_n)$ with respect to the variable x_i is defined by

$$D_{p,q,x_i} f(\vec{x}) = \frac{f(x_1, x_2, \ldots, px_i, \ldots, x_n) - f(x_1, x_2, \ldots, qx_i, \ldots, x_n)}{(p-q)x_i}.$$

Note that when $p = 1$, the (p,q)-calculus reduces to the q-calculus. For more on basic properties of (p,q)-calculus, one may refer to [2], [19].

2.3. **Reduced q-differential Transforms.** Here and throughout, we shall consider all the functions of two variables to be of the product type, e.g., the function $u(x,t)$ is of the type $u_1(x)u_2(t)$.

Suppose that all q-differentials of a function $u(x,t)$ exist in some neighbourhood of $t = a$. The reduced q-differential transform (R$_q$DT) [7] of a function $u(x,t)$ at a point $t = a$ is defined by

$$U_k(x) := \frac{1}{[k]_q!} \left[D^k_{q,t} u(x,t) \right]_{t=a}$$

so that the q-differential inverse transform of $U_k(x)$ is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-a)^k_q.$$

Some properties of R$_q$DT are collected in the following table:

Function	R$_q$DT at $t = 0$
$\alpha u(x,t) \pm v(x,t)$	$\alpha U_k(x) \pm V_k(x)$
$x^m t^n$	$x^m \delta(k-n)$
$D_{q,x} u(x,t)$	$D_q U_k(x)$
$D^r_{q,t} u(x,t)$	$[k+1]_q [k+2]_q \ldots [k+r]_q U_{k+r}(x)$
$u(x,t)v(x,t)$	$\sum_{n=0}^{k} U_{k-n}(x) V_n(x)$

3. Reduced (p,q)-Differential Transforms

In this section, we introduce and study certain properties of reduced (p,q)-differential transform ($R_{p,q}$DT).

Definition 3.1. *Suppose that all (p,q)-differentials of $u(x,t)$ exist in some neighborhood of $t = ap^{-k}$. The* $R_{p,q}$DT *of $u(x,t)$ at a point $t = ap^{-k}$ is defined by*

$$U_k(x) = \frac{1}{[k]_{p,q}!}\left[D^k_{p,q,t}u(x,t)\right]_{t=ap^{-k}} \tag{3.1}$$

so that the (p,q)-differential inverse transform of $U_k(x)$ is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t - ap^{-k})^k_{p,q}p^{-\binom{k}{2}}. \tag{3.2}$$

Note. If there is no ambiguity likely to occur we are using the same symbol $U_k(x)$ to denote the R_qDT as well as $R_{p,q}$DT. Even in the literature, the same symbol has been used to denote the classical reduced differential transform (RDT). The usage will be clear according to the context.

Remark. *When $a = 0$, the (p,q)-differential inverse transform of $U_k(x)$ given in* (3.2) *has a simplified expression. Indeed, we have*

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(x)(t - ap^{-k})^k_{p,q}p^{-\binom{k}{2}} \\ &= \sum_{k=0}^{\infty} U_k(x)t^k p^{\binom{k}{2}}p^{-\binom{k}{2}} \\ &= \sum_{k=0}^{\infty} U_k(x)t^k. \end{aligned} \tag{3.3}$$

Throughout, $R_{p,q}$DT will be considered at $a = 0$. As an example, we obtain the $R_{p,q}$DT of a specific function in the following theorem:

Theorem 3.2. *If $w(x,t) = x^m t^n$ then the corresponding* $R_{p,q}$DT *is given by*

$$W_k(x) = x^m\delta(n-k),$$

where

$$\delta(k) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0. \end{cases}$$

Proof. From the definition, we have

$$\begin{aligned} W_k(x) &= \frac{1}{[k]_{p,q}!}\left[D^k_{p,q,t}(x^m t^n)\right]_{t=0} \\ &= \frac{x^m}{[k]_{p,q}!}\left[D^k_{p,q,t}(t^n)\right]_{t=0} \\ &= \begin{cases} x^m \dfrac{[k]_{p,q}!}{[k]_{p,q}!}, & k=n \\ x^m[n]_{p,q}[n-1]_{p,q}\dots[n-k-1]_{p,q}t^{n-k}, & k<n \\ 0, & k>n \end{cases} \\ &= x^m\delta(n-k) \end{aligned}$$

and we are done. □

In the following theorem, we provide some of the properties of $\mathrm{R}_{p,q}$DT:

Theorem 3.3. (a) *If* $w(x,t) = D_{p,q,x}u(x,t)$ *then* $W_k(x) = D_{p,q}U_k(x)$.
(b) *If* $w(x,t) = D^r_{p,q,t}u(x,t)$ *then*

$$W_k(x) = [k+1]_{p,q}[k+2]_{p,q}\dots[k+r]_{p,q}U_{k+r}(x).$$

(c) *If* $w(x,t) = u(x,t)v(x,t)$ *then* $W_k(x) = \sum_{n=0}^{k} U_{k-n}(x)V_n(x)$.
(d) *If* $w(x,t) = \alpha u(x,t) \pm v(x,t)$ *then* $W_k(x) = \alpha U_k(x) \pm V_k(x)$.

Proof. (a) We have

$$\begin{aligned} W_k(x) &= \frac{1}{[k]_{p,q}!}\left[D^k_{p,q,t}(D_{p,q,x}u(x,t))\right]_{t=0} \\ &= \frac{1}{[k]_{p,q}!}\left[D_{p,q,x}(D^k_{p,q,t}u(x,t))\right]_{t=0} \\ &= D_{p,q,x}\left[\frac{1}{[k]_{p,q}!}D^k_{p,q,t}u(x,t)\right]_{t=0} \\ &= D_{p,q}U_k(x). \end{aligned}$$

(b) We have

$$\begin{aligned} W_k(x) &= \frac{1}{[k]_{p,q}!}\left[D^k_{p,q,t}(D^r_{p,q,t}u(x,t))\right]_{t=0} \\ &= \frac{[k+r]_{p,q}!}{[k]_{p,q}!}\frac{1}{[k+r]_{p,q}!}\left[(D^{k+r}_{p,q,t}u(x,t))\right]_{t=0} \\ &= [k+r]_{p,q}\dots[k+1]_{p,q}U_{k+r}(x). \end{aligned}$$

(c) We have, in view of Theorem A

$$
\begin{aligned}
W_k(x) &= \frac{1}{[k]_{p,q}!}\Big[D^k_{p,q,t}\Big(u(x,t)v(x,t)\Big)\Big]_{t=0}\\
&= \frac{1}{[k]_{p,q}!}\sum_{n=0}^{k}\begin{bmatrix}k\\ n\end{bmatrix}\left[D^n_{p,q,t}u(x,tp^{k-n})D^{k-n}_{p,q,t}v(x,q^nt)\right]_{t=0}\\
&= \sum_{n=0}^{k}\frac{1}{[k-n]_{p,q}![n]_{p,q}!}\left[\Big(\sum_{i=0}^{k-n}\begin{bmatrix}k-n\\ i\end{bmatrix}t^ip^{\binom{i}{2}}(p-q)^iD^{n+i}_{p,q,t}u(x,t)\Big)\right.\\
&\qquad\left.\times\Big(\sum_{j=0}^{n}(-1)^j\begin{bmatrix}n\\ j\end{bmatrix}t^jq^{\binom{j}{2}}(p-q)^jD^{k-n+j}_{p,q,t}v(x,t)\Big)\right]_{t=0}.
\end{aligned}
$$

At $t=0$, all the terms of the inner series vanish except for $i=0$ and $j=0$ so that we have

$$
\begin{aligned}
W_k(x) &= \sum_{n=0}^{k}\frac{1}{[k-n]_{p,q}![n]_{p,q}!}D^{k-n}_{p,q,t}u(x,t)D^n_{p,q,t}v(x,t)\\
&= \sum_{n=0}^{k}U_{k-n}(x)V_n(x).
\end{aligned}
$$

(d) Straight forward. □

4. Applications to (p,q)-partial differential equations

We solve some (p,q)-partial differential equations using $\mathrm{R}_{p,q}$DT. In particular we solve (p,q)-diffusion equation, (p,q)-wave equations, (p,q)-KdV equation and (p,q)-Burgers equation as test problems.

Example 4.1. (p,q)-***Diffusion Equation:*** *Consider the* (p,q)*-diffusion equation*

$$D_{p,q,t}u(x,t) = D^2_{p,q,x}u(x,t), \tag{4.1}$$

subject to the initial condition

$$u(x,0) = \tilde{e}_{p,q}(x). \tag{4.2}$$

Applying $\mathrm{R}_{p,q}$DT *to* (4.1)*, we get*

$$[k+1]_{p,q}U_{k+1}(x) = D^2_{p,q}U_k(x) \qquad k=0,1,2,\ldots. \tag{4.3}$$

Also, the initial condition (4.2)*, gives*

$$U_0(x) = \tilde{e}_{p,q}(x). \tag{4.4}$$

For $k = 0, 1, 2\ldots$, we obtain from (4.3) *and* (4.4) *successively the following:*

$$U_1(x) = \frac{1}{[1]_{p,q}!}\tilde{e}_{p,q}(x)$$
$$U_2(x) = \frac{1}{[2]_{p,q}}U_1(x) = \frac{1}{[1]_{p,q}[2]_{p,q}}\tilde{e}_{p,q}(x) = \frac{1}{[2]_{p,q}!}\tilde{e}_{p,q}(x)$$
$$U_3(x) = \frac{1}{[3]_{p,q}}U_2(x) = \frac{1}{[3]_{p,q}!}\tilde{e}_{p,q}(x)$$
$$\vdots$$
$$U_k(x) = \frac{1}{[k]_{p,q}!}\tilde{e}_{p,q}(x)$$
$$\vdots$$

Now, applying the inverse transform (3.3), *we get*

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{[k]_{p,q}!}\tilde{e}_{p,q}(x)t^k = \tilde{e}_{p,q}(x)\tilde{e}_{p,q}(t) \tag{4.5}$$

which is the required solution of the given diffusion equation.

In Figures 1 and 2, we plot the graphs of the solution (4.5). *In Figure 1, we take $p = 1, q = 0.3$ and $k = 20$ which corresponds to the case of q-calculus. This has been compared with (p,q)-analogue by taking $p = 0.8$, $q = 0.6$ and $k = 20$ in Figure 2.*

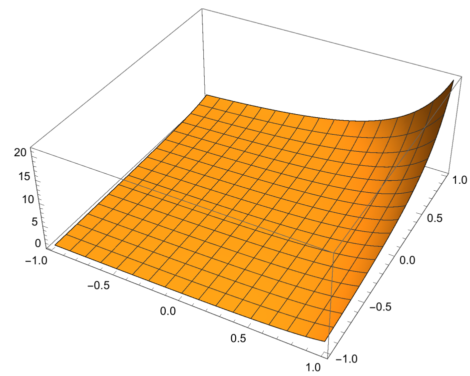

FIGURE 1. Graph of (4.5) for $p = 1$, $q = 0.3$, $k = 20$

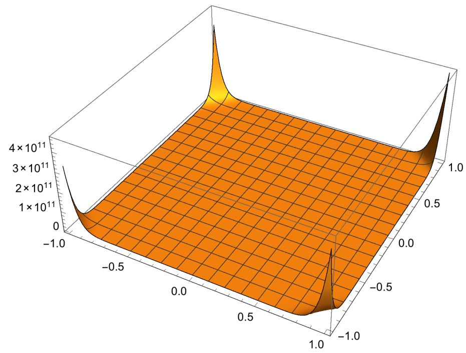

FIGURE 2. Graph of (4.5) for $p = 0.8$, $q = 0.6$, $k = 20$

Example 4.2. *Consider the following nonlinear (p, q)-partial differential equation*

$$D_{p,q,t}u(x,t) = u^2(x,t) + D_{p,q,x}u(x,t), \tag{4.6}$$

subject to the initial condition

$$u(x,0) = a + bx. \tag{4.7}$$

Applying $\mathrm{R}_{p,q}\mathrm{DT}$ *to* (4.6)*, we have for* $k = 0, 1, 2, \ldots$

$$[k+1]_{p,q}U_{k+1}(x) = \sum_{r=0}^{k} U_{k-r}(x)U_r(x) + D_{p,q}U_k(x). \tag{4.8}$$

From the initial condition (4.7)*, we have*

$$U_0(x) = a + bx. \tag{4.9}$$

Solving Equation (4.8) *with initial condition* (4.9)*, we successively get values of* $U_k(x)$ *as follows: For* $k = 0$,

$$[1]_{p,q}U_1(x) = U_0^2(x) + D_{p,q}U_0(x)$$

which gives

$$U_1(x) = a^2 + 2abx + b^2x^2 + b.$$

For $k = 1$

$$[2]_{p,q}U_2(x) = 2U_0(x)U_1(x) + D_{p,q}U_1(x)$$

so that

$$U_2(x) = \frac{1}{[1]_{p,q}[2]_{p,q}}\Big(2a^3 + 6a^2bx + 6ab^2x^2 + 2b^3x^3 + 2b^2x + [2]_{p,q}b^2x + 4ab\Big)$$

and so on. The inverse transform (3.3) *gives the solution of* (4.6) *as*

$$u(x,t) = a + bx + \big(a^2 + 2abx + b^2x^2 + b\big)t + \frac{1}{[2]_{p,q}!}\big(2a^3 + 6a^2bx + 6ab^2x^2 + 2b^2x + [2]_{p,q}b^2x + 2b^3x^3 + 4ab\big)t^2 + \ldots$$

Example 4.3. *Consider the following nonlinear (p, q)-partial differential equation*

$$D_{p,q,t}u(x,t) = D^2_{p,q,x}u(x,t) + D_{p,q,x}\big(xu(x,t)\big) \tag{4.10}$$

subject to the initial condition

$$u(x,0) = x^2.$$

Applying $\mathrm{R}_{p,q}\mathrm{DT}$ *to* (4.10), *we have for* $k=0,1,2,...$

$$[k+1]_{p,q}U_{k+1}(x)=D^2_{p,q}U_k(x)+D_{p,q}\big(xU_k(x)\big), \tag{4.11}$$

where

$$U_0(x)=x^2.$$

Solving (4.11), *we successively get the value of* $U_k(x)$ *as follows:*

$$U_1(x)=\frac{[2]_{p,q}+[3]_{p,q}x^2}{[1]_{p,q}},$$

$$U_2(x)=\frac{[3]_{p,q}[2]_{p,q}+[2]_{p,q}+[3]_{p,q}[3]_{p,q}x^2}{[1]_{p,q}[2]_{p,q}}$$

$$\vdots$$

Substituting all $U_k(x)$ *in* (3.3), *we obtain the series solution as*

$$u(x,t)=x^2+([2]_{p,q}+[3]_{p,q}x^2)\frac{t}{[1]_{p,q}}+\frac{[3]_{p,q}[2]_{p,q}+[2]_{p,q}+[3]_{p,q}[3]_{p,q}x^2}{[1]_{p,q}[2]_{p,q}}t^2+\dots$$

Example 4.4. (p,q)***-2D-Wave Equation:*** *Here we consider the following* (p,q)*-version of the 2D-wave equation given by*

$$D^2_{p,q,t}u(x,t)=D^2_{p,q,x}u(x,t)-3u(x,t),\qquad 0<x<\pi,\ t>0 \tag{4.12}$$

with boundary and initial conditions

$$u(0,t)=\widetilde{\sin}_{p,q}(2t),u(\pi,t)=\widetilde{\cos}_{p,q}(\pi)\widetilde{\sin}_{p,q}(2t)$$
$$u(x,0)=0,\ D_{p,q,t}u(x,0)=2\,\widetilde{\cos}_{p,q}(x). \tag{4.13}$$

Applying $\mathrm{R}_{p,q}\mathrm{DT}$ *to* (4.12), *we have*

$$[k+1]_{p,q}[k+2]_{p,q}U_{k+2}(x)=D^2_{p,q}U_k(x)-3U_k(x). \tag{4.14}$$

From initial conditions (4.13), *we have*

$$U_0(x)=0,\qquad U_1(x)=2\,\widetilde{\cos}_{p,q}(x)$$

so that the values of $U_k(x)$ *from* (4.14) *are obtained successively as follows:*

$$U_2(x)=0,\ U_3(x)=\frac{-8\,\widetilde{\cos}_{p,q}(x)}{[3]_{p,q}!},\ U_4(x)=0,\ U_5(x)=\frac{32\,\widetilde{\cos}_{p,q}(x)}{[5]_{p,q}!}$$
$$U_6(x)=0,\ U_7(x)=\frac{-128\,\widetilde{\cos}_{p,q}(x)}{[7]_{p,q}!},\dots.$$

In general,

$$U_k(x)=\begin{cases}\dfrac{(-1)^{\frac{k-1}{2}}2^k\,\widetilde{\cos}_{p,q}(x)}{[k]_{p,q}!}, & \text{for } k \text{ is odd}\\ 0, & \text{for } k \text{ is even.}\end{cases}$$

Now, applying, the inverse transform, we obtain the solution of (4.12) *as*

$$u(x,t)=\widetilde{\cos}_{p,q}(x)\sum_{k=0}^{\infty}\frac{(-1)^k2^{2k+1}t^{2k+1}}{[2k+1]_{p,q}!}=\widetilde{\cos}_{p,q}(x)\,\widetilde{\sin}_{p,q}(2t). \tag{4.15}$$

In Figure 3 and 4, we plot the graphs of the solution (4.15). Figure 3 corresponds to q analogue solution where we have $p = 1$, $q = 0.3$ and $k = 20$ while in Figure 4, we have taken $p = 0.8$, $q = 0.6$ and $k = 20$.

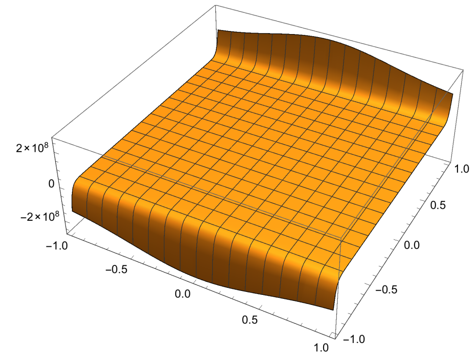

FIGURE 3. Graph of (4.15) for $p = 1$, $q = 0.3$, $k = 20$

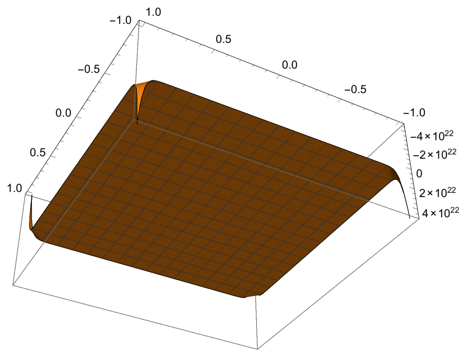

FIGURE 4. Graph of (4.15) for $p = 0.8$, $q = 0.6$, $k = 20$

Example 4.5. (p,q)-***K-dV Equation:*** *A variant of the wave equation, the so called Korteweg-de Vries (K-dV) equation* [14] *(see also* [13, 18]*) is the following:*

$$u_t(x,t) + (m+1)(m+2)u_x(x,t)u^m(x,t) + u_{xxx}(x,t) = g(x,t).$$

We discuss the (p,q)-version of the above equation. For simplicity we take $m = 1$ and $g(x,t) = x^2t^2$, i.e., we consider the equation

$$D_{p,q,t}u(x,t) + 6u(x,t)D_{p,q,x}u(x,t) + D^3_{p,q,x}u(x,t) = x^2t^2 \tag{4.16}$$

with the initial condition

$$u(x,0) = 1. \tag{4.17}$$

Applying $\mathrm{R}_{p,q}$DT *to* (4.16)*, we have*

$$[k+1]_{p,q}U_{k+1}(x) = -6\sum_{r=0}^{k} U_r(x)D_{p,q}U_{k-r}(x) - D^3_{p,q}U_k(x) + \delta(k-2)x^2, \tag{4.18}$$

where the initial condition (4.17) *gives*

$$U_0(x) = 1.$$

Now, solving (4.18)*, we successively get the value of* $U_k(x)$ *as follows:*

$$U_1(x) = 0, \qquad U_2(x) = 0, \qquad U_3(x) = \frac{x^2}{[3]_{p,q}},$$

$$U_4(x) = -6\frac{[2]_{p,q}x}{[3]_{p,q}[4]_{p,q}}, \qquad U_5(x) = 36\frac{[2]_{p,q}}{[3]_{p,q}[4]_{p,q}[5]_{p,q}}, \qquad U_6(x) = 0,$$

$$U_7(x) = -6\frac{[2]_{p,q}x^4}{[3]_{p,q}[3]_{p,q}[7]_{p,q}}, \dots$$

Finally, applying the inverse transform (3.3)*, we obtain the solution of* (4.16) *as*

$$u(x,t) = 1 + \frac{x^2}{[3]_{p,q}}t^3 - 6\frac{[2]_{p,q}x}{[3]_{p,q}[4]_{p,q}}t^4 + 36\frac{[2]_{p,q}}{[3]_{p,q}[4]_{p,q}[5]_{p,q}}t^5 + \dots$$

Example 4.6. (p,q)***-Burgers equation:*** *Consider the* $(n+1)$ *dimensional* (p,q)*-Burgers equation*

$$D_{p,q,t}u(\vec{x},t) = D^2_{p,q,x_1}u(\vec{x},t) + D^2_{p,q,x_2}u(\vec{x},t) + \dots + D^2_{p,q,x_n}u(\vec{x},t) + u(\vec{x},t)D_{p,q,x_1}u(\vec{x},t) \tag{4.19}$$

with initial condition

$$u(\vec{x},0) = u(x_1, x_2, \dots x_n, 0) = x_1 + x_2 + \dots + x_n. \tag{4.20}$$

Applying $\mathrm{R}_{p,q}\mathrm{DT}$ *to* (4.19)*, we have for* $k = 0, 1, 2, \dots$

$$[k+1]_{p,q}U_{k+1}(\vec{x}) = \sum_{i=1}^{n} D^2_{p,q,x_i}U_k(\vec{x}) + \sum_{r=0}^{k} U_r(\vec{x})D_{p,q,x_1}U_{k-r}(\vec{x}). \tag{4.21}$$

Now, using (4.20)*, we obtain*

$$U_0(\vec{x}) = \sum_{i=1}^{n} x_i.$$

Using which in (4.21) *we successively get the values of* $U_k(\vec{x})$ *as follows*

$$U_1(\vec{x}) = \sum_{i=1}^{n} x_i$$

$$U_2(\vec{x}) = \frac{2}{[2]_{p,q}} \sum_{i=1}^{n} x_i$$

$$U_3(\vec{x}) = \frac{4 + [2]_{p,q}}{[2]_{p,q}[3]_{p,q}} \sum_{i=1}^{n} x_i$$

$$U_4(\vec{x}) = \frac{8 + 2[2]_{p,q} + 4[3]_{p,q}}{[2]_{p,q}[3]_{p,q}[4]_{p,q}} \sum_{i=1}^{n} x_i$$

$$\vdots$$

Finally, applying the inverse transform, we get the solution of (4.19) as

$$u(\vec{x},t) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i t + \frac{2}{[2]_{p,q}} \sum_{i=1}^{n} x_i t^2 + \frac{4 + [2]_{p,q}}{[2]_{p,q}[3]_{p,q}} \sum_{i=1}^{n} x_i t^3 + \dots.$$

5. Application to (p,q)-Ito system

In this section, for the sake of simplicity, we shall write u, v, w, z for respectively, $u(x,t)$, $v(x,t)$, $w(x,t)$, $z(x,t)$.

Ito [5] considered the following system of PDE

$$\left.\begin{aligned} u_t &= v_x \\ v_t &= -2v_{xxx} - 6(uv)_x \end{aligned}\right\} \tag{5.1}$$

which is now-a-days usually referred to as Ito system. Many people have contributed in this direction giving various generalizations of it. There are several methods to solve this system and its variants. One may refer to [11], [12], [21] in this direction. In [11], the so called coupled system has been considered which is given by

$$\left.\begin{aligned} u_t &= v_x \\ v_t &= -2v_{xxx} - 6(uv)_x - 12ww_x \\ w_t &= w_{xxx} + 3uw_x \end{aligned}\right\} \tag{5.2}$$

and in [12], the following generalized Ito system has been considered.

$$\left.\begin{aligned} u_t &= v_x \\ v_t &= -2v_{xxx} - 6(uv)_x + aww_x + bzw_x \\ &\quad + cwz_x + dzz_x + fw_x + gz_x \\ w_t &= w_{xxx} + 3uw_x \\ z_t &= z_{xxx} + 3uz_x, \end{aligned}\right\} \tag{5.3}$$

where a, b, c, d, f and g are arbitrary constants. Al-Sawoor and Al-Amr [1] used the RDT method to (5.3). In this section, we consider the (p,q)-version of the system (5.2) and (5.3), and solve them using $\mathrm{R}_{p,q}$DT method.

Example 5.1. *Consider the following coupled (p,q)-Ito system*

$$\left.\begin{aligned} D_{p,q,t}u &= D_{p,q,x}v \\ D_{p,q,t}v &= -2D^3_{p,q,x}v - 6D_{p,q,x}(uv) - 12wD_{p,q,x}w \\ D_{p,q,t}w &= D^3_{p,q,x}w + 3uD_{p,q,x}w \end{aligned}\right\} \tag{5.4}$$

with initial conditions

$$u(x,0) = \frac{ax}{3c}, \qquad v(x,0) = \frac{-a^2x^2}{2c^2}, \qquad w(x,0) = 0. \tag{5.5}$$

Applying the $\mathrm{R}_{p,q}$DT *to the system* (5.4)*, we have*

$$\left.\begin{aligned} [k+1]_{p,q}U_{k+1}(x) &= D_{p,q}V_k(x) \\ [k+1]_{p,q}V_{k+1}(x) &= D^3_{p,q}V_k(x) - 6D_{p,q}\left(\sum_{r=0}^{k} U_r(x)V_{k-r}(x)\right) \\ &\quad - 12\sum_{r=0}^{k} W_r(x)D_{p,q}W_{k-r}(x) \\ [k+1]_{p,q}W_{k+1}(x) &= D^3_{p,q}W_k(x) + 3\sum_{r=0}^{k} U_r(x)D_{p,q}W_{k-r}(x), \end{aligned}\right\} \tag{5.6}$$

where the initial conditions (5.5) *gives*

$$U_0(x) = \frac{ax}{3c}, \qquad V_0(x) = \frac{-a^2x^2}{2c^2}, \qquad W_0(x) = 0$$

using which we successively get the values of $U_k(x), V_k(x)$ *and* $W_k(x)$ *in* (5.6) *as follow:*

$$U_1(x) = \frac{-[2]_{p,q}a^2x}{2c^2}, \qquad V_1(x) = \frac{[3]_{p,q}a^3x^2}{c^3}, \qquad W_1(x) = 0.$$
$$U_2(x) = \frac{[3]_{p,q}a^3x}{c^3} \qquad V_2(x) = \frac{-a^4x^2[3]_{p,q}}{[2]_{p,q}2c^4}\Big(4[3]_{p,q} + 3[2]_{p,q}\Big), \qquad W_2(x) = 0\dots.$$

Finally, applying the inverse transform (3.3) *we obtain the solution of* (5.4)

$$\begin{aligned} u(x,t) &= \frac{ax}{3c} - \frac{[2]_{p,q}a^2x}{2c^2}t + \frac{[3]_{p,q}a^3x}{c^3}t^2 + \dots \\ v(x,t) &= \frac{-a^2x^2}{2c^2} + \frac{[3]_{p,q}a^3x^2}{c^3}t - \frac{a^4x^2[3]_{p,q}}{[2]_{p,q}2c^4}\Big(4[3]_{p,q} + 3[2]_{p,q}\Big)t^2 + \dots\dots \\ w(x,t) &= 0. \end{aligned}$$

Example 5.2. *Consider the following generalized* (p,q)*-Ito system*

$$\left.\begin{aligned} D_{p,q,t}u &= v_x \\ D_{p,q,t}v &= -2v_{xxx} - 6(uv)_x - 6zw_x - 6wz_x \\ D_{p,q,t}w &= w_{xxx} + 3uw_x \\ D_{p,q,t}z &= z_{xxx} + 3uz_x, \end{aligned}\right\} \tag{5.7}$$

with initial conditions

$$\left.\begin{aligned} u(x,0) &= \frac{-b_2 + 4\mu^4}{6\mu^2} - 2\mu^2\tanh^2(\mu\mathrm{x}) \\ v(x,0) &= \frac{-b_2^2 + 4f_1t_1\mu^2 - 8b_2\mu^4}{8\mu^4} + b_2\tanh^2(\mu\mathrm{x}) \\ w(x,0) &= -\frac{f_1t_0}{t_1} + f_1\tanh(\mu\mathrm{x}) \\ z(x,0) &= t_0 + t_1\tanh(\mu\mathrm{x}) \end{aligned}\right\} \tag{5.8}$$

where μ, b_2, t_0, t_1 *and* f_1 *are arbitrary constants. Applying the* $\mathrm{R}_{p,q}\mathrm{DT}$ *to the system* (5.7), *we have*

$$\left.\begin{aligned}
[k+1]_{p,q}U_{k+1}(x) =& V_k^{'}(x)\\
[k+1]_{p,q}V_{k+1}(x) =& -2V_k^{'''}(x) - 6\sum_{r=0}^{k} U_r(x)V_{k-r}^{'}(x) - 6\sum_{r=0}^{k} V_r(x)U_{k-r}^{'}(x)\\
& - 6\sum_{r=0}^{k} Z_r(x)W_{k-r}^{'}(x) - 6\sum_{r=0}^{k} W_r(x)Z_{k-r}^{'}(x)\\
[k+1]_{p,q}W_{k+1}(x) =& W_k^{'''}(x) + 3\sum_{r=0}^{k} U_r(x)W_{k-r}^{'}(x)\\
[k+1]_{p,q}Z_{k+1}(x) =& Z_k^{'''}(x) + 3\sum_{r=0}^{k} U_r(x)Z_{k-r}^{'}(x).
\end{aligned}\right\} \tag{5.9}$$

Now, the initial condition (5.8), *gives*

$$\left.\begin{aligned}
U_0(x) &= \frac{-b_2+4\mu^4}{6\mu^2} - 2\mu^2\tanh^2(\mu\mathrm{x})\\
V_0(x) &= \frac{-b_2^2+4f_1t_1\mu^2-8b_2\mu^4}{8\mu^4} + b_2\tanh^2(\mu\mathrm{x})\\
W_0(x) &= -\frac{f_1t_0}{t_1} + f_1\tanh(\mu\mathrm{x})\\
Z_0(x) &= t_0 + t_1\tanh(\mu\mathrm{x}).
\end{aligned}\right\}$$

Using which we successively get the values of $U_k(x)$, $V_k(x)$, $W_k(x)$ *and* $Z_k(x)$ *in* (5.9) *as*

$$U_1(x) = 2b_2\mu\tanh(\mu x)\,\mathrm{sech}^2(\mu x), \qquad V_1(x) = -\frac{b_2^2}{\mu}\tanh(\mu x)\,\mathrm{sech}^2(\mu x),$$

$$W_1(x) = -\frac{b_2f_1}{2\mu^2}\,\mathrm{sech}^2(\mu x), \qquad Z_1(x) = -\frac{b_2t_1}{2\mu}\,\mathrm{sech}^2(\mu x), \ldots$$

Finally, applying the inverse transform (3.2) *we obtain the solution of* (5.7)

$$\begin{aligned}
u(x,t) &= \frac{-b_2+4\mu^4}{6\mu^2} - 2\mu^2\tanh^2(\mu x) + 2b_2\mu\tanh(\mu x)\,\mathrm{sech}^2(\mu x)t + \ldots\\
v(x,t) &= \frac{-b_2^2+4f_1t_1\mu^2-8b_2\mu^4}{8\mu^4} + b_2\tanh^2(\mu\mathrm{x}) - \frac{\mathrm{b}_2^2}{\mu}\tanh(\mu\mathrm{x})\,\mathrm{sech}^2(\mu\mathrm{x})\mathrm{t} + \ldots\\
w(x,t) &= -\frac{f_1t_0}{t_1} + f_1\tanh(\mu x) - \frac{b_2f_1}{2\mu^2}\,\mathrm{sech}^2(\mu x)t + \ldots\\
z(x,t) &= t_0 + t_1\tanh(\mu x) - \frac{b_2t_1}{2\mu}\,\mathrm{sech}^2(\mu x)t + \ldots
\end{aligned}$$

6. Conclusion

In this paper, Reduced Differential Transform method in the framework of (p,q)-calculus, denoted by $\mathrm{R}_{p,q}\mathrm{DT}$, has been introduced and applied in solving a variety of partial differential equations and initial value problems. In particular, the following have been considered:

(i) Diffusion equation
(ii) 2D-Wave equation
(iii) K-dV equation
(iv) Burgers equation
(v) Ito system.

While the diffusion equation has been studied for the special case $p = 1$, i.e., in the framework of q-calculus in [20], the other equations have not been studied even in q-calculus.

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Pankaj Jain
Department of Mathematics South Asian University Akbar Bhawan, Chanakya Puri, New Delhi-110021, India
E-mail address: pankaj.jain@sau.ac.in, pankajkrjain@hotmail.com

Chandrani Basu
Department of Mathematics South Asian University Akbar Bhawan, Chanakya Puri, New Delhi-110021, India
E-mail address: chandrani.basu@gmail.com

Vivek Panwar
Department of Mathematics South Asian University Akbar Bhawan, Chanakya Puri, New Delhi-110021, India
E-mail address: vivek.pan1992@gmail.com