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CESÁRO PARANORMED SEQUENCE SPACE BASED INTUITIONISTIC FUZZY DISTANCE MEASURE

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Abstract. In machine learning, distance measure plays an important role in defining the similarity between two data-items. In the paper, we discuss some of the drawbacks of distance measures (metrics) with their possibly induced clustering algorithms. Further, to overcome the drawbacks, we propose a novel intuitionistic fuzzy distance measure associated with generalized cesáro paranormed sequence space $Ces_p^q(\mathcal{F})$. We also discuss some geometric properties of $Ces_p^q(\mathcal{F})$. Moreover, the proposed distance measure is utilized in k-mean clustering algorithm to propose fuzzy c-mean clustering algorithm for $Ces_p^q(\mathcal{F})$.

1. Introduction

Clustering is a widely used technique of organizing data objects into different groups known as clusters, such that elements of a group are similar to each other and are dissimilar to others. It is a phenomenon that often arises in machine learning and computational intelligence. It is utilized in various fields such as supply chain management, image processing, artificial intelligence, reliability engineering, data mining, pattern recognition, etc. The clustering technique is mainly classified into two approaches: (i) hierarchical and (ii) non-hierarchical (partitioning) approach. A tree-shaped structure is employed in hierarchical clustering to arrange data-set into nested clusters by using the dendrogram tree. On the other hand, the division of data-set into k number of non-overlapping clusters such that each data object is in exactly one group is known as a non-hierarchical or partitional clustering approach. The main difference between these two clustering approaches is that k is unknown in the hierarchical approach however is known in the partitioning approach.

The widely known and fastest partitional clustering technique is a k-means clustering algorithm. The k-means clustering algorithm is a fundamental task in unsupervised machine learning with very diverse scientific and industrial applications due to its easy implementation, efficiency, and empirical success. In general, for the assigned input data $X = \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^n$, where n represents the total amount of objects and n is the attribute value or dimensions. The classical

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k-means clustering algorithm classifies the given data points X into k number of distinct clusters $c_1, c_2, \dots, c_k$ such that $c_i \cap c_j = \phi,\ 1 \leq i \neq j \leq k$. The objective of the k-means clustering is to minimize the sum of squared error (SSE) function. In other words, the algorithm iteratively forming the clusters with the data points until the minimum SSE criterion cannot be achieved. The SSE criterion function is mathematically calculated through the metric space, particularity Euclidean distance measure among the data points and their respective cluster centroids. Drineas et al. [42] proved that determining a suitable choice of initial seeds and distance measure for the k-means clustering algorithm belongs to an NP-hard class. Therefore, it requires an appropriate selection technique of seeding and measure function. Several variants of the k-means algorithm were proposed and in almost all the variants, seeding are randomly chosen. Moreover, the clusters induced by the k-means algorithm vary on the variation of distance measure. To date, there is no robust procedure to evaluate optimal seeds and distance measure as per the clustering requirement of datasets.

In literature, various distance measures were used to introduce different variants of clustering algorithm. One of the most commonly uses measure is l_p- metric (see [43],[44],[45],[46]). A spherical or ball-shaped geometry is formed around the clusters when Euclidean distance measure is used. The topology of boundary around a cluster source plays a crucial role in enhancing clustering results. Each distance measure has a well defined geometrical structure, so rigidity is found in the shape of clusters. In order to reduce rigidity in the shape of the clusters, adaptive distance measure are incorporated for clustering. The adaptive distance measures only modifies the size of clusters without changing their original shape. If length of a point equals to zero, it does not imply that the point itself is zero. In such case, normed space becomes a paranormed space (see Def. 1.1). The flexibility in the paranorm can be efficiently used to modify the shape of the clusters as per requirement of the dataset.

The major contribution in the field of partitional clustering is due to introduction of k-means clustering algorithm by MacQueen [24]. The k-means clustering algorithm was introduced by MacQueen [24]. Let $\{x_1, x_2, \dots\}$ is a dataset to be clustered into k different clusters (say C_m, $m = 1, 2, \dots, k$). The k-means clustering work as to determine the partition of the data in such a way that the squared error between the empirical mean and points of the cluster is minimum. Suppose z_m is the mean of the m_{th} cluster C_m, then, the squared error between z_m and points in C_m is defined as

$$J(C_m) = \sum_{x_i \in C_m} \|x_i - z_m\|^2$$

The main purpose of the k-means clustering algorithm is to minimize the overall squared error for all clusters k, that is to minimize the following function (objective function).

$$J(C) = \sum_{m=1}^{k} \sum_{x_i \in C_m} \|x_i - z_m\|^2$$

which is based on minimum distance of the points from the center (to know more about k-mean and its variance, please see [8, 37, 6]). Thus, this algorithm yields different results on varying distance measure, so clustering results obtained through it can be further enhanced by choosing an appropriate distance measure. Therefore, distance measure has a vital role in the clustering process.

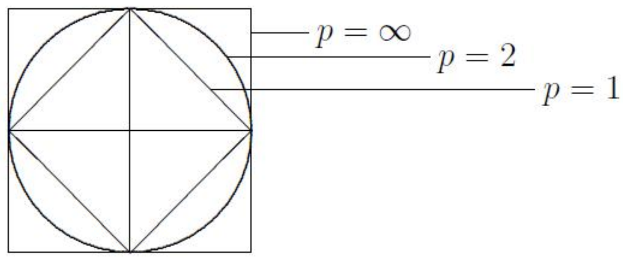

FIGURE 1. Geometry of l_p distance measure.

Clustering process is usually carried out through l_2 distance measure [15], but, due to its trajectory some time it fails to offer good results. Suppose two points x and y are selected on the boundary of the square (case $p = 1$) and let z be the center (see Figure 1), then l_1 will fail to distinguish x and y, but; these points may be distinguished by l_2. If points x and y lie on the circumference of the circle. then l_2 will fail to distinguish them. Moreover, l_p $(p \geq 1)$ distance measures are not flexible, so, these can not be modified as per the need of the clustering problem. Hence, clustering results derived through distance dependent algorithms basically depend upon two properties of distance measure: (1) trajectory, (2) flexibility. Till now, we have not come across to any distance measure which offers good result over every kind of clustering problems. Thus other variants of l_p distance measures can also be used for the clustering. For example, the distance measure of sequence space $l^{p,q}$, $1 \leq p, q \leq \infty$ introduced by Kellogg [20] and further studied by Jovanovic and Rakocevic [18], Oscar and Carme [5] and Ivana et al. [12] offer more flexibility in comparison to distance measure of l_p due to involvement of one more parameter q. In [32], Sencimen and Pehlivan discussed the boundedness of the sequence in probabilistic norm space. Sargent [33] introduced another interesting sequence spaces $m(\phi)$ and $n(\phi)$ closely related to l_p. Some useful extensions of $m(\phi)$ and $n(\phi)$ sequence spaces were proposed by Tripathy and Sen [36], Mursaleen [27, 28], Vakeel [22] etc. Malkowsky et al. defined matrix mapping into strong cesaro sequence space [10] and studied modulus function [26]. Recently, Khan et al. [21] have used distance measure of double sequence for first time to cluster the objects. Başar and Şever [?], defined double sequence space $\mathcal{L}_p$ which is a general form of usual l_p sequence space. In [13], Esi and Hazarika proposed double sequence spaces of interval numbers defined by Orlicz function. The Orlicz function was used by Savas and Richard [34, 26] for proposing some double sequence spaces. The double sequence spaces are also studied in fuzzy setting in [16, 35]. In this paper, we proposed a new intuitionistic fuzzy distance measure associated with generalized cesaro paranormed sequence space $Ces_p^q(\mathcal{F})$ in order to bring more flexibility.

2. Preliminaries

1.1. Intuitionistic Fuzzy Set (IFS) [1]
An Atanassov intuitionistic fuzzy set A in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ is of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \tag{2.1}$$

Here μ_A: $X \to$ [0,1] and ν_A: $X \to$ [0,1] simultaneously assigns membership value and non-membership value respectively to each element $x \in X$ with respect to A, if

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{2.2}$$

Hence, the membership value lies in the interval $[\mu_A(x),\ \mu_A(x) + \pi_A(x)]$. If $\pi_A(x) = 0$, then $\nu_A(x) = 1 - \mu_A(x)$ and in this case, A reduces to fuzzy set (FS).

1.2. Paranorm [26] Let X be a vector space, a paranorm P is a function $P : X \to \mathbb{R}$ which satisfy the following property:

(i) $P(0) \geq 0$.
(ii) $P(x) \geq 0 \ \forall \ x \ \in X$.
(iii) $P(-x) = P(x) \ \forall \ x \ \in X$.
(iv) $P(x + y) \leq P(x) + P(y) \ \forall \ x, y \ \in X$.
(v) let $\{a_i\}$ be a sequence of scalars such that $a_i \to a$ and let x_i be a sequence of vectors with $\lim_{i\to\infty} P(x_i - x) \to 0$, then $\lim_{i\to\infty} P(a_i x_i - ax) \to 0$.

1.3. Modulus Function [30]: A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

(i) $f(x) = 0$ if and only if $x = 0$.
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$ and $y \geq 0$.
(iii) f is increasing.
(iv) f is continuous from the right of 0.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (iv) that f is cotinuous on $[0, \infty)$. Furthermore, we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$ from condition (ii), and so $f(x) = f(nx\frac{1}{n}) \leq nf(x/n)$, hence

$$\frac{1}{n} f(x) \leq f(x/n)$$

for all $n \in \mathbb{N}$. A modulus function may be bounded or unbounded. For example, $f(x) = x^p$ for $0 < p \leq 1$ is unbounded, but $f(x) = x/(1 + x)$ is bounded. Now for a sequence space w. The sequence space $w(f)$ is defined as $w(f) = \{x = (x_k) : (f(|x_k|)) \in w\}$ for a modulus f [25, 31].

An extension of $w(f)$ has been done in literature by considering the sequence of modulus function $\mathcal{F} = (f_k)$, i.e. $w(\mathcal{F})= \{x = (x_k) : (f_k(|x_k|)) \in w\}$.

1.4. Uniformly Convexity of paranorm [7]: A paranorm P is said to be *uniformly convex* (UC) if, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for $x, y \in S(X)$, $P(x - y) > \epsilon$ implies

$$P\left(\frac{1}{2}(x + y)\right) < 1 - \delta$$

1.5. ϵ-separated sequence [19]: A sequence (x_n) is said to be ϵ-*separated sequence*, if for some $\epsilon > 0$,

$$sep(x_n) = \inf\{P(x_n - x_m) : n \neq m\} > \epsilon.$$

1.6. Property β [19]: A paranormed space X is said to have *property* β if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $sep(x_n) \geq \epsilon$, there is an index k such that

$$P\left(\frac{x + x_k}{2}\right) \geq 1 - \delta, \text{ for some } k \in \mathbb{N}.$$

In [19], a new geometric constant connected with the packing constant (see [23]) and with the Banach-Saks property was defined as follows:

$$C(X) = \sup\{A(x_n) : (x_n) \text{ is a weakly null sequence in } S(X)\}.$$

where $A(x_n) = \lim_{n\to\infty} \inf\{P(x_i + x_j) : i, j \geq n, i \neq j\}$ for a sequence $(x_n) \subset X$.

3. Generalized Cesáro Paranormed Sequence Space

In this section, we define generalized Cesáro paranormed space using sequence of moduli $(Ces_p^q(\mathcal{F}))$ as follows:

Let $p \in [1, \infty)$ and $q = \{q_k\}$ be a bounded sequence of positive real numbers such that

$$Q_n = \sum_{k=0}^{n} q_k, \ n \in \mathbb{N},$$

$$Ces_p^q(\mathcal{F}) = \left\{x = (x_i) : \left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_i|)\right)^p\right)^{\frac{1}{p}} < \infty\right\}.$$

Theorem 3.1. $Ces_p^q(\mathcal{F})$ is a paranormed space with respect to paranorm.

$$P(x) = \left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_i|)\right)^p\right)^{\frac{1}{p}}$$

Proof. By using the conditions of the paranormed and modulas function, it is easy to verify that P satisfies all the conditions of Def. (1.2).

3.1. **Some geometric properties of $Ces_p^q(\mathcal{F})$. Lemma 3.1.** Let $x, y \in Ces_p^q(\mathcal{F})$. Then for any $\epsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that

$$|P(x + y) - P(x)| < \epsilon$$

whenever

$$P(x) \leq L \text{ and } P(y) \leq \delta.$$

Proof. For any fix $\epsilon > 0$ and $L > 0$, take $\beta = 2^{-1}L^{-1}\epsilon$ and $\delta = 2^{-1}\epsilon$. Then for any $x, y \in Ces_p^q(\mathcal{F})$ with $P(x) \leq L$ and $P(y) \leq \delta$, we have

$$
\begin{aligned}
P(x+y) =&\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_i+y_i|)\Bigg)^p\Bigg)^{\frac{1}{p}}\\
\leq&\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n}\Bigg(q_i f_i(|x_i|)+q_i f_i(|y_i|)\Bigg)\Bigg)^p\Bigg)^{\frac{1}{p}}\\
=&\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n}\Bigg((1-\beta)q_i f_i(|x_i|)+\beta\Bigg(q_i f_i(|x_i|)+\frac{q_i f_i(|y_i|)}{\beta}\Bigg)\Bigg)\Bigg)^p\Bigg)^{\frac{1}{p}}\\
=&\Bigg(\sum_{n=1}^{\infty}\Bigg((1-\beta)\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_i|)+\beta\frac{1}{Q_n}\sum_{i=1}^{n}\Bigg(q_i f_i(|x_i|)+\frac{q_i f_i(|y_i|)}{\beta}\Bigg)\Bigg)^p\Bigg)^{\frac{1}{p}}\\
\leq&(1-\beta)\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_i|)\Bigg)^p\Bigg)^{\frac{1}{p}}+\beta\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n}\Bigg(q_i f_i(|x_i|)+\frac{q_i f_i(|y_i|)}{\beta}\Bigg)\Bigg)^p\Bigg)^{\frac{1}{p}}\\
\leq&\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_i|)\Bigg)^p\Bigg)^{\frac{1}{p}}+\frac{\beta}{2}\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n} 2q_i f_i(|x_i|)\Bigg)^p\Bigg)^{\frac{1}{p}}+\frac{\beta}{2}\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n} \frac{2q_i f_i(|y_i|)}{\beta}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
\leq&p(x)+\frac{\epsilon}{2}+\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|y_i|)\Bigg)^p\Bigg)^{\frac{1}{p}}\\
\leq&P(x)+\epsilon.
\end{aligned}
$$

□

Theorem 3.2. The space $Ces_p^q(\mathcal{F})$ satisfy the property β.

Proof. Let if possible, $Ces_p^q(\mathcal{F})$ does not have the property β. Then there exists $\epsilon_\circ > 0$ such that, for any $\delta \in \left(0, \frac{\epsilon_\circ}{(1+2^{p+1})}\right)$, there is a sequence $(x_n) \subset S(Ces_p^q(\mathcal{F}))$ with $sep(x_n) > \epsilon_\circ^{1/p}$ and an element $x_\circ \in S(Ces_p^q(\mathcal{F}))$ such that

$$P\left(\frac{x_n + x_\circ}{2}\right) > 1 - \delta, \text{ for any } n \in \mathbb{N}.$$

Fix $\delta \in \left(0, \frac{\epsilon_\circ}{(1+2^{p+1})}\right)$. We want to show that

$$\lim_{j\to\infty}\sup_k \sum_{n=j+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_k^i|)\right)^p \leq \frac{2^{1+p}\delta}{2^p-1}. \tag{3.1.a}$$

Otherwise, without loss of generality, we can assume that there exists a sequence (j_k) such that $j_k \to \infty$ as $k \to \infty$ and

$$\sum_{n=j_k+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_k^i|)\right)^p > \frac{2^{1+p}\delta}{2^p-1} \text{ for every } k \in \mathbb{N}. \tag{3.1.b}$$

Let $\delta > 0$ be a real number corresponding to $\epsilon = \delta$ and $L = 1$ in Lemma 3.1. By absolute continuity of the distance of $x_\circ$, there exists a positive integer n_1 such that

$$P(x_\circ \chi_{n_1, n_1+1, n_1+2, \cdots})^p = \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_\circ^i|)\right)^p < \delta_1$$

Choose k so large that $j_k > n_1$. By the Lemma 3.1 and (3.1.b), we have

$$
\begin{aligned}
1-\delta <& \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}\frac{q_if_i(|x_k^i+x_\circ^i|)}{2}\right)^p \\
=& \sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}\frac{q_if_i(|x_k^i+x_\circ^i|)}{2}\right)^p + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}\frac{q_if_i(|x_k^i+x_\circ^i|)}{2}\right)^p \\
\leq& \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|x_\circ^i|)\right)^p + \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|x_k^i|)\right)^p + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}\frac{q_if_i(|x_k^i|)}{2}\right)^p + \delta \\
\leq& \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|x_k^i|)\right)^p + \frac{1}{2^p}\sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|x_k^i|)\right)^p + \delta \\
\leq& \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|x_k^i|)\right)^p - \frac{2^p-1}{2^p}\sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|x_k^i|)\right)^p + \delta \\
<& 1-2\delta+\delta = 1-\delta
\end{aligned}
$$

That is a contradiction. Hence (3.1.a) must hold. Since

$$
\left(\frac{1}{Q_{n_1}}\sum_{i=1}^{n_1}q_if_i(|x_k^i|)\right)^p \leq \sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|x_k^i|)\right)^p \leq 1,
$$

we have $q_if_i(x_k^i) \leq Q_{n_1}$ for $k \in \mathbb{N}$ and $1, 2, \cdots, n_1$. Hence there is a subsequence (z_n) of (x_n) and a sequence (a_n) of real numbers such that

$$
\lim_{k\to\infty} q_if_i(|z_k^i|) = a_i, \text{ for } i = 1, 2, \cdots, n_1.
$$

Therefore,

$$
\sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_if_i(|z_k^i|) - q_if_i(|z_m^i|)|\right)^p < \delta \text{ for } n, m \text{ sufficiently large.}
$$

Consequently,

$$
\begin{aligned}
P(z_k - z_m) =& \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_if_i(|z_k^i|) - q_if_i(|z_m^i|)|\right)^p \\
=& \sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_if_i(|z_k^i|) - q_if_i(|z_m^i|)|\right)^p + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_if_i(|z_k^i|) - q_if_i(|z_m^i|)|\right)^p \\
\leq& \sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_if_i(|z_k^i|) - q_if_i(|z_m^i|)|\right)^p + 2^p\left(\sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|z_k^i|)\right)^p\right) \\
&+ \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_if_i(|z_m^i|)\right)^p \\
\leq& \delta + 2^{p+1}\delta < \epsilon_\circ
\end{aligned}
$$

i.e., $sep(x_n) \leq sep(z_n) < \epsilon_\circ^{\frac{1}{p}}$. This is a contradiction. Therefore $Ces_p^q(\mathcal{F}_b)$ must satisfy the property (β).

Theorem 3.3. $C(Ces_p^q(\mathcal{F})) = 2^{\frac{1}{p}}$.

Proof. Let

$$K = \sup\{A(u_n) : u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n^i e_i \in S(Ces_p^q(\mathcal{F}))\},$$

where

$$0 = i_\circ < i_1 < i_2 < \cdots, u_n \overset{w}{\to} 0.$$

Then $C(Ces_p^q(\mathcal{F}_b)) \geq K$. Moreover, for any $\epsilon > 0$, there is a sequence $(x_n) \subset S(Ces_p^q(\mathcal{F}_b))$ with $x_n \overset{w}{\to} 0$ such that

$$A(x_n) + \epsilon > C(Ces_p^q(\mathcal{F}_b)).$$

By the definition of $A(x_n)$, there exists a subsequence (y_n) of (x_n) such that

$$P(y_n + y_m) + 2\epsilon > C(Ces_p^q(\mathcal{F}_b)) \tag{3.2.a}$$

for any $n, m \in \mathbb{N}$ with $m \neq n$. Take $v_1 = y_1$. Then, by the absolute continuity of the paranorm of y_1, there exists $i_1 \in \mathbb{N}$ such that

$$p\left(\sum_{i_1+1}^{\infty} v_1^i e_i\right) < \epsilon.$$

Putting $z_1 = \sum_{i=1}^{i_1} v_1^i e_i$, we have

$$P(z_1 + y_m) = P\left(y_1 + y_m - \sum_{i_1+1}^{\infty} v_1^i e_i\right) P(y_1 + y_m) - \epsilon \text{ for any } m > 1.$$

Hence by (3.2.a), we have

$$P(z_1 + y_m) + 3\epsilon > C(Ces_p^q(\mathcal{F}_b)) \text{ for any } m > 1.$$

Since $y_n^i \to 0$ for $i = 1, 2, \cdots$, there exists $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that

$$P\left(\sum_{i=1}^{i_1} y_n^i e_i\right) < \epsilon \text{ whenever } n \geq n_2.$$

Define $v_2 = y_{n_2}$. Then there is $i_2 > i_1$ such that

$$P\left(\sum_{i=i_2+1}^{\infty} v_2^i e_i\right) < \epsilon.$$

Taking $z_2 = \sum_{i=i_1+1}^{i_2} v_2^i e_i$, we obtain

$$\begin{aligned} P(z_1 + z_2) &= P\left(y_1 - \sum_{i_1+1}^{\infty} v_1^i e_i + y_{n_2} - \sum_{i=1}^{i_1} v_2^i e_i - \sum_{i_2+1}^{\infty} v_2^i e_i\right) \\ &\geq P(y_1 + y_{n_2}) - 3\epsilon. \end{aligned}$$

Hence by (3.2.a), we immediately obtain

$$P(z_1 + z_2) + 5\epsilon > C(Ces_p^q(\mathcal{F}_b)).$$

Suppose that increasing sequences $(i_j)_{j=1}^{k-1}$, $(n_j)_{j=1}^{k-1}$ of natural numbers and a sequence $(z_j)_{j=1}^{k-1}$ of elements of $C(Ces_p^q(\mathcal{F}_b))$ are already defined and

$$P(z_1 + z_2) + 6\epsilon > C(Ces_p^q(\mathcal{F}_b)) \text{ for } m, n \in \{1, 2, \cdots, k-1\},\ m \neq n.$$

since $y_n^i \to 0$ for $i = 1, 2, \cdots$, there exists a natural number $n_k > n_{k-1}$ such that

$$P\left(\sum_{i=1}^{i_{k-1}} y_n^i e_i\right) < \epsilon$$

provided $n \geq n_k$. Put $v = y_{n_k}$. Then there is $i_k > i_{k-1}$ such that

$$P\left(\sum_{i=i_k+1}^{\infty} v_k^i e_i\right) < \epsilon.$$

Defining $z_k = \sum_{i=i_{k-1}+1}^{i_k} v_k^i e_i$, we obtain

$$P(z_j + z_k) = P\left(y_{n_j} - \sum_{i=1}^{i_{j-1}} v_j^i e_i - \sum_{i_{j+1}}^{\infty} v_j^i e_i + y_{n_k} - \sum_{i=1}^{i_{k-1}} v_k^i e_i - \sum_{i_{k+1}}^{\infty} v_k^i e_i\right)$$

$\geq P(y_{n_j} + y_{n_k}) - 4\epsilon$ for $j = 1, 2, \cdots, k-1$.

Hence, by (3.2.a), we obtain

$P(z_j + z_k) + 6\epsilon > C(Ces_p^q(\mathcal{F}_b))$ for $j = 1, 2, \cdots, k-1$.

Using the induction principle, we can find a sequence (z_n) satisfying the following conditions:

(1) $z_n = \sum_{i=i_{n-1}+1} v_n^i e_i$, where $0 = i_\circ < i_1 < i_2 \cdots$;
(2) $P(z_n + z_m) + 6\epsilon > C(Ces_p^q(\mathcal{F}_b))$ for $m, n \in \mathbb{N}$, $m \neq n$;
(3) $P(z_n) \leq 1$ for $n = 1, 2, \cdots$ and;
(4) $z_n \overset{w}{\to} 0$ as $n \to \infty$.

Define $u_n = z_n / P(z_n)$ for each $n \in \mathbb{N}$. Then every $u_n \in S(Ces_p^q(\mathcal{F}_b))$ and

$$P(u_n + u_m) = P\left(\frac{z_n}{P(z_n)} + \frac{z_m}{P(z_m)}\right) \geq P(z_n + z_m) \geq C(Ces_p^q(\mathcal{F}_b)) - 6\epsilon$$

for any $m, n \in \mathbb{N}$, $m \neq n$. By the arbitrariness of ϵ, we have $C(Ces_p^q(\mathcal{F}_b)) = K$. Let $\epsilon > 0$ be given. Take $n_\epsilon \in \mathbb{N}$ such that

$$\sum_{k=i_{n_\epsilon}+1}^{\infty} \left(\frac{a}{Q_k}\right)^p < \epsilon,$$

where

$$a = \sum_{i=i_{n_\epsilon - 1}+1}^{i_{n_\epsilon}} q_i f_i(|u_{n_\epsilon}^i|).$$

Hence for any $m > n_\epsilon$, we have

$$\begin{aligned}
P(u_{n_\epsilon}+u_m)^p &= \sum_{k=i_{n_\epsilon-1}+1}^{i_{m-1}} \left(\frac{1}{Q_k}\sum_{i=1}^{k} q_i f_i(|u^i_{n_\epsilon}|)\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{Q_k}(a+\sum_{i=1}^{k} q_i f_i(|u^i_m|))\right)^p \\
&\geq \sum_{k=i_{n_\epsilon-1}+1}^{i_{m-1}} \left(\frac{1}{Q_k}\sum_{i=1}^{k} q_i f_i(|u^i_{n_\epsilon}|)\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{Q_k}\sum_{i=1}^{k} q_i f_i(|u^i_m|)\right)^p \\
&= \sum_{k=i_{n_\epsilon-1}+1}^{\infty} \left(\frac{1}{Q_k}\sum_{i=1}^{k} q_i f_i(|u^i_{n_\epsilon}|)\right)^p - \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{a}{Q_k}\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{Q_k}\sum_{i=1}^{k} q_i f_i(|u^i_m|)\right)^p \\
&> 1-\epsilon+1 = 2-\epsilon
\end{aligned}$$

i.e. $A(u_n) \geq (2-\epsilon) \geq (2-\epsilon)^{1/p}$.

On the other hand, for ϵ mentioned above, by Lemma 3.1, there exists $\delta > 0$ such that

$$|P(x+y) - P(x)| < \epsilon$$

whenever $P(x) \leq 1$ and $P(y) \leq \delta$. Take $n_\delta \in \mathbb{N}$ such that

$$\sum_{k=i_{n_\delta+1}}^{\infty} \left(\frac{a}{Q_k}\right)^p < \delta, \text{ and } a = \sum_{i=i_{n_\delta-1}+1}^{i_{n_\delta}} q_i f_i(|u^i_{n_\delta}|).$$

Hence for any $m > n_\delta$, we have

$$\begin{aligned}
P(u_{n_\delta}+u_m)^p &= \sum_{k=i_{n_\delta-1}+1}^{i_{m-1}} \left(\frac{1}{Q_k}\sum_{i=1}^{k} q_i f_i(|u^i_{n_\delta}|)\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{Q_k}(a+\sum_{i=1}^{k} q_i f_i(|u^i_m|))\right)^p \\
&\leq \sum_{k=i_{n_\delta-1}+1}^{\infty} \left(\frac{1}{Q_k}\sum_{i=1}^{k} q_i f_i(|u^i_{n_\delta}|)\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{Q_k}(a+\sum_{i=1}^{k} q_i f_i(|u^i_m|))\right)^p \\
&= P(u^i_{n_\delta})^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{Q_k}(a+\sum_{i=1}^{k} q_i f_i(|u^i_m|))\right)^p - \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{Q_k}\sum_{i=1}^{k} q_i f_i(|u^i_m|)\right)^p + P(u^i_m)^p \\
&< 2+\epsilon
\end{aligned}$$

i.e. $A((u_n)) \leq (2+\epsilon)^{1/p}$. □

4. Cesáro Paranormed Sequence Space ($Ces^q_p(\mathcal{F})$) based Intuitionistic fuzzy distance measure

Obviously, $Ces^q_p(\mathcal{F})$ is a metric space with distance measure between two points x and y defined as follows:

$$d(x,y) = \left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_i - y_i|)\right)^p\right)^{\frac{1}{p}} \tag{4.1}$$

In this distance measure, the new involved parameters p, q and modulus functions f_i incorporate higher order flexibility in the measure, which help us to modify it as per need of the problems.

Cesàro Paranormed Sequence Space ($Ces_p^q(\mathcal{F})$) based Intuitionistic fuzzy distance measure between two intuitionistic fuzzy sets A and B we define as;

$$d_{Ces_p^q(\mathcal{F})}(A,B) = \left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i f_i(|x_i - y_i|)\right)^p\right)^{\frac{1}{p}} \tag{4.2}$$

where $x_i = (\mu_{x_i}, \nu_{x_i}) \in A$ and $y_i = (\mu_{y_i}, \nu_{y_i}) \in B$ are intuitionistic fuzzy numbers.

Application: In any c-means clustering algorithm, the Euclidean distance measure is exploited to compute the distances of cluster centers with each data-item. Here, we define c-mean clustering algorithm using our proposed distance measure $d_{Ces_p^q(\mathcal{F})}$: (1) Select first k data points as cluster center $x_k = \{x_1, x_2, \cdots, x_k\}$ (where k is the number of cluster); (2) By using the proposed distance measure $d_{Ces_p^q(\mathcal{F})}$, find distances between each data point and cluster center; (3) Put data point into that cluster whose $d_{Ces_p^q(\mathcal{F})}$-distance with its center is minimum; (4) Redefine cluster centers for newly evolved clusters due to the above steps, the new cluster centers are computed as $c_i = \frac{1}{k_i}\sum_{j=1}^{k_i} x_i$, where k_i denote the number of points in the ith cluster; (5) Repeat this process until the difference between two consecutive cluster center becomes less than a desired small number. Similarly, various variants of fuzzy c-means clustering algorithms can be modified using our proposed distance measure.

5. Conclusion

In this paper, we have proposed a new intuitionistic fuzzy distance measure that utilizes paranorm sequence space $Ces_p^q(\mathcal{F})$. We have also studied some geometric properties of $Ces_p^q(\mathcal{F})$. The parameters p, q, f involved in $Ces_p^q(\mathcal{F})$ gives three degree of freedom to proposed distance measure $d_{Ces_p^q(\mathcal{F})}$. The flexibility in distance measure can be judiciously used for the clustering the real world data sets. Using $d_{Ces_p^q(\mathcal{F})}$, we have modified the standard c-mean clustering algorithm by replacing the Euclidean distance measure with the proposed distance. In similar fashion, other distance measure based clustering algorithms can also be modified. In the future work, we will study the effectiveness of $d_{Ces_p^q(\mathcal{F})}$ in clustering the intuitionistic fuzzy datasets.

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