Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 12 Issue 4(2021), Pages 25-32.
DOI:10.54379/jiasf-2021-4-3

FURTHER INEQUALITIES FOR THE EUCLIDEAN OPERATOR RADIUS

HASSAN RANJBAR, ASADOLLAH NIKNAM

Abstract. By use of some non-negative Hermitian forms defined for n-tuple of bounded linear operators on the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ we establish new numerical radius and operator norm inequalities for sum of products of operators.

1. Introduction

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The numerical range of an operator A is the subset of the complex numbers $\mathbb{C}$ given by:

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

The numerical radius $\omega(A)$ of an operator A on $\mathcal{H}$ is given by:

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

It is well known that $\omega(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators $A{:}\mathcal{H} \to \mathcal{H}$. This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\frac{1}{2}\|A\| \le \omega(A) \le \|A\|. \tag{1.1}$$

For other results concerning the numerical range and radius of bounded linear operators on aHilbert space, see [3, 4, 6, 7, 8].

In [9], the author has introduced the following norm on the Cartesian product $\mathcal{B}^{(n)}(\mathcal{H}) \equiv \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \times \ldots \times \mathcal{B}(\mathcal{H})$,

$$\|(T_1, T_2 \ldots, T_n)\|_{n,e} \equiv \sup_{(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathcal{A}^{(n)}} \|\alpha_1 T_1 + \alpha_2 T_2 + \ldots + \alpha_n T_n\|$$

where $(T_1, T_2, \ldots, T_n) \in \mathcal{B}^{(n)}(\mathcal{H})$ and $\mathcal{A}^{(n)} \equiv \left\{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n : \sum_{i=1}^n |\alpha_i|^2 \le 1\right\}$ is the Euclidean closed ball in $\mathbb{C}^n$. Of course, $\|\cdot\|_{n,e}$ is a norm on $\mathcal{B}^{(n)}(\mathcal{H})$, and

$$\|(T_1, T_2, \ldots, T_n)\|_{n,e} = \|(T_1^*, T_2^*, \ldots, T_n^*)\|_{n,e}$$

2000 *Mathematics Subject Classification.* 47A63, 47A64, 47B15, 15A45.
Key words and phrases. Numerical radius; norm, inequality.

Submitted June 7, 2021. Published November 4, 2021.
Communicated by S.S. Dragomir.

where T^* is used for the adoint of the operator T. In the same paper, the following inequalities are proved,

$$\frac{1}{n}\left\|T_1T_1^*+T_2T_2^*+\ldots+T_nT_n^*\right\|\le\left\|(T_1,T_2,\ldots,T_n)\right\|_{n,e}^2\le\left\|T_1T_1^*+T_2T_2^*+\ldots+T_nT_n^*\right\| \tag{1.2}$$

and

$$\frac{1}{4n}\left\|T_1T_1^*+T_2T_2^*+\ldots+T_nT_n^*\right\|\le w_{n,e}^2\left(T_1,T_2,\ldots,T_n\right)\le\left\|T_1T_1^*+T_2T_2^*+\ldots+T_nT_n^*\right\|. \tag{1.3}$$

where the Euclidean operator radius of an n-tuple of operators $(T_1, T_2, \ldots, T_n)$ is defined by

$$w_{n,e}\left(T_1,T_2,\ldots,T_n\right)\equiv\sup_{\|x\|=1}\left(\sum_{i=1}^{n}\left|\langle T_ix,x\rangle\right|^2\right)^{\frac{1}{2}}.$$

This paper aims to present extensions of the above-mentioned inequalities by using certain convex and concave functions. Meanwhile, by employing some non-negative Hermitian forms defined for n-tuple of bounded linear operators on the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, we present some new numerical radius and operator norm inequalities for sum of products of operators.

2. Related Inequalities

To present our main results we need the following lemmas. The first lemma was obtained by Bourin and Uchiyama [2] (see also [5, Corollary 3.5]):

Lemma 2.1. *Let $T_i \geq 0\,(i = 1, \ldots, n)$. Then for every non-negative concave function f on $[0,\infty)$,*

$$\left\|f\left(T_1+T_2+\ldots+T_n\right)\right\|\le\left\|f\left(T_1\right)+f\left(T_2\right)+\ldots+f\left(T_n\right)\right\|.$$

The above inequality is reversed if f is non-negative convex function on $[0,\infty)$ with $f(0) = 0$.

The second lemma was obtained by Aujla and Silva [1]:

Lemma 2.2. *Let $T_i \geq 0\,(i = 1, \ldots, n)$ and let $p_1, p_2, \ldots, p_n$ be positive scalars with $\sum_{i=1}^n p_i = 1$. Then for every non-negative convex function f on $[0,\infty)$,*

$$\left\|f\left(p_1T_1+p_2T_2+\ldots+p_nT_n\right)\right\|\le\left\|p_1f\left(T_1\right)+p_2f\left(T_2\right)+\ldots+p_nf\left(T_n\right)\right\|.$$

The above inequality is reversed if f is non-negative concave function

Theorem 2.3. *Let $(T_1, T_2, \ldots, T_n) \in \mathcal{B}^{(n)}(\mathcal{H})$, and let f be a non-negative function on $[0,\infty)$ such that $g(t) = f\left(\sqrt{t}\right)$ is concave. Then*

$$\begin{aligned}&\frac{1}{n}\left\|f\left(|T_1|\right)+f\left(|T_2|\right)+\ldots+f\left(|T_n|\right)\right\|\\&\quad\le f\left(\left\|(T_1,T_2,\ldots,T_n)\right\|_{n,e}\right)\\&\quad\le\left\|f\left(|T_1|\right)+f\left(|T_2|\right)+\ldots+f\left(|T_n|\right)\right\|\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&\frac{1}{4n}\left\|f\left(|T_1|\right)+f\left(|T_2|\right)+\ldots+f\left(|T_n|\right)\right\|\\&\quad\le f\left(w_{n,e}\left(T_1,T_2,\ldots,T_n\right)\right)\\&\quad\le\left\|f\left(|T_1|\right)+f\left(|T_2|\right)+\ldots+f\left(|T_n|\right)\right\|.\end{aligned} \tag{2.2}$$

Proof. We prove the RHS of (2.1). We have

$$
\begin{aligned}
f\left(\|(T_1,T_2,\ldots,T_n)\|_{n,e}\right) &= g\left(\|(T_1,T_2,\ldots,T_n)\|_{n,e}^2\right) && (2.3)\\
&\le g\left(\|T_1T_1^*+T_2T_2^*+\ldots+T_nT_n^*\|\right) && (2.4)\\
&= g\left(\left\||T_1|^2+|T_2|^2+\ldots+|T_n|^2\right\|\right)\\
&= \left\|g\left(|T_1|^2+|T_2|^2+\ldots+|T_n|^2\right)\right\| && (2.5)\\
&\le \left\|g\left(|T_1|^2\right)+g\left(|T_2|^2\right)+\ldots+g\left(|T_n|^2\right)\right\| && (2.6)\\
&= \|f(|T_1|)+f(|T_2|)+\ldots+f(|T_n|)\| && (2.7)
\end{aligned}
$$

where the equalities (2.3) and (2.7) follows from the assumption $g(t)=f\left(\sqrt{t}\right)$, the inequality (2.4) follows from the fact that g is increasing function (since g is non-negative convave function on $[0,\infty)$), the equality (2.5) is based on the continuous functional calculus, and the inequality (2.6) is due to Lemma 2.1.

For the LHS of (2.1),

$$
\begin{aligned}
f\left(\|(T_1,T_2,\ldots,T_n)\|_{n,e}\right) &= g\left(\|(T_1,T_2,\ldots,T_n)\|_{n,e}^2\right)\\
&\ge g\left(\frac{1}{n}\left\||T_1|^2+|T_2|^2+\ldots+|T_n|^2\right\|\right)\\
&= \left\|g\left(\frac{|T_1|^2+|T_2|^2+\ldots+|T_n|^2}{n}\right)\right\|\\
&\ge \frac{1}{n}\left\|g\left(|T_1|^2\right)+g\left(|T_2|^2\right)+\ldots+g\left(|T_n|^2\right)\right\| && (2.8)\\
&= \frac{1}{n}\|f(|T_1|)+f(|T_2|)+\ldots+f(|T_n|)\|
\end{aligned}
$$

where (2.8) follows from Lemma 2.2 for concave functions.

The RHS of (2.2) is completely similar to the LHS of (2.1), so we omit its proof. For the RHS of (2.2) we have

$$
\begin{aligned}
f\left(w_{n,e}(T_1,T_2,\ldots,T_n)\right) &= g\left(w_{n,e}^2(T_1,T_2,\ldots,T_n)\right)\\
&\ge g\left(\frac{1}{4n}\left\||T_1|^2+|T_2|^2+\ldots+|T_n|^2\right\|\right)\\
&\ge \frac{1}{4}\left\|g\left(\frac{|T_1|^2+|T_2|^2+\ldots+|T_n|^2}{n}\right)\right\| && (2.9)\\
&\ge \frac{1}{4n}\left\|g\left(|T_1|^2\right)+g\left(|T_2|^2\right)+\ldots+g\left(|T_n|^2\right)\right\|\\
&= \frac{1}{4n}\|f(|T_1|)+f(|T_2|)+\ldots+f(|T_n|)\|
\end{aligned}
$$

where to obtain (2.9) we used the fact that each concave function satisfies the inequality $f(\alpha t)\ge \alpha f(t)$ where $\alpha\le 1$. □

Corollary 2.4. *Let $(T_1,T_2,\ldots,T_n)\in\mathcal{B}^{(n)}(\mathcal{H})$, then for any $0<r\le 2$,*

$$
\begin{aligned}
\frac{1}{n}\left\||T_1|^r+|T_2|^r+\ldots+|T_n|^r\right\| &\le \|(T_1,T_2,\ldots,T_n)\|_{n,e}^r\\
&\le \left\||T_1|^r+|T_2|^r+\ldots+|T_n|^r\right\|
\end{aligned}
$$

and

$$\frac{1}{4n}\left\| |T_1|^r + |T_2|^r + \ldots + |T_n|^r \right\| \le w_{n,e}^r (T_1, T_2, \ldots, T_n) \\ \le \left\| |T_1|^r + |T_2|^r + \ldots + |T_n|^r \right\|.$$

Theorem 2.5. *Let* $(T_1, T_2, \ldots, T_n) \in \mathcal{B}^{(n)}(\mathcal{H})$, *and let* f *be a non-negative function on* $[0, \infty)$ *such that* $g(t) = f\left(\sqrt{t}\right)$ *is convex, then*

$$f\left(\|(T_1, T_2, \ldots, T_n)\|_{n,e}\right) \le \frac{1}{n}\left\| f\left(\sqrt{n}\,|T_1|\right) + f\left(\sqrt{n}\,|T_2|\right) + \ldots + f\left(\sqrt{n}\,|T_n|\right)\right\| \tag{2.10}$$

and

$$f\left(w_{n,e}(T_1, T_2, \ldots, T_n)\right) \le \frac{1}{n}\left\| f\left(\sqrt{n}\,|T_1|\right) + f\left(\sqrt{n}\,|T_2|\right) + \ldots + f\left(\sqrt{n}\,|T_n|\right)\right\|.$$

Proof. To obtain the inequality (2.10), we can write

$$\begin{aligned} f\left(\|(T_1, T_2, \ldots, T_n)\|_{n,e}\right) &= g\left(\|(T_1, T_2, \ldots, T_n)\|_{n,e}^2\right) \\ &\le g\left(\left\| |T_1|^2 + |T_2|^2 + \ldots + |T_n|^2 \right\|\right) \\ &= \left\| g\left(|T_1|^2 + |T_2|^2 + \ldots + |T_n|^2\right)\right\| \\ &= \left\| g\left(\frac{n|T_1|^2 + n|T_2|^2 + \ldots + n|T_n|^2}{n}\right)\right\| \\ &\le \left\| \frac{g\left(n|T_1|^2\right) + g\left(n|T_2|^2\right) + \ldots + g\left(n|T_n|^2\right)}{n}\right\| \qquad (2.11) \\ &= \frac{1}{n}\left\| g\left(n|T_1|^2\right) + g\left(n|T_2|^2\right) + \ldots + g\left(n|T_n|^2\right)\right\| \\ &= \frac{1}{n}\left\| f\left(\sqrt{n}\,|T_1|\right) + f\left(\sqrt{n}\,|T_2|\right) + \ldots + f\left(\sqrt{n}\,|T_n|\right)\right\| \end{aligned}$$

where the inequality (2.11) follows from Lemma 2.2 for convex functions. □

Corollary 2.6. *Let* $(T_1, T_2, \ldots, T_n) \in \mathcal{B}^{(n)}(\mathcal{H})$, *then for any* $r \ge 2$,

$$\|(T_1, T_2, \ldots, T_n)\|_{n,e}^r \le n^{\frac{r-2}{2}} \left\| |T_1|^r + |T_2|^r + \ldots + |T_n|^r \right\|$$

and

$$w_{n,e}^r (T_1, T_2, \ldots, T_n) \le n^{\frac{r-2}{2}} \left\| |T_1|^r + |T_2|^r + \ldots + |T_n|^r \right\|.$$

3. Norm and numerical radius inequalities for non-negative Hermitian forms

Let $\mathcal{X}$ be a linear space over the real or complex number field $\mathbb{K}$. A mapping $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ is said to be a positive hermitian form if the following conditions are satisfied:

(1) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$ for all $x, y, z \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{K}$.
(2) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all $x, y \in \mathcal{X}$.
(3) $\langle x, x\rangle \ge 0$ for all $x \in \mathcal{X}$.

Let $(T_1, \cdots, T_n) \in \mathbb{B}^{(n)}(\mathcal{H})$ be an n-tuple of bounded linear operators on the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and $(p_1, \cdots, p_n) \in \mathbb{R}_+^{*n}$ an n-tuple of non-negative weights not all of them equal to zero.

For an $x \in \mathcal{H}, x \neq 0$ we define

$$\langle \mathsf{T}, \mathsf{V} \rangle_{p,x} = \sum_{j=1}^{n} p_j \langle T_j x, V_j x \rangle = \left\langle \left(\sum_{j=1}^{n} p_j V_j^* T_j \right) x, x \right\rangle,$$

where $\mathsf{T} = (T_1, \cdots, T_n), \mathsf{V} = (V_1, \cdots, V_n) \in \mathbb{B}^{(n)}(\mathcal{H})$.

We can then state the following result:

Lemma 3.1. *For any $x \in \mathcal{H}, x \neq 0$ and $(p_1, \cdots, p_n) \in \mathbb{R}_+^{*n}$ we have that $\langle \cdot, \cdot \rangle_{p,x}$ is a non-negative Hermitian form on $\mathbb{B}^{(n)}(\mathcal{H})$.*

Proof. We have that

$$\langle \mathsf{T}, \mathsf{T} \rangle_{p,x} = \left\langle \left(\sum_{j=1}^{n} p_j T_j^* T_j \right) x, x \right\rangle = \left\langle \left(\sum_{j=1}^{n} p_j |T_j|^2 \right) x, x \right\rangle \geq 0$$

for any $\mathsf{T} = (T_1, \cdots, T_n) \in \mathbb{B}^{(n)}(\mathcal{H})$, where the operator modulus is defined by $|A|^2 = A^* A$, $A \in \mathbb{B}(\mathcal{H})$.

The functional $\langle \cdot, \cdot \rangle_{p,x}$ is linear in the first variable and

$$\begin{aligned}
\langle \mathsf{V}, \mathsf{T} \rangle_{p,x} &= \overline{\left\langle \left(\sum_{j=1}^{n} p_j T_j^* V_j \right) x, x \right\rangle} \\
&= \left\langle x, \left(\sum_{j=1}^{n} p_j T_j^* V_j \right) x \right\rangle \\
&= \left\langle \left(\sum_{j=1}^{n} p_j T_j^* V_j \right)^* x, x \right\rangle \\
&= \left\langle \left(\sum_{j=1}^{n} p_j V_j^* T_j \right) x, x \right\rangle = \langle \mathsf{T}, \mathsf{V} \rangle_{p,x}
\end{aligned}$$

for any $\mathsf{T} = (T_1, \cdots, T_n), \mathsf{V} = (V_1, \cdots, V_n) \in \mathbb{B}^{(n)}(\mathcal{H})$. □

Before proceeding we note that if $p = (1, \cdots, 1)$, then we denote $\langle \cdot, \cdot \rangle_{p,x}$ by $\langle \cdot, \cdot \rangle_x$.

Theorem 3.2. *Let $\mathsf{T} = (T_1, \cdots, T_n), \mathsf{V} = (V_1, \cdots, V_n) \in \mathbb{B}^{(n)}(\mathcal{H})$, then*

$$\omega^2 \left(\sum_{j=1}^{n} V_j^* T_j \right) \leq \frac{1}{p} \left\| \sum_{j=1}^{n} |T_j|^2 \right\|^p + \frac{1}{q} \left\| \sum_{j=1}^{n} |V_j|^2 \right\|^q \tag{3.1}$$

for all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $x \in \mathcal{H}, x \neq 0$. Writing the Schwarz inequality for the non-negative Hermitian form $\langle \cdot, \cdot \rangle_x$ we have

$$\begin{aligned}
&\left| \left\langle \left(\sum_{j=1}^{n} V_j^* T_j \right) x, x \right\rangle \right|^2 \\
&\leq \left\langle \left(\sum_{j=1}^{n} |T_j|^2 \right) x, x \right\rangle \left\langle \left(\sum_{j=1}^{n} |V_j|^2 \right) x, x \right\rangle \\
&\leq \frac{1}{p} \left\langle \left(\sum_{j=1}^{n} |T_j|^2 \right) x, x \right\rangle^p + \frac{1}{q} \left\langle \left(\sum_{j=1}^{n} |V_j|^2 \right) x, x \right\rangle^q
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we deduce the desired result (3.1). □

Remark. *If we choose $p = q = 2$ in Theorem 3.2*

$$\omega \left(\sum_{j=1}^{n} V_j^* T_j \right) \leq \sqrt{\frac{1}{2} \left(\left\| \sum_{j=1}^{n} |T_j|^2 \right\|^{\frac{1}{2}} + \left\| \sum_{j=1}^{n} |V_j|^2 \right\|^{\frac{1}{2}} \right)}. \tag{3.2}$$

Let $V_j^* = T_j$ in (3.2)

$$\omega \left(\sum_{j=1}^{n} T_j^2 \right) \leq \sqrt{\frac{1}{2} \left(\left\| \sum_{j=1}^{n} |T_j|^2 \right\|^{\frac{1}{2}} + \left\| \sum_{j=1}^{n} |T_j^*|^2 \right\|^{\frac{1}{2}} \right)}.$$

If we take $T_j := \frac{A_j + A_j^*}{2}$ and $V_j := \frac{A_j - A_j^*}{2}$ in (3.2)

$$\omega \left(\sum_{j=1}^{n} |A_j|^2 \right) \leq \sqrt{\frac{1}{2} \left(\left\| \sum_{j=1}^{n} \left(\frac{A_j + A_j^*}{2} \right) \right\|^{\frac{1}{2}} + \left\| \sum_{j=1}^{n} \left(\frac{A_j - A_j^*}{2} \right) \right\|^{\frac{1}{2}} \right)}.$$

Theorem 3.3. *Let $\mathsf{T} = (T_1, \cdots, T_n), \mathsf{V} = (V_1, \cdots, V_n) \in \mathbb{B}^{(n)}(\mathcal{H})$, then*

$$\omega^2 \left(\sum_{j=1}^{n} V_j^* T_j \right) \leq 2^{\frac{r-1}{r}} \left\| \sum_{j=1}^{n} |T_j|^2 + |V_j|^2 \right\|^{\frac{1}{r}} \tag{3.3}$$

for all $r \geq 1$.

Proof. Let $x \in \mathcal{H}$,$x \neq 0$.

$$\begin{aligned}
&\left|\left\langle \left(\sum_{j=1}^{n} V_j^* T_j\right) x, x \right\rangle\right|^2 \\
&\leq \left\langle \left(\sum_{j=1}^{n} |T_j|^2\right) x, x \right\rangle^{\frac{1}{2}} + \left\langle \left(\sum_{j=1}^{n} |V_j|^2\right) x, x \right\rangle^{\frac{1}{2}} \\
&\leq 2^{\frac{r-1}{r}} \left(\left\langle \left(\sum_{j=1}^{n} |T_j|^2\right) x, x \right\rangle^{\frac{r}{2}} + \left\langle \left(\sum_{j=1}^{n} |V_j|^2\right) x, x \right\rangle^{\frac{r}{2}} \right)^{\frac{1}{r}} \\
&= 2^{\frac{r-1}{r}} \left\langle \left(\sum_{j=1}^{n} |T_j|^2 + |V_j|^2\right) x, x \right\rangle^{\frac{1}{r}}
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we deduce the desired result (3.3). □

Remark. *If we take $r = 1$ in Theorem 3.3*

$$\omega\left(\sum_{j=1}^{n} V_j^* T_j\right) \leq \left\|\sum_{j=1}^{n} |T_j|^2 + |V_j|^2\right\|^{\frac{1}{2}}.$$

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] J.S. Aujla, F.C. Silva, *Weak majorization inequalities and convex functions*, Linear Algebra Appl. **369** (2003) 217–233.

[2] J.C. Bourin, M. Uchiyama, *A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$*, Linear Algebra Appl. **423** (2007) 512–518.

[3] S. S. Dragomir, *Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces*, Linear Algebra Appl. **419** (2006) 256–264.

[4] F. Kittaneh, H. R. Moradi, *Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*, Math. Inequal. Appl. **233** (2020) 1117-1125.

[5] T. Kosem, *Inequalities between $\|f(A+B)\|$ and $\|f(A)+f(B)\|$*, Linear Algebra Appl. **418** (2006) 153–160.

[6] H. R. Moradi, M. Sababheh, *New estimates for the numerical radius*, accepted in: Filomat. arXiv preprint arXiv:2010.12756 (2020).

[7] M. E. Omidvar, H. R. Moradi, *Better bounds on the numerical radii of Hilbert space operators*, Linear Algebra Appl. **604** (2020) 265–277.

[8] M. E. Omidvar, H. R. Moradi, *New estimates for the numerical radius of Hilbert space operators*, Linear Multilinear Algebra. **69 5** (2021) 946–956.

[9] G. Popescu, *Unitary invariants in multivariable operator theory*. Mem. Amer. Math. Soc. 200 (2009), no. 941, vi+91 pp. ISBN: 978-0-8218-4396-3.

Hassan Ranjbar
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: hassan.ranjbar514@gmail.com

Asadollah Niknam
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iranl
E-mail address: dassamankin@yahoo.co.uk