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ŁOJASIEWICZ INEQUALITY IN P-MINIMAL STRUCTURES

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Abstract. Th purpose of this paper is to extend the Łojasiewicz inequality for functions definable in some subclass of P-minimal structures. More precisely, we prove that the Łojasiewicz inequality holds for functions definable in p-optimal expansions of $\mathbb{Q}_p$. It is also shown that the Łojasiewicz exponent is a rational number in such p-optimal expansions.

1. Introduction

Let S be a compact semi-algebraic subset of $\mathbb{R}^n$ and $f, g\colon S \longrightarrow \mathbb{R}$ two continuous semi-algebraic functions such that $g^{-1}(0) \subset f^{-1}(0)$. The well-known *Łojasiewicz inequality* in semi-algebraic geometry (see for instance [2]) states that there exist a positive integer $\rho \geqslant 1$ and a constant $c > 0$ such that

$$|f(x)|^\rho \leqslant c|g(x)| \tag{1.1}$$

for all $x \in S$. It was introduced by Hörmander [15] and Łojasiewicz [18] and [19], and has found rather striking applications in various branches of mathematics : theory of ordinary and partial differential equations, dynamical systems, optimization, and so on.

The origin of the Łojasiewicz inequality lies in the distribution theory. More precisely, in the problem of the division of a distribution by a function posed by L. Schwartz [23]. In the solution of this problem (in the full generality by Łojasiewicz [18]), the main difficulty was to explain the structure of real analytic sets (i.e. subsets of $\mathbb{R}^n$ described by systems of real analytic equations). From this description of analytic sets fundamental Łojasiewicz inequality follows, which is the main fact used in solution of the division problem.

The best exponent ρ in the Łojasiewicz inequality (1.1) i.e. the the smallest one is called the *Łojasiewicz exponent* of f with respect to g on S, and denoted by $\ell_S(f, g)$. It is shown that (see [10]) $\ell_S(f, g)$ is a rational number. Moreover, inequality (1.1) holds with exponent $\ell_S(f, g)$ and some constant $c > 0$.

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The existence of the Łojasiewicz exponent was first established by Łojasiewicz [19] for *semi-analytic* functions, and by Hironaka [14] (using the resolution of singularities) in the *subanalytic* context. The rationality of the Łojasiewicz exponent for *subanalytic* functions was showed by Bochnak-Risler [1]. This result has been adapted to the p-adic subanalytic case by Denef-Dries [7], and to the p-adic semi-algebraic functions by Fekak-Srhir [11]. Note also that the case of real semi-algebraic functions was treated by Fekak [10]. More generally, see [12] and [17] for the o-minimal structures case.

The purpose of this paper is to transpose these results to the *P-minimal* case. More precisely, we shall prove the Łojasiewicz inequality for definable functions in p-optimal expansions of the field of p-adic numbers $\mathbb{Q}_p$. We also show the rationality of the Łojasiewicz exponent for definable functions in such p-optimal expansions. The arguments used in the o-minimal case are mainly the tameness and the simple structure of their definable sets. We will adapt her those arguments to the P-minimal case.

The content of the present paper is organized as follows. In section 2 we recall some basic definitions and facts from *P-minimality*. In particular, *p-optimal fields* are presented in detail as we are going to use it extensively throughout the paper. Section 3 is devoted to the Łojasiewicz inequality. In Section 4 our second main result (the rationality of the Łojasiewicz exponent) is stated and proved.

2. P-minimal structures

The notion of *P-minimality* was first introduced by D. Haskell and H-D. Macpherson [13] as a p-adic analogue of the notion of *o*-minimal structures [8]. They managed to show that these structures have several similarities with the o-minimal case. In particular, they have shown that these structures have a good notion of dimension.

Let us first fix some notations and terminologies that will be used here. Throughout this paper, p denotes a fixed prime number and $\mathbb{Q}_p$ the field of p-adic numbers.

Recall that a *p-valued field* is a pair (K, v) with K is a field of characteristic 0 and v a valuation of K that satisfies the following conditions:

- $v(p) = \min\{0 < v(x) \mid x \in K \setminus \{0\}\}$,
- $k \simeq \mathbb{F}_p$ (the finite field with p elements).

A p-valued field is said to be *p-adically closed* if it does not admit any proper algebraic extension to a p-valued field. A p-adically closed field is also characterized as being a Henselian p-valued fields with a $\mathbb{Z}$-group value group. For example, both the p-adic number field $\mathbb{Q}_p$ and the p-adic algebraic numbers $\widehat{\mathbb{Q}}_p$ are p-adically closed fields. We denote by $\mathcal{L}_{\text{Mac}}$ Macintyre's language for p-valued fields defined by

$$\mathcal{L}_{\text{Mac}} = \{+, -, \cdot, V, (P_n)_{n \geqslant 2}, 0, 1\},$$

where V and P_n are unary relation symbols. If K is a p-adically closed field, then V is interpreted as its valuation ring, and for each $n \geqslant 2$, P_n as

$$K \models \forall\, x\, \big(P_n(x) \;\longleftrightarrow\; \exists\, y\; (x = y^n)\big)\cdot$$

A subset of K^m is called *semi-algebraic* if it is a boolean combination of subsets of the form

$$\{x \in K^m \mid \exists\, y \in K,\; f(x) = y^n\},$$

where $f\colon K^m \longrightarrow K$ is a polynomial function. The most remarkable result concerning model theory of p-adically closed fields is :

Theorem 2.1 (Macintyre [20]). *The theory pCF of p-adically closed fields admits quantifier elimination in the language $\mathcal{L}_{\mathrm{Mac}}$.*

Macintyre's theorem is a powerful result and has many important consequences. Here are some of these consequences:

- the definable subsets are precisely semi-algebraic sets.
- every p-adically closed field is elementarily equivalent to $\mathbb{Q}_p$.
- the theory pCF is decidable.

We refer the reader to the excellent reference [22] by A. Prestel and P. Roquette for a generalization of this result and more details about the basic notions from model theory of p-adically closed fields used here.

Let $\widehat{\mathcal{L}}$ be any language extending $\mathcal{L}_{\mathrm{Mac}}$. We briefly recall now the definition of a P-minimal structure (see also [13]):

Definition 2.2. Let $\mathcal{K}$ be an $\widehat{\mathcal{L}}$-structure. We say that $\mathcal{K}$ is *P-minimal* if for every $\mathcal{K}'$ elementary equivalent to $\mathcal{K}$, every definable subset of K' is quantifier free definable by an $\mathcal{L}_{\mathrm{Mac}}$-formula.

Note that an $\widehat{\mathcal{L}}$-structure $\mathcal{K}$ is called *p-minimal* if every definable subset of K is quantifier free definable by an $\mathcal{L}_{\mathrm{Mac}}$-formula. Recall that a function $f\colon X \subseteq K^m \longrightarrow K$ is said to be *definable* if its graph $\{(x, f(x)) \mid x \in X\}$ is definable.

Example 2.1. 1) Every p-adically closed field is P-minimal (Macintyre's theorem).

2) Let $\mathcal{L}_{\mathrm{an}}$ denote the language $\mathcal{L}_{\mathrm{Mac}}$ enriched with the field inverse $^{-1}$ (with $0^{-1} = 0$), and for each convergent power series $f\colon \mathbb{Z}_p^m \longrightarrow \mathbb{Q}_p$, a function symbol for the restricted analytic function from $\mathbb{Q}_p^m$ to $\mathbb{Q}_p$ defined by

$$x \longmapsto \begin{cases} f(x) & \text{if } x \in \mathbb{Z}_p^m, \\ 0 & \text{otherwise.} \end{cases}$$

Then the $\mathcal{L}_{\mathrm{an}}$-structure $\mathbb{Q}_p$ is P-minimal (see [9] for more details).

In view of the similarity between o-minimality and P-minimality, it is natural to ask whether any P-minimal structure admits cell decomposition as in the o-minimal case. This question was already raised in [13]. A first partial answer was given by Mourgues in [21], where she proved the following result :

Theorem 2.3 (Mourgues [21]). *Let $(K, \widehat{\mathcal{L}})$ be a P-minimal field. Then the following properties are equivalent:*

1) *$(K, \widehat{\mathcal{L}})$ admits cell decomposition;*
2) *$(K, \widehat{\mathcal{L}})$ admits definable Skolem functions.*

Recall that a structure $\mathcal{M}$ has definable Skolem functions if for every definable subset $X \subseteq M^{m+1}$, there is a definable function $g\colon \pi(X) \longrightarrow M$ such that $(x, g(x)) \in X$ for all $x \in \pi(X)$, where $\pi\colon M^{m+1} \longrightarrow M^m$ is the projection map. We also say that $\mathcal{M}$ admits *definable selection* or admits *definable choice.* Note that an example of a P-minimal structure without definable Skolem functions was given later by Cubides Kovacsics-Nguyen [5].

One way to deal with this lack of the existence of cell decomposition theorem is to introduce a more restrictive notion of P-minimality (explicitly by adding the existence of definable Skolem functions as a condition). An example of this approach is the recent attempt of Darnière-Halupczok [6], who suggest a notion of so-called p-optimal structures.

Let us first recall that a function $f\colon K^m \longrightarrow K$ is said to be *basic* if it is polynomial in the variable x_m with coefficients which are global definable functions in $(x_1, \dots, x_{m-1})$. A subset of K^m is called *basic* if it is a boolean combination of subsets of the form

$$\{x \in K^m \mid \exists\, y \in K,\ f(x) = y^n\},$$

where $f\colon K^m \longrightarrow K$ is a basic function. We can now give the definition of p-optimal field (as introduced in [6]) :

Definition 2.4. Let $(K, \widehat{\mathcal{L}})$ be an expansion of a p-adically closed field K. We say that $\mathcal{K}$ is *p-optimal* if for each $m \geqslant 1$, every definable subset of K^m is a finite boolean combination of basic sets.

Example 2.2. 1) Every p-adically closed field is p-optimal.
2) The $\mathcal{L}_{\text{an}}$-structure $\mathbb{Q}_p$ is p-optimal.

They also managed to give the following characterization of p-optimal fields :

Theorem 2.5 (Darnière-Halupczok [6])**.** *Let $(K, \widehat{\mathcal{L}})$ be an expansion of a p-adically closed field K. Then the following properties are equivalent:*

1) $(K, \widehat{\mathcal{L}})$ *is p-optimal;*
2) $(K, \widehat{\mathcal{L}})$ *admits Denef's cell decomposition;*
3) $(K, \widehat{\mathcal{L}})$ *is P-minimal and admits definable Skolem functions.*

In what follows, $(\mathbb{Q}_p, \widehat{\mathcal{L}})$ denotes a fixed p-optimal field. Recall that a subset is said to be *locally closed* if it is the intersection of an open and a closed subset. Then we have:

Proposition 2.6. *Let D be a locally closed definable subset of $\mathbb{Q}_p^m$. Then there is a definable homeomorphism from D onto a closed definable subset of $\mathbb{Q}_p^{m+1}$.*

Proof. The proof of Bochnak, Coste and Roy in the real case can easily be adapted here. (For more details see Proposition 2.2.9 of [2] pp. 29-30). □

Now here is another result which will be very useful afterwards :

Theorem 2.7. *Let D be a closed and bounded definable set of $\mathbb{Q}_p^m$ and $f\colon D \longrightarrow \mathbb{Q}_p$ a continuous definable function. Then $f(D)$ is a closed and bounded definable subset of $\mathbb{Q}_p$.*

Proof. The proof is of course easy since Heine-Borel Theorem holds for $\mathbb{Q}_p$. Nevertheless, we give another proof which can be generalized for any p-adically closed field, and based on the existence of P-minimal monotonicity theorem [16] and definable Skolem functions in p-optimal fields. It is adapted from the o-minimal case (see for instance Lemma 1.9 of [8] p. 95; see also Theorem 3.4 of [3] pp. 26-27). To obtain a contradiction, suppose that

$$\forall\, t \in \mathbb{Q}_p,\ \exists\, x \in D,\ |f(x)|_p > |t|_p\,.$$

By definable choice, there is a definable map $g\colon \mathbb{Q}_p \longrightarrow D$ such that $|f(g(t))|_p > |t|_p$ for all t in $\mathbb{Q}_p$. Since D is closed and bounded definable set it follows from the p-adic local monotonicity theorem (see Theorem 1.10 of [16]), applied to the m coordinates of g, that there is a definable set A of $\mathbb{Q}_p$ such that $\lim_{|t|_p \to +\infty} g|_A(t) = x$ exists and belongs to D. So $f(x) = f(\lim\limits_{|t|_p \to +\infty} g|_A(t)) = \lim\limits_{|t|_p \to +\infty} f(g|_A(t))$, but the last limit cannot exist in $\mathbb{Q}_p$, since $|f(g(t))|_p > |t|_p$ for all t. Contradiction. $\square$

Remark 2.8. Note that contrary to the o-minimal case, it is shown in [6] (Lemma 4.3 p. 11) that every P-minimal expansion of $\mathbb{Q}_p$ is polynomially bounded. That is, for every definable function $f\colon \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$, there is $M > 0$ and non-zero integer d and such that

$$|f(x)|_p \leqslant |x|_p^d$$

for all $|x|_p > M$, where $|.|_p$ denotes the p-dic norm on $\mathbb{Q}_p$. More generally, Cubides Kovacsics and Delon have shown in [4] that every P-minimal field is polynomially bounded.

3. Łojasiewicz inequality

In the rest of the paper we work in a fixed but arbitrary p-optimal expansion $(\mathbb{Q}_p, \widehat{\mathcal{L}})$ of the p-adically closed field $\mathbb{Q}_p$. Recall that if $x = (x_1, \ldots, x_m) \in \mathbb{Q}_p^m$, then $\|x\|_p$ will denote the maximum norm of x. That is, $\|x\|_p = \max\{|x_1|_p, \ldots, |x_m|_p\}$.

The following proposition shows that the growth of a continuous definable function with values in $\mathbb{Q}_p$ is bounded by a polynomial :

Proposition 3.1. *Let D be a closed definable subset of $\mathbb{Q}_p^m$ and $f\colon D \longrightarrow \mathbb{Q}_p$ a continuous definable function. Then there exist a positive integer d and $c > 0$ such that for all $x \in D$*

$$|f(x)|_p \leqslant c(1 + \|x\|_p^2)^d.$$

Proof. Let us put for $r \in \mathbb{Q}_p$,

$$D_r = \{x \in D : \ \|x\|_p = |r|_p\}.$$

Then D_r is a bounded and closed definable subset of $\mathbb{Q}_p^m$. According to Theorem 2.7, we can define a function $\theta\colon \mathbb{Q}_p \longrightarrow |\mathbb{Q}_p|_p$ by

$$\theta(r) = \begin{cases} \sup\{|f(x)|_p : \ x \in D_r\} & \text{if } D_r \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The function θ is definable since its graph is given by

$$\Big\{(r,t) \in \mathbb{Q}_p \times |\mathbb{Q}_p| : \ \Big(\exists\, x \in D_r,\ |t|_p = |f(x)|_p \text{ and } \forall\, y \in D_r,\ |f(y)|_p \leqslant |t|_p\Big) \text{ or } (D_r = \emptyset \text{ and } t = 0)\Big\}.$$

Therefore there exist a positive integer d and a constant $M > 0$ such that

$$|\theta(r)|_p \leqslant |r|_p^d$$

for all $|r|_p > M$. Let $c_0 = \sup\{|\theta(r)|_p : \ |r|_p \leqslant M\}$ and $c = \max\{1, c_0\}$. It follows that

$$|f(x)|_p \leqslant c(1 + \|x\|_p^2)^d$$

for all $x \in D$. This completes the proof. $\square$

Proposition 3.2. *Let D be a locally closed definable subset of $\mathbb{Q}_p^m$ and $f\colon D \longrightarrow \mathbb{Q}_p$ a continuous definable function. Let $g\colon \{x \in D:\ f(x) \neq 0\} \longrightarrow \mathbb{Q}_p$ be a continuous definable function. Then there exists a positive integer $d \geqslant 1$ such that the function $x \longmapsto f^d(x)g(x)$ can be continuously extended to D by 0 when $f(x) = 0$.*

Proof. According to Proposition 2.6, we may assume that D is a closed definable subset of $\mathbb{Q}_p^m$. For $x \in D$ and $t \in \mathbb{Q}_p^*$, consider

$$D_{x,t} = \left\{ y \in D:\ \|y - x\|_p \leqslant 1 \text{ and } |tf(y)|_p = 1 \right\}.$$

Then $D_{x,t}$ is a bounded and closed definable subset of $\mathbb{Q}_p^m$. Let us define

$$\theta(x,t) = \begin{cases} \sup\left\{ |g(y)|_p :\ y \in D_{x,t} \right\} & \text{if } D_{x,t} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\theta\colon D \times \mathbb{Q}_p^* \longrightarrow \mathbb{Q}_p$ is well-defined and definable. Let $x_0 \in D$ such that $f(x_0) = 0$. Then there is a positive integer k (independent of x_0 by a compactness argument) and $r(x_0) > 0$ such that

$$|\theta(x_0,t)|_p \leqslant |t|_p^k$$

for all $|t|_p > r(x_0)$. This means that

$$|f(y)|_p^k\, |g(y)|_p \leqslant 1 \quad \text{on} \quad \left\{ y \in D:\ f(y) \neq 0 \text{ and } \|y - x_0\|_p \leqslant 1 \right\}$$

for $|f(y)|_p$ sufficiently small. The function $f^{k+1}g$, extended by 0, is thus continuous at x_0. This is precisely the assertion of the proposition. □

Theorem 3.3. *Let D be a locally closed definable subset of $\mathbb{Q}_p^m$ and $f, g\colon D \longrightarrow \mathbb{Q}_p$ two continuous definable functions such that $g^{-1}(0) \subset f^{-1}(0)$. Then there exist a positive integer $d \geqslant 1$ and a continuous definable function $h\colon D \longrightarrow \mathbb{Q}_p$ such that $f^d = hg$ on D.*

Proof. The function $1/g$ is continuous definable on $\{x \in D:\ f(x) \neq 0\}$. By Proposition 3.2, there exists a positive integer $d \geqslant 1$ such that the function $h\colon D \longmapsto \mathbb{Q}_p$ defined by

$$h(x) = \begin{cases} \frac{f^d(x)}{g(x)} & \text{if } f(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

is continuous, definable and $f^d = hg$ on D. □

As a corollary one gets the Łojasiewicz inequality (which is the first main result):

Corollary 3.4 (Łojasiewicz inequality)**.** *Let D be a closed and bounded definable subset of $\mathbb{Q}_p^m$ and $f, g\colon D \longrightarrow \mathbb{Q}_p$ two continuous definable functions such that $g^{-1}(0) \subset f^{-1}(0)$. Then there exist a positive integer $d \geqslant 1$ and $c > 0$ such that for any $x \in D$*

$$|f(x)|_p^d \leqslant c\, |g(x)|_p .$$

Proof. We use theorem 3.3 with $c = \sup\left\{ |h(x)|_p :\ x \in D \right\}$. □

Corollary 3.5. *Let D be a closed and bounded definable subset of $\mathbb{Q}_p^m$ and $f\colon D \longrightarrow \mathbb{Q}_p$ a continuous definable function. Then there exist a positive integer $d \geqslant 1$ and $c > 0$ such that for any $x \in D$*

$$|f(x)|_p^d \leqslant c\,\mathrm{d}(x, f^{-1}(0))\cdot$$

4. Łojasiewicz exponent

Note that since $(\mathbb{Q}_p, \widehat{\mathcal{L}})$ satisfies the *extreme value propery* of [6], its field of exponents is the rational numbers field $\mathbb{Q}$. That is, for every definable function $f\colon \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$ (which is not identically nul for all sufficiently large $|x|_p$), there exist a positive real number c and $q \in \mathbb{Q}$ such that

$$|f(x)|_p \sim c\ |x|_p^q$$

for $|x|_p$ sufficiently large. Here are some examples:

Example 4.1. 1) The p-optimal field $(\mathbb{Q}_p, \mathcal{L}_{\mathrm{Mac}})$.
2) The p-optimal field $(\mathbb{Q}_p, \mathcal{L}_{\mathrm{an}})$.

According to Corollary 3.4, we can state the following definition:

Definition 4.1. Let $D \subset \mathbb{Q}_p^m$ be a closed and bounded definable set and $f, g\colon D \longrightarrow \mathbb{Q}_p$ two continuous definable functions such that $g^{-1}(0) \subset f^{-1}(0)$. The *Łojasiewicz exponent* of f with respect to g on D, denoted by $\ell_D(f, g)$, is defined as

$$\ell_D(f,g) = \inf\left\{\theta > 0 :\ |f(x)|_p^\theta \leqslant c\,|g(x)|_p \quad \text{for some } c > 0 \text{ and all } x \in D\right\}\cdot$$

The next theorem shows the rationality of the Łojasiewicz exponent for definable functions defined over definable bounded and closed subsets:

Theorem 4.2. *Let $D \subset \mathbb{Q}_p^m$ be a closed and bounded definable set and $f, g\colon D \longrightarrow \mathbb{Q}_p$ two continuous definable functions such that $g^{-1}(0) \subset f^{-1}(0)$. Then the Łojasiewicz exponent $\ell_D(f, g)$ of f with respect to g on D is a rational number. Moreover, if $\ell_D(f, g) \neq 0$ then there exists a constant $c > 0$ such that the Łojasiewicz inequality*

$$|f(x)|_p^{\ell_D(f,g)} \leqslant c\,|g(x)|_p$$

holds for all $x \in D$.

Proof. For $v \in \mathbb{Q}_p$, let us put $D_v = \{x \in D :\ |f(x)|_p = |v|_p\}\cdot$ Define $\rho\colon \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$ by

$$\rho(v) = \begin{cases} \inf\left\{|g(x)|_p :\ x \in D_v\right\} & \text{if } D_v \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Now consider the subset Δ of $\mathbb{Q}_p^2$ defined by

$$\Delta = \left\{(v, w) \in \mathbb{Q}_p \times \mathbb{Q}_p :\ |w|_p = |\rho(v)|_p\right\}\cdot$$

It is easy to see that Δ is a definable subset of $\mathbb{Q}_p^2$. Since $(\mathbb{Q}_p, \widehat{\mathcal{L}})$ admits definable Skolem functions, there exists a definable function $h\colon \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$ such that

$$|h(v)|_p = \inf_{\substack{x \in D \\ |f(x)|_p = |x|_p}} |g(x)|_p\,\cdot$$

Since the field of exponents of $(\mathbb{Q}_p, \widehat{\mathcal{L}})$ is $\mathbb{Q}$, there exist a rational number $r/q \in \mathbb{Q}$ and a constant $c_0 > 0$ such that

$$|h(v)|_p \sim c_0 \, |v|_p^{\frac{r}{q}} \quad \text{as } |v|_p \to 0. \tag{$*$}$$

From $(*)$, it follows that

$$|h(v)|_p \geqslant \frac{c_0}{2} \; |v|_p^{\frac{r}{q}}$$

for $|v|_p$ going to 0. This last inequality gives

$$|g(x)|_p \geqslant \frac{c_0}{2} \; |f(x)|_p^{\frac{r}{q}} .$$

for $|f(x)|_p$ going to 0. This means that for some $\delta > 0$ sufficiently small, we have

$$|f(x)|_p^{\frac{r}{q}} \leqslant \frac{2}{c_0} \, |g(x)|_p \tag{4.1}$$

for $|f(x)|_p < \delta$. Notice that $0 < r/q$, by continuity of f and g on the bounded and closed definable set D and using theorem 2.7. Moreover, $g(x)$ does not vanish for $|f(x)|_p \geqslant \delta$ since $g^{-1}(0) \subset f^{-1}(0)$. It follows that the function $x \mapsto f(x)^{\frac{r}{q}}/g(x)$ is continuous on the bounded and closed definable subset $\{x \in D \mid \, |f(x)|_p \geqslant \delta\}$. By theorem 2.7, there is $c_1 > 0$ such that

$$|f(x)|_p^{\frac{r}{q}} \leqslant c_1 \, |g(x)|_p \tag{4.2}$$

for $|f(x)|_p \geqslant \delta$. Let $c = \max\{2/c_0, c_1\}$. Then combining the inequalities (4.1) and (4.2), we get

$$|f(x)|_p^{\frac{r}{q}} \leqslant c \, |g(x)|_p$$

for all $x \in D$. Therefore

$$\ell_D(f, g) \leqslant \frac{r}{q}.$$

On the other hand, assume that $\ell_D(f, g) \neq 0$ and let $\theta > 0$ such that the inequality

$$|f(x)|_p^{\theta} \leqslant c_0' \, |g(x)|_p$$

holds for some constant $c_0' > 0$ and all $x \in D$. From this last inequality, we deduce that

$$|v|_p^{\theta} \leqslant c_0' \, |h(v)|_p$$

for $|v|_p$ going to 0. Now using $(*)$ once again, we can find a constant $c_1' > 0$ such that

$$|h(v)|_p \leqslant c_1' \, |v|_p^{\frac{r}{q}}$$

for $|v|_p$ going to 0. Combining these two last inequalities, we conclude that $|v|_p^{\theta} \leqslant c_0' c_1' \, |v|_p^{\frac{r}{q}}$ for $|v|_p$ going to 0. A passage to the limit as $|v|_p \to 0$ in this last inequality implies that $r/q \leqslant \theta$. Hence $r/q \leqslant \ell_D(f, g)$. This completes the proof of the theorem. □

Remark 4.3. Note that our results still hold true in p-optimal expansions of an arbitrary p-adically closed field K provided that they satisfy the *extreme value propery* of [6]. We have to pay attention that when working with general p-adically closed fields, it is possible that the value group $|K^{\times}|_p$ (considered as a multiplicative group) is not contained in $\mathbb{R}^{\times}$. In this case the *Łojasiewicz exponent* for example can still be defined as in 4.1, the only difference being that θ will be elements of $|K^{\times}|_p$ rather than $\mathbb{R}_+^{\times}$.

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