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HIGHER ORDER APOSTOL-FROBENIUS-TYPE POLY-GENOCCHI POLYNOMIALS WITH PARAMETERS a, b AND c

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Abstract. In this paper, the Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c are defined using the concept of polylogarithm. These polynomials possess numerous properties including recurrence relations, explicit formulas and certain differential identity. Consequently, these higher order Apostol-Frobenius-type poly-Genocchi polynomials are classified as Appell polynomials and inherit some properties of Appell polynomials. A symmetrized generalization of these polynomials is constructed that satisfies a kind of double generating function. Lastly, the paper has been concluded by introducing the type 2 Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c which possess several interesting identities.

1. Introduction

The Genocchi numbers have been generalized in different ways. Two of these are the Genocchi polynomials and Genocchi polynomials of higher order, which are given as follows:

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}, \quad |t| < \pi, \tag{1}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{xt}. \tag{2}$$

Other generalizations are done by mixing the concept of some known polynomials like mixing with the Apostol polynomials, which yields the Apostol-Genocchi polynomials, and Apostol-Genocchi polynomials of higher order, which are respectively

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defined as follows:

$$\sum_{n=0}^{\infty} G_n(x,\lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t+1}e^{xt}, \tag{3}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^k e^{xt}, \tag{4}$$

where $|t| < \pi$ when $\lambda = 1$ and $|t| < \log(-\lambda)$ when $\lambda \neq 1, \lambda \in \mathbb{C}$. Also, mixing with Frobenius polynomials yields the so-called Frobenius-Genocchi polynomials, which are given by

$$\sum_{n=0}^{\infty} G_n^F(x;u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t-u}e^{xt}, \tag{5}$$

(see [8, 9, 18, 19, 26, 27, 29, 37]). Another generalization of Genocchi numbers was constructed by mixing these numbers with the following definition of polylogarithm $\mathrm{Li}_k(z)$

$$\mathrm{Li}_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^k}, k \in \mathbb{Z}. \tag{6}$$

These constructed polynomials are called the poly-Genocchi polynomials, which were first introduced by Kim et al. [25]. More precisely, the poly-Genocchi polynomials are defined as follows

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{x^n}{n!} = \frac{2Li_k(1-e^t)}{e^t+1}e^{xt}. \tag{7}$$

Moreover, a modified poly-Genocchi polynomials, denoted by $G_{n,2}^{(k)}(x)$, were defined by Kim et al. [25] as follows

$$\sum_{n=0}^{\infty} G_{n,2}^{(k)}(x)\frac{x^n}{n!} = \frac{Li_k(1-e^{-2t})}{e^t+1}e^{xt}. \tag{8}$$

Note that, when $k = 1$, equations (7) and (8) give the Genocchi polynomials in (1). That is,

$$G_n^{(1)}(x) = G_{n,2}^{(1)}(x) = G_n(x).$$

Kim et. al [25] obtained several properties of these polynomials.

On the other hand, Kurt [30] defined two forms of generalized poly-Genocchi polynomials with parameters a, b, and c, as follows

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x;a,b,c)\frac{x^n}{n!} \tag{9}$$

$$\frac{2Li_k(1-(ab)^{-2t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} G_{n,2}^{(k)}(x;a,b,c)\frac{x^n}{n!}, \tag{10}$$

which are motivated by the definitions in (7) and (8), respectively. Kurt [30] also derived several properties parallel to those of poly-Genocchi polynomials by Kim et al. [25]. Note that, when $x = 0$, (7) reduces to

$$\frac{2Li_k(1-e^t)}{e^t+1} = \sum_{n=0}^{\infty} G_n^{(k)}\frac{x^n}{n!}, \tag{11}$$

where $G_n^{(k)}$ are called the poly-Genocchi numbers.

In this paper, a new variation of poly-Genocchi polynomials with parameters a, b and c is constructed by mixing the definitions of Apostol and Frobenius polynomials, namely, the Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c. Section 2 introduces the definition of this new variation and enumerates some special cases. Section 3 provides some identities that contain a number of relations of this new variation with some Genocchi-type polynomials. Section 4 devotes its discussion on some identities that link the Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c to Appell polynomials. Section 5 focuses on the connections of these higher order Apostol-Frobenius-type poly-Genocchi polynomials to Stirling numbers of the second kind and different variations of higher order Bernoulli-type polynomials. Section 6 constructs another form of generalization of these higher order Apostol-Frobenius-type poly-Genocchi polynomials, which is called the symmetrized generalization. Section 7 introduces the type 2 Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c using the concept of polyexponential function [24]. Section 8 contains the conclusion of the paper.

2. Definition and Some Special Cases

Now, a new variation of poly-Genocchi polynomials called the Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a b and c, will be introduced.

Definition 2.1. The Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c, denoted by $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)$, are defined as follows

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}c^{xt},\quad |t|<\frac{\sqrt{(\ln\left(\frac{\lambda}{u}\right))^2+4\pi^2}}{|\ln a+\ln b|}. \tag{12}$$

When $\alpha=1$, (12) yields

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}c^{xt},\quad |t|<\frac{\sqrt{(\ln\left(\frac{\lambda}{u}\right))^2+4\pi^2}}{|\ln a+\ln b|}, \tag{13}$$

where $\mathcal{G}_n^{(k)}(x;\lambda,u,a,b,c)=\mathcal{G}_n^{(k,1)}(x;\lambda,a,b,c)$ denotes the Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b and c.

The following are some special cases of the Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c:

(1) When $c=e$, equation (12) reduces to

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,e)\frac{t^n}{n!}=\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}e^{xt}. \tag{14}$$

For convenience, we use $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b)$ to denote $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,e)$. That is,

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b)\frac{t^n}{n!}=\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}e^{xt}. \tag{15}$$

(2) Using the fact that

$$Li_1(z)=-\ln(1-z),$$

when $k=1$, (12) yields

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\left(\frac{(1-u)t\ln ab}{\lambda b^t-ua^{-t}}\right)^{\alpha}c^{xt}, \tag{16}$$

where the polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda,u,a,b,c)=\mathcal{G}_n^{(1,\alpha)}(x;\lambda,u,a,b,c)$ are called the Apostol-Frobenius-type Genocchi polynomials of higher order with parameters a b and c. When $\alpha=1$, (16) yields

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(1)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\frac{(1-u)t\ln ab}{\lambda b^t-ua^{-t}}c^{xt}, \tag{17}$$

where the polynomials $\mathcal{G}_n^{(1)}(x;\lambda,u,a,b,c)$ are called the Apostol-Frobenius-type Genocchi polynomials with parameters a b and c.

(3) When $a=1, b=e$, (15) will reduce to

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,1,e)\frac{t^n}{n!}=\left(\frac{Li_k(1-e^{-(1-u)t})}{\lambda e^t-u}\right)^{\alpha}e^{xt}. \tag{18}$$

We may use the notations

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u)=\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,1,e)\ \text{ and }\ \mathcal{G}_n^{(k)}(x;\lambda,u)=\mathcal{G}_n^{(k)}(x;\lambda,u,1,e)$$

and call them Apostol-Frobenius-type poly-Genocchi polynomials of higher order and Apostol-Frobenius-type poly-Genocchi polynomials, respectively.

(4) When $\lambda=1$, (18) gives

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;1,u)\frac{t^n}{n!}=\left(\frac{Li_k(1-e^{-(1-u)t})}{e^t-u}\right)^{\alpha}e^{xt}. \tag{19}$$

which is the higher order version of equation (8), i.e. the higher order version of the modified poly-Genocchi polynomials of Kim et al.[25]. We may use $G_{n,2}^{(k,\alpha)}(x)$ to denote $\mathcal{G}_n^{(k,\alpha)}(x;1)$.

(5) When $k=1$, (18) gives

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(1,\alpha)}(x;\lambda,u)\frac{t^n}{n!}=\left(\frac{(1-u)t}{\lambda e^t-u}\right)^{\alpha}e^{xt}, \tag{20}$$

and when $\lambda=1$, (20) gives

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(1,\alpha)}(x;1,u)\frac{t^n}{n!}=\left(\frac{(1-u)t}{e^t-u}\right)^{\alpha}e^{xt},$$

where $\mathcal{G}_n^{(1,\alpha)}(x;\lambda,u) = \mathcal{G}_n^{(\alpha)}(x;\lambda,u)$ and $\mathcal{G}_n^{(1,\alpha)}(x;1,u) = \mathcal{G}_n^{(\alpha)}(x;u)$ are called the Apostol-Frobenius-type Genocchi polynomials and Frobenius-Genocchi polynomials of higher order in (4) and (2), respectively. Furthermore, when $\alpha = 1$, we have

$$\sum_{n=0}^{\infty} \mathcal{G}_n(x;\lambda,u)\frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t - u}e^{xt}, \tag{21}$$

and

$$\sum_{n=0}^{\infty} \mathcal{G}_n(x;u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t - u}e^{xt},$$

where $\mathcal{G}_n(x;\lambda,u)$ and $\mathcal{G}_n(x;u)$ are called the Apostol-Frobenius-type Genocchi polynomials and Frobenius-Genocchi polynomials in (4) and (2), respectively.

3. Some Identities in Connection with Genocchi-type Polynomials

In this section, some relations for $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ will be established which are expressed in terms of some Genocchi-type polynomials.

First, certain recurrence relation for $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ will be established.

Theorem 3.1. *The Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a,b,c satisfy the recurrence relation*

$$\mathcal{G}_n^{(k,\alpha)}(x+1;\lambda,u,a,b,c) = \sum_{r=0}^{n}\binom{n}{r}(\ln c)^r \mathcal{G}_{n-r}^{(k,\alpha)}(x;\lambda,u,a,b,c). \tag{22}$$

Proof. Equation (12) can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x+1;\lambda,u,a,b,c)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} e^{xt\ln c}e^{t\ln c}\\ &= \left\{\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}\right\}\left\{\sum_{n=0}^{\infty}\frac{(t\ln c)^n}{n!}\right\}\\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \mathcal{G}_{n-r}^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^{n-r}}{(n-r)!}\frac{(\ln c)^r t^r}{r!}\\ &= \sum_{n=0}^{\infty}\left\{\sum_{r=0}^{n}\binom{n}{r}(\ln c)^r \mathcal{G}_{n-r}^{(k,\alpha)}(x;\lambda,u,a,b,c)\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem. □

Consider a special case of (15) by taking $x = 0$. This gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(0;\lambda,u,a,b,c)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha}. \tag{23}$$

We use the notation $\mathcal{G}_i^{(k,\alpha)}(\lambda,u,a,b) = \mathcal{G}_i^{(k,\alpha)}(0;\lambda,u,a,b)$ and call them the Apostol-Frobenius-type poly-Genocchi numbers of higher order with parameters a and b. The following theorem contains an identity that expresses $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)$ as polynomial in x, which involves $\mathcal{G}_i^{(k,\alpha)}(\lambda,u,a,b)$ as coefficients.

Theorem 3.2. *The Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b, c satisfy the relation,*

$$\mathcal{G}_n^{(k,\alpha)}(x; \lambda, u, a, b, c) = \sum_{i=0}^{n} \binom{n}{i} (\ln c)^{n-i} \mathcal{G}_i^{(k,\alpha)}(\lambda, u, a, b) x^{n-i}. \tag{24}$$

Proof. Equation (12) can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x; \lambda, u, a, b, c) \frac{t^n}{n!} &= \left(\frac{Li_k(1 - (ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}} \right)^{\alpha} c^{xt} = e^{xt \ln c} \sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(\lambda, u, a, b) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{(xt \ln c)^{n-i}}{(n-i)!} \mathcal{G}_i^{(k,\alpha)}(\lambda, u, a, b) \frac{t^i}{i!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} \binom{n}{i} (\ln c)^{n-i} \mathcal{G}_i^{(k,\alpha)}(\lambda, u, a, b) x^{n-i} \right) \frac{t^n}{n!}\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

The next identity gives the relation between $\mathcal{G}_n^{(k,\alpha)}(x; \lambda, u, a, b, c)$ and $\mathcal{G}_n^{(k,\alpha)}(x; \lambda, u)$.

Theorem 3.3. *The Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b, c satisfy the relation,*

$$\mathcal{G}_n^{(k,\alpha)}(x; \lambda, u, a, b, c) = (\ln a + \ln b)^n \mathcal{G}_n^{(k,\alpha)} \left(\frac{x \ln c + \alpha \ln a}{\ln a + \ln b}; \lambda, u \right). \tag{25}$$

Proof. Using (12), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x; \lambda, a, b, c) \frac{t^n}{n!} &= \left(\frac{Li_k(1 - (ab)^{-(1-u)t})}{a^{-t}(\lambda (ab)^t - u)} \right)^{\alpha} e^{xt \ln c} \\ &= e^{\frac{x \ln c + \alpha \ln a}{\ln ab} t \ln ab} \left(\frac{Li_k(1 - e^{-(1-u)t \ln ab})}{\lambda e^{t \ln ab} - u} \right)^{\alpha} \\ &= \sum_{n=0}^{\infty} (\ln a + \ln b)^n \mathcal{G}_n^{(k,\alpha)} \left(\frac{x \ln c + \alpha \ln a}{\ln a + \ln b}; \lambda, u \right) \frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

4. Classification as Appell Polynomials

The following theorem contains a differential identity that can be used to classify Apostol-type poly-Genocchi polynomials as Appell polynomials [31, 33, 36]

Theorem 4.1. *The Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b, c satisfy the relation,*

$$\frac{d}{dx} \mathcal{G}_{n+1}^{(k,\alpha)}(x; \lambda, u, a, b, c) = (n+1)(\ln c) \mathcal{G}_n^{(k,\alpha)}(x; \lambda, u, a, b, c). \tag{26}$$

Proof. Applying the first derivative to equation (12), we have

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=t(\ln c)\left(\frac{Li_k(1-(ab)^{-2t})}{(\lambda b^t-ua^{-t})}\right)^{\alpha}e^{xt\ln c}$$

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^{n-1}}{n!}=\sum_{n=0}^{\infty}(\ln c)\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}$$

Hence,

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{d}{dx}\mathcal{G}_{n+1}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}(\ln c)\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}$$

.
Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

Remark 4.1. When $c=e$, equation (26) reduces to

$$\frac{d}{dx}\mathcal{G}_{n+1}^{(k,\alpha)}(x;\lambda,u,a,b)=(n+1)\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b), \tag{27}$$

which is one of the property for the polynomial to be classified as Appell polynomial.

Being classified as Appell polynomials, the Apostol-Frobenius-type poly-Genocchi polynomials $\mathcal{G}_n^{(k)}(x;\lambda,u,a,b)$ must possess the following properties

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b)=\sum_{i=0}^{n}\binom{n}{i}c_ix^{n-i}$$

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b)=\left(\sum_{i=0}^{\infty}\frac{c_i}{i!}D^i\right)x^n,$$

for some scalar $c_i\neq 0$. It is then necessary to find the sequence $\{c_n\}$. However, by using (24) with $c=e$, $c_i=\mathcal{G}_i^{(k,\alpha)}(\lambda,a,b)$. This implies the following corollary.

Corollary 4.2. *The Apostol-Frobenius-type poly-Genocchi polynomials with parameters a,b,c satisfy the formula,*

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b)=\left(\sum_{i=0}^{\infty}\frac{\mathcal{G}_i^{(k,\alpha)}(\lambda,u,a,b)}{i!}D^i\right)x^n.$$

For example, when $n=3$, we have

$$\begin{aligned}\mathcal{G}_3^{(k,\alpha)}(x;\lambda,u,a,b)&=\left(\sum_{i=0}^{\infty}\frac{\mathcal{G}_i^{(k,\alpha)}(\lambda,u,a,b)}{i!}D^i\right)x^3\\&=\frac{\mathcal{G}_0^{(k,\alpha)}(\lambda,u,a,b)}{0!}x^3+\frac{\mathcal{G}_1^{(k,\alpha)}(\lambda,u,a,b)}{1!}D^1x^3+\frac{\mathcal{G}_2^{(k,\alpha)}(\lambda,u,a,b)}{2!}D^2x^3\\&\quad+\frac{\mathcal{G}_3^{(k,\alpha)}(\lambda,u,a,b)}{3!}D^3x^3\\&=\mathcal{G}_0^{(k,\alpha)}(\lambda,u,a,b)x^3+3\mathcal{G}_1^{(k,\alpha)}(\lambda,u,a,b)x^2+3\mathcal{G}_2^{(k,\alpha)}(\lambda,u,a,b)x+\mathcal{G}_3^{(k,\alpha)}(\lambda,u,a,b).\end{aligned}$$

The next corollary immediately follows from equation (27) and the characterization of Appell polynomials [31, 33, 36].

Corollary 4.3. *The Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b, c satisfy the addition formula*

$$\mathcal{G}_n^{(k,\alpha)}(x+y;\lambda,u,a,b) = \sum_{i=0}^{\infty} \binom{n}{i} \mathcal{G}_i^{(k,\alpha)}(x;\lambda,u,a,b) y^{n-i}. \tag{28}$$

Taking $x = 0$ in formula (28) and using the fact $\mathcal{G}_n^{(k)}(0;\lambda,a,b) = \mathcal{G}_n^{(k)}(\lambda,a,b)$, Corollary 4.3 gives formula (24) in Theorem 3.2 with $c = e$.

An extension of this addition formula can be derived as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x+y;\lambda,u,a,b,c)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{(\lambda b^t - ua^{-t})}\right)^{\alpha} c^{xt} e^{yt\ln c} \\ &= \left(\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y\ln c)^n \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_i^{(k,\alpha)}(x;\lambda,u,a,b,c)(y\ln c)^{n-i}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives the following theorem.

Theorem 4.4. *The Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c satisfy the addition formula,*

$$\mathcal{G}_n^{(k,\alpha)}(x+y;\lambda,u,a,b,c) = \sum_{i=0}^{\infty}\binom{n}{i}(\ln c)^{n-i}\mathcal{G}_i^{(k,\alpha)}(x;\lambda,u,a,b,c)y^{n-i}. \tag{29}$$

By taking $x = 0$, equation (29) exactly gives (24).

5. Connections with Some Special Numbers and Polynomials

In this section, some connections of the higher order Apostol-type poly-Genocchi polynomials $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ with other well-known special numbers and polynomials will be established.

The next theorem contains an identity that relates the Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c to Stirling numbers of the second kind $\left\{ n \atop m \right\}$ defined in [17] by

$$\sum_{n=m}^{\infty}\left\{ n \atop m \right\}\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{30}$$

Here, it is important to note that if $(c_0, c_1, \ldots, c_j, \ldots)$ is any sequence of numbers and l is a positive integer, then

$$\begin{aligned}\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!}\right)^l &= \prod_{i=1}^{l}\left(\sum_{n_i=0}^{\infty}\frac{c_{n_i}}{n_i!}t^{n_i}\right) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{n_1+n_2+\ldots+n_\alpha=n}\prod_{i=1}^{l} c_{n_i}\binom{n}{n_1,n_2,\ldots,n_\alpha}\right\}\frac{t^n}{n!}. \end{aligned}\tag{31}$$

(see [17]).

Theorem 5.1. *The Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c satisfies the relation,*

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{\alpha} \mathcal{G}_{n-j}^{(\alpha)} \left(\frac{x\ln c + \alpha \ln a}{\ln a + \ln b}; \lambda, u \right) d_j, \qquad (32)$$

where

$$d_j = \sum_{n_1+n_2+\ldots+n_\alpha = j} \prod_{i=1}^{\alpha} c_{n_i} \binom{j}{n_1, n_2, \ldots, n_\alpha} \ \ \textit{and} \ \ c_j = \sum_{m=0}^{j} (-1)^{m+j+1} \frac{((1-u)\ln ab)^j m! \left\{ {j+1 \atop m+1} \right\}}{(j+1)(m+1)^{k-1}}.$$

Proof. Now, (12) can be written as

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \frac{c^{xt}}{(\lambda b^t - ua^{-t})^{\alpha}} \left(\sum_{m=1}^{\infty} \frac{(1-e^{-(1-u)t\ln ab})^m}{m^k} \right)^{\alpha}$$

$$= \frac{c^{xt}}{(\lambda b^t - ua^{-t})^{\alpha}} \left(\sum_{m=0}^{\infty} \frac{(1-e^{-(1-u)t\ln ab})^{m+1}}{(m+1)^k} \right)^{\alpha}$$

$$= \frac{c^{xt}}{(\lambda b^t - ua^{-t})^{\alpha}} \left(\sum_{m=0}^{\infty} \frac{m!}{(m+1)^{k-1}} \frac{(1-e^{-(1-u)t\ln ab})^{m+1}}{(m+1)!} \right)^{\alpha}$$

$$= \frac{c^{xt}}{(\lambda b^t - ua^{-t})^{\alpha}} \left(\sum_{m=0}^{\infty} \frac{(-1)^{m+1} m!}{(m+1)^{k-1}} \sum_{j=m+1}^{\infty} \left\{ {j \atop m+1} \right\} \frac{(-(1-u)t\ln ab)^j}{j!} \right)^{\alpha}$$

$$= (-1)^{\alpha} c^{xt} \left(\frac{(1-u)t\ln ab}{\lambda b^t - ua^{-t}} \right)^{\alpha} \left(\sum_{m=0}^{\infty} \sum_{j=m}^{\infty} \frac{(-1)^{m+1} m! \left\{ {j+1 \atop m+1} \right\}}{(j+1)(m+1)^{k-1}} \frac{(-(1-u)t\ln ab)^j}{j!} \right)^{\alpha}.$$

Using (16), we get

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} (-1)^{\alpha} \mathcal{G}_n^{(\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!} \right)^{\alpha},$$

where

$$c_j = \sum_{m=0}^{j} (-1)^{m+j+1} \frac{((1-u)\ln ab)^j m! \left\{ {j+1 \atop m+1} \right\}}{(j+1)(m+1)^{k-1}}.$$

Note that, using (31), $\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!}\right)^{\alpha}$ can be expressed as

$$\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!} \right)^{\alpha} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!},$$

where

$$d_n = \sum_{n_1+n_2+\ldots+n_\alpha = n} \prod_{i=1}^{\alpha} c_{n_i} \binom{n}{n_1, n_2, \ldots, n_\alpha}.$$

It follows that

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}(-1)^{\alpha}\mathcal{G}_{n-j}^{(\alpha)}(x;\lambda,u,a,b,c)d_j\right\}\frac{t^n}{n!}.$$

Comparing the coefficients and using equation (25) complete the proof of the theorem. □

Remark 5.1. When $\alpha=1$, $d_j=c_j$.

The identities in the following theorem are derived using the fact that the Apostol-type poly-Genocchi polynomials of higher order $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b)$ with parameters a and b satisfy the relation in (14).

Theorem 5.2. *The Apostol-type poly-Genocchi polynomials of higher order $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ with parameters a,b,c satisfy the following explicit formulas:*

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)=\sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{{l \atop m}\right\}\binom{n}{l}(\ln c)^l\mathcal{G}_{n-l}^{(k,\alpha)}(-m\ln c;\lambda,u,a,b)(x)^{(m)} \tag{33}$$

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)=\sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{{l \atop m}\right\}\binom{n}{l}(\ln c)^l\mathcal{G}_{n-l}^{(k,\alpha)}(\lambda,u,a,b)(x)_m \tag{34}$$

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b)\sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\left\{{l+s \atop s}\right\}\frac{\binom{n-l}{m}}{\binom{l+s}{s}}\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,u,a,b)B_m^{(s)}(x\ln c;\lambda) \tag{35}$$

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b)=\sum_{m=0}^{\infty}\frac{\binom{n}{m}}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\mathcal{G}_{n-m}^{(k,\alpha)}(j;\lambda,u,a,b)F_m^{(s)}(x;\mu), \tag{36}$$

where $(x)^{(n)}=x(x+1)\cdots(x+n-1)$, $(x)_n=x(x-1)\cdots(x-n+1)$, the rising and falling factorials of x of degree n, $B_n^{(\alpha)}(x;\lambda)$, the Apostol-Bernoulli polynomials of higher order in [32], *defined by*

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x;\lambda)\frac{x^n}{n!}, \tag{37}$$

and $F_n^{(s)}(x;\mu)$, the Frobenius polynomials of higher order [32], *defined by*

$$\left(\frac{1-\mu}{e^t-\mu}\right)^{s}e^{xt}=\sum_{n=0}^{\infty}F_n^{(s)}(x;\mu)\frac{t^n}{n!}.$$

Proof. Proving relations (33)-(35) makes use of the definition of Stirling numbers of the second kind in (30). To prove (33), using the generalized Binomial Theorem, (12) may be written as

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}\sum_{m=0}^{\infty}\binom{x+m-1}{m}(1-e^{-t\ln c})^m$$

$$= \sum_{m=0}^{\infty} (x)^{(m)} \frac{(e^{t\ln c}-1)^m}{m!} \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} e^{-mt\ln c}$$
$$= \sum_{m=0}^{\infty} (x)^{(m)} \left(\sum_{n=0}^{\infty} \left\{ n \atop m \right\} \frac{(t\ln c)^n}{n!}\right) \left(\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(-m\ln c;\lambda,u,a,b)\frac{t^n}{n!}\right)$$
$$= \sum_{m=0}^{\infty} (x)^{(m)} \sum_{n=0}^{\infty}\sum_{l=0}^{n} \left\{ l \atop m \right\} (\ln c)^l \frac{t^l}{l!} \mathcal{G}_{n-l}^{(k,\alpha)}(-m\ln c;\lambda,u,a,b)\frac{t^{n-l}}{(n-l)!}$$
$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty}\sum_{l=m}^{n} \left\{ l \atop m \right\} \binom{n}{l} (\ln c)^l \mathcal{G}_{n-l}^{(k,\alpha)}(-m\ln c;\lambda,u,a,b)(x)^{(m)} \right\} \frac{t^n}{n!}.$$

Comparing coefficients completes the proof of (33). To prove identity (34), (12) may be written as

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} \sum_{m=0}^{\infty} \binom{x}{m} (e^{t\ln c}-1)^m$$
$$= \sum_{m=0}^{\infty} (x)_m \frac{(e^{t\ln c}-1)^m}{m!} \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha}$$
$$= \sum_{m=0}^{\infty} (x)_m \left(\sum_{n=0}^{\infty} \left\{ n \atop m \right\} \frac{(t\ln c)^n}{n!}\right) \left(\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(0;\lambda,u,a,b)\frac{t^n}{n!}\right)$$
$$= \sum_{m=0}^{\infty} (x)_m \sum_{n=0}^{\infty}\sum_{l=0}^{n} \left\{ l \atop m \right\} (\ln c)^l \frac{t^l}{l!} \mathcal{G}_{n-l}^{(k,\alpha)}(\lambda,u,a,b)\frac{t^{n-l}}{(n-l)!}$$
$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty}\sum_{l=m}^{n} \left\{ l \atop m \right\} \binom{n}{l} (\ln c)^l \mathcal{G}_{n-l}^{(k,\alpha)}(\lambda,u,a,b)(x)_m \right\} \frac{t^n}{n!}.$$

Again, comparing coefficients completes the proof of (34). To prove relation (35), using (37), (12) may be expressed as

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!} = \left(\frac{(e^t-1)^s}{s!}\right)\left(\frac{t^s e^{xt\ln c}}{(e^t-1)^s}\right)\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} \frac{s!}{t^s}$$

$$= \left(\sum_{n=0}^{\infty} \left\{ n+s \atop s \right\} \frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty} B_m^{(s)}(x\ln c;\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(0;\lambda,u,a,b)\frac{t^m}{m!}\right)\frac{s!}{t^s}$$
$$= \left(\sum_{n=0}^{\infty} \left\{ n+s \atop s \right\} \frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} B_m^{(s)}(x\ln c;\lambda)\mathcal{G}_{n-m}^{(k,\alpha)}(\lambda,u,a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}$$
$$= \left(\sum_{n=0}^{\infty}\sum_{l=0}^{n} \left\{ l+s \atop s \right\} \frac{t^{l+s}}{(l+s)!} \sum_{m=0}^{n-l} \binom{n-l}{m} B_m^{(s)}(x\ln c;\lambda)\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,u,a,b)\frac{t^{n-l}}{(n-l)!}\right)\frac{s!}{t^s}.$$

This can further be written as

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}$$
$$=\left(\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{m=0}^{n-l}\left\{\begin{matrix}l+s\\ s\end{matrix}\right\}\frac{l!s!}{(l+s)!}\binom{n-l}{m}B_m^{(s)}(x\ln c;\lambda)\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,u,a,b)\frac{n!}{(n-l)!l!}\frac{t^n}{n!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\left\{\begin{matrix}l+s\\ s\end{matrix}\right\}\frac{\binom{n-l}{m}}{\binom{l+s}{s}}B_m^{(s)}(x\ln c;\lambda)\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,u,a,b)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (35). To prove relation (36), (12) may be expressed as

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\left(\frac{(1-\mu)^s}{(e^t-\mu)^s}e^{xt\ln c}\right)\left(\frac{(e^t-\mu)^s}{(1-\mu)^s}\right)\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}$$

$$=\frac{1}{(1-\mu)^s}\left(\sum_{n=0}^{\infty}F_n^{(s)}(x\ln c;\mu)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}e^{jt}\right)$$
$$=\frac{1}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\left(\sum_{n=0}^{\infty}F_n^{(s)}(x\ln c;\mu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(j;\lambda,u,a,b)\frac{t^n}{n!}\right)$$
$$=\frac{1}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m}^{(k,\alpha)}(j;\lambda,u,a,b)F_m^{(s)}(x\ln c;\mu)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\mathcal{G}_{n-m}^{(k,\alpha)}(j;\lambda,u,a,b)F_m^{(s)}(x\ln c;\mu)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (36). □

6. Symmetrized Generalization of $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)$

Definition 6.1. *For $m,n\geq 0$, we define the symmetrized generalization of multi poly-Genocchi polynomials with parameters a, b and c as follows,*

$$\mathcal{S}_n^{(m,\alpha)}(x,y;\lambda,u,a,b,c)=\sum_{k=0}^{m}\binom{m}{k}\frac{\mathcal{G}_n^{(-k,\alpha)}(x;\lambda,u,a,b,c)}{(\ln a+\ln b)^n}\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)^{m-k}. \tag{38}$$

The following theorem contains the double generating function for $\mathcal{S}_n^{(m)}(x,y;a,b,c)$.

Theorem 6.2. *For $n,m\geq 0$, we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{S}_n^{(m,\alpha)}(x,y;\lambda,u,a,b,c)\frac{t^n}{n!}\frac{z^m}{m!}=\frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)z}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}e^{(1-u)t}}{(1+\lambda e^t)(e^{2t}-e^{(1-u)t+z}+e^z)}. \tag{39}$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{S}_n^{(m,\alpha)}(x,y;\lambda,u,a,b,c)\frac{t^n}{n!}\frac{z^m}{m!}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{\mathcal{G}_n^{(-k,\alpha)}(x;\lambda,u,a,b,c)}{(\ln a+\ln b)^n}\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)^{m-k}\frac{t^n}{n!}\frac{z^m}{k!(m-k)!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=k}^{\infty} \frac{\mathcal{G}_n^{(-k,\alpha)}(x;\lambda,u,a,b,c)}{(\ln a+\ln b)^n}\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)^{m-k}\frac{t^n}{n!}\frac{z^m}{k!(m-k)!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\mathcal{G}_n^{(-k,\alpha)}(x;\lambda,u,a,b,c)}{(\ln a+\ln b)^n}\frac{t^n}{n!}\frac{z^k}{k!}\sum_{l=0}^{\infty}\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)^{l}\frac{z^l}{l!}\\
&= e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)z}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mathcal{G}_n^{(-k,\alpha)}(\lambda,u,a,b,c)}{(\ln a+\ln b)^n}\frac{t^n}{n!}\frac{z^k}{k!}.
\end{aligned}
$$

Applying (25) yields

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{S}_n^{(m,\alpha)}(x,y;\lambda,a,b,c)\frac{t^n}{n!}\frac{z^m}{m!} = e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)z}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{G}_n^{(-k,\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda,u\right)\frac{t^n}{n!}\frac{z^k}{k!}.
$$

Now, using (18), we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{S}_n^{(m,\alpha)}(x,y;\lambda,u,a,b,c)\frac{t^n}{n!}\frac{z^m}{m!} &= e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)z}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}\sum_{k=0}^{\infty}\frac{L_{i_{(-k)}}(1-e^{-(1-u)t})}{u+\lambda e^t}\frac{z^k}{k!}\\
&= \frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)z}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}}{u+\lambda e^t}\sum_{k=0}^{\infty}L_{i_{(-k)}}(1-e^{-(1-u)t})\frac{z^k}{k!}.
\end{aligned}
$$

Employing the definition of polylogarithm yields

$$
\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{S}_n^{(m,\alpha)}(x,y;\lambda,u,a,b,c)\frac{t^n}{n!}\frac{z^m}{m!} &= \frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)z}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}}{1+\lambda e^t}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(1-e^{-(1-u)t}\right)^m}{m^{-k}}\frac{z^k}{k!}\\
&= \frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)z}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}}{(1+\lambda e^t)(1-((1-e^{-(1-u)t})e^z))}\\
&= \frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)z}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}e^{(1-u)t}}{(1+\lambda e^t)(e^{2t}-e^{(1-u)t+z}+e^z)}.
\end{aligned}
$$

□

The Apostol-Frobenius-type poly-Genocchi polynomials discussed above will be referred to as type 1 Apostol-Frobenius-type poly-Genocchi polynomials. Type 2 of these polynomials will be introduced in the next section.

7. Type 2 Higher Order Apostol-Frobenius-type Poly-Genocchi Polynomials

In this section, we will consider another variation of Genocchi polynomials using the concept of polyexponential function [24] defined by

$$
e_k(z) = \sum_{m=1}^{\infty}\frac{z^m}{(m-1)!m^k}. \tag{40}
$$

Note that when $k=1$, $e_1(z)=e^z-1$. Hence, if $z=\log(1+2t)$,

$$
e_1(z) = e_1(\log(1+2t)) = e^{\log(1+2t)}-1 = 2t.
$$

Definition 7.1. The type 2 Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c, denoted by $\mathcal{G}_{n,2}^{(k)}(x;\lambda,a,b,c)$, are defined as follows,

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\left(\frac{e_k(\log(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}c^{xt},\quad |t|<\frac{\sqrt{(\ln\left(\frac{\lambda}{u}\right))^2+4\pi^2}}{|\ln a+\ln b|}. \tag{41}$$

The following are some special cases of $\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,a,b,c)$:

(1) When $x=0$, we use $\mathcal{G}_{n,2}^{(k,\alpha)}(\lambda,u,a,b)$ to denote $\mathcal{G}_{n,2}^{(k,\alpha)}(0;\lambda,a,b,c)$, the type 2 Apostol-Frobenius-type poly-Genocchi numbers with parameters a, b. That is,

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(\lambda,u,a,b)\frac{t^n}{n!}=\left(\frac{e_k(\log(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}. \tag{42}$$

(2) When $a=1, b=c=e$, (41) yields

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u)\frac{t^n}{n!}=\left(\frac{e_k(\log(1+(1-u)t))}{\lambda e^t-u}\right)^{\alpha}e^{xt}, \tag{43}$$

where the polynomials $\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u)=\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,1,e,e)$ are called the type 2 Apostol-Frobenius-type poly-Genocchi polynomials.

(3) When $k=1$, (41) gives

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\left(\frac{(1-u)t\ln ab}{\lambda b^t-ua^{-t}}\right)^{\alpha}c^{xt}, \tag{44}$$

where the polynomials $\mathcal{G}_{n,2}^{(\alpha)}(x;\lambda,u,a,b,c)=\mathcal{G}_{n,2}^{(1,\alpha)}(x;\lambda,u,a,b,c)$ are exactly the same as the type 1 Apostol-Frobenius-type Genocchi polynomials with parameters a, b and c. That is,

$$\mathcal{G}_{n,2}^{(1,\alpha)}(x;\lambda,u,a,b,c)=\mathcal{G}_{n}^{(1,\alpha)}(x;\lambda,u,a,b,c).$$

Rewrite (41) as follows

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\left(\frac{e_k(\log(1+(1-u)t\ln ab))}{a^{-t}(\lambda(ab)^t-u)}\right)^{\alpha}c^{xt},$$

$$=e^{\frac{x\ln c+\alpha\ln a}{\ln ab}t\ln ab}\left(\frac{e_k(\log(1+(1-u)t\ln ab))}{\lambda(ab)^t-u}\right)^{\alpha}.$$

Using (43), we get

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}(\ln a+\ln b)^n\mathcal{G}_{n,2}^{(k,\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda,u\right)\frac{t^n}{n!}.$$

Comparing the coefficients yields the following theorem.

Theorem 7.2. *The type 2 Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b and c satisfies the relation,*

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)=(\ln a+\ln b)^n\mathcal{G}_{n,2}^{(k,\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda,u\right). \tag{45}$$

When $k=1$, (45) reduces to the following relation

$$\mathcal{G}_{n,2}^{(\alpha)}(x;\lambda,a,b,c)=(\ln a+\ln b)^{n-j}\mathcal{G}_{n,2}^{(\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda,u\right). \tag{46}$$

The next theorem contains an identity that relates the type 2 Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a,b and c to Stirling numbers of the first kind $\begin{bmatrix} n \\ m \end{bmatrix}$ defined by

$$\sum_{n=m}^{\infty}\begin{bmatrix} n \\ m \end{bmatrix}\frac{t^n}{n!}=\frac{(\log(1+t))^m}{m!}. \tag{47}$$

Theorem 7.3. *The type 2 Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a,b and c satisfies the relation,*

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,a,b,c)=\sum_{j=0}^{n}\binom{n}{j}(\ln a+\ln b)^{n-j}\mathcal{G}_{n-j,2}^{(\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda\right)d_j, \tag{48}$$

where

$$d_j=\sum_{n_1+n_2+\ldots+n_\alpha=j}\prod_{i=1}^{\alpha}c_{n_i}\binom{j}{n_1,n_2,\ldots,n_\alpha}\ \text{ and }\ c_j=\sum_{m=0}^{j}\frac{(2\ln ab)^j\begin{bmatrix} j+1 \\ m+1 \end{bmatrix}}{(j+1)(m+1)^{k-1}}.$$

Proof. Applying the definition of polyexponential function (40), (41) may be written as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}&=\frac{c^{xt}}{(a^{-t}+\lambda b^t)^\alpha}\left(\sum_{m=1}^{\infty}\frac{(\log(1+(1-u)t\ln ab))^m}{(m-1)!m^k}\right)^\alpha\\&=\frac{c^{xt}}{(a^{-t}+\lambda b^t)^\alpha}\left(\sum_{m=0}^{\infty}\frac{(\log(1+(1-u)t\ln ab))^{m+1}}{m!(m+1)^k}\right)^\alpha\\&=\frac{c^{xt}}{(a^{-t}+\lambda b^t)^\alpha}\left(\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\frac{\log(1+(1-u)t\ln ab))^{m+1}}{(m+1)!}\right)^\alpha.\end{aligned}$$

This can further be written, using (47), as follows

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}&=\frac{c^{xt}}{(a^{-t}+\lambda b^t)^\alpha}\left(\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\sum_{j=m+1}^{\infty}\begin{bmatrix} j \\ m+1 \end{bmatrix}\frac{((1-u)t\ln ab)^j}{j!}\right)^\alpha\\&=c^{xt}\left(\frac{(1-u)t\ln ab}{\lambda b^t-ua^{-t}}\right)^\alpha\left(\sum_{m=0}^{\infty}\sum_{j=m}^{\infty}\frac{\begin{bmatrix} j+1 \\ m+1 \end{bmatrix}}{(j+1)(m+1)^{k-1}}\frac{((1-u)t\ln ab)^j}{j!}\right)^\alpha.\end{aligned}$$

Applying (44) yields

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha, \tag{49}$$

where

$$c_j = \sum_{m=0}^{j} \frac{((1-u)\ln ab)^j \begin{bmatrix} j+1 \\ m+1 \end{bmatrix}}{(j+1)(m+1)^{k-1}}.$$

Note that, using (31), equation (49) can be expressed as

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}\mathcal{G}_{n-j,2}^{(\alpha)}(x;\lambda,u,a,b,c)d_j\right\}\frac{t^n}{n!},$$

where

$$d_j = \sum_{n_1+n_2+\ldots+n_\alpha=j}\prod_{i=1}^{\alpha} c_{n_i}\binom{j}{n_1,n_2,\ldots,n_\alpha}.$$

This immediately gives (48) by comparing the coefficients and using equation (45). □

Remark 7.1. It is left to the reader to prove the following identities which can be done parallel to the proofs of the corresponding identities in sections 3, 4 and 5 for type 1 Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c:

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}\mathcal{G}_{i,2}^{(k,\alpha)}(\lambda,a,b)x^{n-i}$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x+1;\lambda,a,b,c) = \sum_{r=0}^{n}\binom{n}{r}(\ln c)^{r}\mathcal{G}_{n-r,2}^{(k,\alpha)}(x;\lambda,a,b,c)$$

$$\frac{d}{dx}\mathcal{G}_{n+1,2}^{(k,\alpha)}(x;\lambda,a,b,c) = (n+1)(\ln c)\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x+y;\lambda,a,b,c) = \sum_{i=0}^{\infty}\binom{n}{i}(\ln c)^{n-i}\mathcal{G}_{i,2}^{(k,\alpha)}(x;\lambda,a,b,c)y^{n-i}$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{{l \atop m}\right\}\binom{n}{l}(\ln c)^{l}\mathcal{G}_{n-l,2}^{(k,\alpha)}(-m\ln c;\lambda,a,b)(x)^{(m)}$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{{l \atop m}\right\}\binom{n}{l}(\ln c)^{l}\mathcal{G}_{n-l,2}^{(k,\alpha)}(\lambda,a,b)(x)_m$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b) = \sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\left\{{l+s \atop s}\right\}\frac{\binom{n-l}{m}}{\binom{l+s}{s}}\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,a,b)B_m^{(s)}(x\ln c;\lambda)$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b) = \sum_{m=0}^{\infty}\frac{\binom{n}{m}}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\mathcal{G}_{n-m,2}^{(k,\alpha)}(j;\lambda,a,b)F_m^{(s)}(x;\mu).$$

8. Conclusion and Recommendations

This paper introduced certain variation of poly-Genocchi polynomials, called the Apostol-Frobenius-type poly-Genocchi polynomials of higher order, also known as type 1 Apostol-Frobenius-type poly-Genocchi polynomials of higher order, using the concept of polylogarithm and Apostol-type polynomials of higher order with parameters a, b and c. Some interesting properties and identities of these polynomials were explored parallel to those of the poly-Euler polynomials and poly-Bernoulli polynomials. Using a differential identity, the type 1 Apostol-Frobenius-type poly-Genocchi polynomials were classified as Appell polynomials, which, consequently, gave some interesting relations. Moreover, these type 1 Apostol-Frobenius-type poly-Genocchi polynomials of higher order were expressed in terms of Stirling numbers of the second kind and Apostol-Frobenius-type poly-Bernoulli polynomials of higher order. Furthermore, the symmetrized generalization of these type 1 Apostol-Frobenius-type poly-Genocchi polynomials of higher order was introduced and a kind of double generating function was established. The paper was concluded by introducing the type 2 Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c. Several identities were established, two of which showed the connections of these polynomials with Stirling numbers of the first kind and the type 2 Apostol-Frobenius-type poly-Bernoulli polynomials.

For future research works, one may try to construct a q-analogue of these Apostol-Frobenius-type poly-Genocchi polynomials using the method employed in [39]. In [34], some partial derivative equations were established relating the Gould-Hopper-Fubini-type polynomials with the Apostol-Genocchi polynomials introduced by Srivastava and Kizilates [35]. It would then be interesting to establish a relation between the Apostol-Frobenius-type poly-Genocchi polynomials and Gould-Hopper-Fubini-type polynomials. Parallel to the construction of certain mixed type special polynomials in [38], it would also be interesting to construct another variation of poly-Genocchi polynomials by mixing these polynomials with Appell polynomials.

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