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OSCILLATION OF SECOND-ORDER DAMPED NONCANONICAL DIFFERENTIAL EQUATIONS WITH SUPERLINEAR NEUTRAL TERM

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Abstract. The aim of this paper is to study the oscillatory behavior of solutions of second-order damped noncanonical differential equations with a superlinear neutral term. The oscillation criteria are obtained via Riccati-type transformations and comparison principles. Our results are new and complement existing results. Examples are provided to illustrate the main results.

1. Introduction

This paper is concerned with the second-order nonlinear differential equation with a superlinear neutral term and a damping term

$$(az')'(t) + d(t)z'(t) + f(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{E}$$

where $z(t) = x(t) + b(t)x^{\alpha}(\tau(t))$ is called the companion function of x. Throughout the paper, we will assume that

- (D_1) α and β are quotients of odd positive integers with $\alpha \geq 1$;
- (D_2) $b, f \in \mathrm{C}([t_0,\infty),[0,\infty))$ are such that $b(t) < 1$ and $f(t)$ is not identically zero for large t;
- (D_3) $a, d \in \mathrm{C}([t_0,\infty),(0,\infty))$ and

$$A(t) := \int_{t_0}^{t} \frac{\mathrm{d}s}{a(s)} \quad \text{is bounded;} \tag{1.1}$$

- (D_4) $\tau, \sigma \in \mathrm{C}([t_0,\infty),\mathbb{R})$ satisfy

$$\tau(t) \geq t, \quad \sigma(t) \leq t, \quad \text{and} \quad \lim_{t\to\infty} \sigma(t) = \infty.$$

By a solution of (E), we mean a function $x \in \mathrm{C}([t_x,\infty),\mathbb{R})$ for some $t_x \geq t_0$ such that $az' \in \mathrm{C}^1([t_x,\infty),\mathbb{R})$, and x satisfies (E) on $[t_x,\infty)$. We consider only those solutions of (E) that exist on some half-line $[t_x,\infty)$ and satisfy the condition

$$\sup\{|x(t)| : T \leq t < \infty\} > 0 \quad \text{for any} \quad T \geq t_x;$$

and further, we tacitly assume that (E) possesses such solutions. Such a solution x of (E) is called *oscillatory* if its set of zeros is unbounded above; otherwise, it

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is called *nonoscillatory*. Equation (E) is called oscillatory if all of its solutions are oscillatory.

The problem of oscillatory behavior of solutions of various classes of second-order neutral differential equations without damping term has received much attention by researchers over the years; for recent contributions, see, for example, [2, 3, 6, 7, 9, 14–18, 21, 22, 27] and the references cited therein. However, from the review of literature, it is clear that results on the oscillatory behavior of second-order differential equations with damping term are very scarce, see, for example, [1, 4, 5, 10, 11, 19, 20, 23–26].

It should be noted that the results obtained in these papers (except [26], in which $b(t) \geq 1$ is assumed) cannot be applied to the case of a superlinear neutral term (i.e., $\alpha > 1$). To the best of authors' knowledge, there are no results for second-order differential equations with a superlinear neutral term and a damping term when $0 \leq b(t) \leq 1$. Therefore, the aim of the present paper is to provide new results, which can easily be extended to more general second-order damped differential equations with a superlinear neutral term to obtain more general oscillation results. It should be noted that in most of the results, the authors assume that $b(t) \geq 1$ and/or $b(t) \to \infty$ as $t \to \infty$ for the case $\alpha > 1$. For this reason, it is our belief that the present paper is a good contribution to the oscillation theory of second-order differential equations with a damping term and a superlinear neutral term.

2. Preliminaries

In this section, we give some auxiliary results that will be used later. When considering oscillation, without loss of generality, we may deal only with eventually positive solutions of (E). For easy reference, we employ the notations

$$E(t) := \exp\left(\int_{t_0}^{t} \frac{d(s)}{a(s)}\mathrm{d}s\right) \quad \text{and} \quad F(t) := \int_{t}^{\infty} \frac{\mathrm{d}s}{E(s)a(s)}.$$

Note that from (1.1), it is easy to see that F is well defined.

Lemma 2.1. *Let x be a positive solution of* (E). *Then the companion function z satisfies one of the following cases for sufficiently large t:*

(P_1) $z(t) > 0$ *and* $z'(t) > 0$;
(P_2) $z(t) > 0$ *and* $z'(t) < 0$.

Proof. Assume that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1 \geq t_0$. Then $z(t) > 0$, and either $z'(t)$ is oscillatory or $z'(t)$ is nonoscillatory for all $t \geq t_1$. Assume that $z'(t)$ is oscillatory. If $z'(t_1) = 0$, then

$$(az')'(t_1) = -f(t_1)x^{\beta}(\sigma(t_1)) \leq 0,$$

from which we can prove that $z'(t)$ cannot have another zero after it vanishes once. Thus, $z'(t)$ has a fixed sign for all sufficiently large t. This completes the proof. □

Lemma 2.2. *Let conditions (D_1)–(D_4) hold. If x is a positive solution of* (E) *such that (P_1) of Lemma 2.1 holds, then*

$$z(t) > A(t)a(t)z'(t), \quad t \geq t_1 \tag{2.1}$$

and z/A is eventually decreasing.

Proof. Since $z'(t) > 0$ for $t \geq t_1$, from (E), we have

$$z(t) > 0, \quad z'(t) > 0, \quad \text{and} \quad (az')'(t) \leq 0 \quad \text{for} \quad t \geq t_1,$$

and so

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{a(s)z'(s)}{a(s)} \mathrm{d}s > A(t)a(t)z'(t),$$

which proves (2.1). From (2.1), one can easily see that

$$\left(\frac{z}{A}\right)'(t) = \frac{z'(t)A(t) - z(t)A'(t)}{A^2(t)} = \frac{a(t)z'(t)A(t) - z(t)}{a(t)A^2(t)} < 0,$$

so z/A is decreasing on $[t_1, \infty)$. This completes the proof. □

To prove our results, we use the additional condition

(D_5) $\lim_{t\to\infty} \frac{b(t)A^{\alpha}(\tau(t))}{A(t)} = 0.$

Lemma 2.3. *Let conditions* (D_1)–(D_5) *hold and suppose* (E) *has a positive solution. If*

$$\int_{t_0}^{\infty} f(t)\mathrm{d}t = \infty, \tag{2.2}$$

then (P_2) *of Lemma 2.1 holds.*

Proof. Let x be a positive solution of (E), say $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for all $t \geq t_1 \geq t_0$. From Lemma 2.1, z satisfies either (P_1) or (P_2). Assume (P_1) holds. Then, from Lemma 2.2, $z(t)/A(t)$ is decreasing for $t \geq t_2 \geq t_1$. Since z is increasing and z/A is decreasing, there exist constants $M_1 > 0$ and $M_2 > 0$ and $t_3 \geq t_2$ such that

$$z(t) \geq M_1 \quad \text{and} \quad \frac{z(t)}{A(t)} \leq M_2 \quad \text{for} \quad t \geq t_3.$$

Let $\varepsilon \in (0, 1)$. It follows from (D_5) that there exists $t_4 \geq t_3$ such that

$$\frac{b(t)A^{\alpha}(\tau(t))}{A(t)} \leq M_2^{1-\alpha}(1-\varepsilon) \quad \text{for} \quad t \geq t_4.$$

Combining these with $z > x$, (D_4), and the definition of z, we see that

$$\begin{aligned}
x(t) &= z(t) - b(t)x^{\alpha}(\tau(t)) \\
&\geq z(t) - b(t)z^{\alpha}(\tau(t)) \\
&= z(t) - b(t)\left(\frac{z(\tau(t))}{A(\tau(t))}\right)^{\alpha} A^{\alpha}(\tau(t)) \\
&\geq z(t) - b(t)\left(\frac{z(t)}{A(t)}\right)^{\alpha} A^{\alpha}(\tau(t)) \\
&= z(t)\left\{1 - \left(b(t)\frac{A^{\alpha}(\tau(t))}{A(t)}\right)\left(\frac{z(t)}{A(t)}\right)^{\alpha-1}\right\} \\
&\geq M_1\left\{1 - M_2^{1-\alpha}(1-\varepsilon)M_2^{\alpha-1}\right\} = M_1\varepsilon =: M > 0
\end{aligned}$$

for $t \geq t_4$. Using this in (E), one gets

$$(az')'(t) + d(t)z'(t) + M^{\beta}f(t) \leq 0.$$

In view of $z'(t) > 0$ and (D_3), this inequality implies that

$$(az')'(t) + M^\beta f(t) \le 0, \quad t \ge t_4.$$

Integrating the latter inequality from t_4 to t, (2.2) yields

$$0 < a(t)z'(t) \le a(t_4)z'(t_4) - M^\beta \int_{t_4}^{t} f(s)\mathrm{d}s \to -\infty$$

as $t \to \infty$, a contradiction. Thus (P_1) cannot hold. The proof is complete. □

Lemma 2.4. *Let conditions* (D_1)–(D_5) *hold and suppose* (E) *has a positive solution* x *such that* (P_2) *of Lemma 2.1 holds. If there exists a function* $\rho \in \mathrm{C}^1([t_0,\infty),\mathbb{R}^+)$ *such that*

$$\rho'(t)a(t) - \rho(t)d(t) \ge 0 \tag{2.3}$$

and

$$\int_{t_0}^{\infty} \frac{1}{a(t)\rho(t)} \left(\int_{t_0}^{t} \rho(s)f(s)\mathrm{d}s \right) \mathrm{d}t = \infty, \tag{2.4}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof. Let x be a positive solution of (E), say $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for all $t \ge t_1 \ge t_0$. Since z is decreasing, there exist a constant $M_3 > 0$ and $t_2 \ge t_1$ such that $z(t) \le M_3$ for $t \ge t_2$. Let $\varepsilon_2 \in (0,1)$. In view of (D_5) and noting that $A(t)$ is increasing and bounded implies

$$\lim_{t\to\infty} b(t) = 0,$$

there exists $t_3 \ge t_2$ such that

$$b(t) \le M_3^{1-\alpha}(1-\varepsilon_2) \quad \text{for} \quad t \ge t_3.$$

Combining these with $z > x$, (D_4), and the definition of z, we see that

$$\begin{aligned} x(t) &= z(t) - b(t)x^\alpha(\tau(t)) \\ &\ge z(t) - b(t)z^\alpha(\tau(t)) \\ &\ge z(t) - b(t)z^\alpha(t) \\ &= z(t)\left\{1 - b(t)z^{\alpha-1}(t)\right\} \\ &\ge z(t)\left\{1 - M_3^{1-\alpha}(1-\varepsilon_2)M_3^{\alpha-1}\right\} \\ &= \varepsilon_2 z(t). \end{aligned}$$

Using the last inequality in (E), we have

$$(az')'(t) + d(t)z'(t) + \varepsilon f(t)z^\beta(\sigma(t)) \le 0 \tag{2.5}$$

for $\varepsilon := \varepsilon_2^\beta$ and $t \ge t_3$. Since z is strictly decreasing, there exists the finite limit

$$\lim_{t\to\infty} z(t) =: \ell \ge 0.$$

Assume $\ell > 0$. Then there exists $t_4 \ge t_3$ such that $z(\sigma(t)) \ge \ell$ for $t \ge t_4$. Thus

$$(az')'(t) + d(t)z'(t) + \ell_1 f(t) \le 0 \quad \text{for} \quad t \ge t_4, \tag{2.6}$$

where $\ell_1 := \varepsilon\ell^\beta > 0$. Define

$$w(t) := \rho(t)a(t)z'(t).$$

Then, using (2.6), we have

$$\begin{aligned} w'(t) &= \rho(t)(az')'(t) + \rho'(t)a(t)z'(t) \\ &\le -\ell_1\rho(t)f(t) + (\rho'(t)a(t) - \rho(t)d(t))z'(t) \\ &\le -\ell_1\rho(t)f(t), \quad t \ge t_4, \end{aligned}$$

where we have used (2.3). Integrating the last inequality from t_4 to t gives

$$w(t) \le w(t_4) - \ell_1 \int_{t_4}^{t} \rho(s)f(s)\mathrm{d}s \le -\ell_1 \int_{t_4}^{t} \rho(s)f(s)\mathrm{d}s,$$

i.e.,

$$z'(t) \le -\frac{\ell_1}{a(t)\rho(t)} \int_{t_4}^{t} \rho(s)f(s)\mathrm{d}s.$$

Again integrating, we obtain

$$0 < z(t) \le z(t_4) - \ell_1 \int_{t_4}^{t} \frac{1}{a(s)\rho(s)} \left(\int_{t_4}^{s} \rho(u)f(u)\mathrm{d}u \right) \mathrm{d}s \to -\infty$$

as $t \to \infty$, a contradiction. Thus, $\ell = 0$. The fact that $\lim_{t\to\infty} x(t) = 0$ simply follows from the definition of z. The proof is complete. □

3. Oscillation Results

In this section, we present some new oscillation criteria for (E).

Theorem 3.1. *Let conditions (D_1)–(D_5) and (2.2) hold. If there exists a function $\rho \in \mathrm{C}^1([t_0,\infty),\mathbb{R}^+)$ such that (2.3) and (2.4) hold, then any solution x of* (E) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Combining Lemmas 2.3 and 2.4 completes the proof. □

Theorem 3.2. *Let conditions (D_1)–(D_5) and (2.2) hold. If*

$$\int_{t_0}^{\infty} \frac{1}{a(t)E(t)} \left(\int_{t_0}^{t} f(s)E(s)F^{\beta}(\sigma(s))\mathrm{d}s \right) \mathrm{d}t = \infty, \tag{3.1}$$

then (E) *is oscillatory.*

Proof. Let x be a positve solution of (E), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ on $[t_1,\infty)$ for some $t_1 \ge t_0$. In view of (2.2), we have from Lemma 2.3 that z satisfies (P_2) of Lemma 2.1 for all $t \ge t_2 \ge t_1$. Proceeding as in the proof of Lemma 2.4, one can obtain (2.5), and it can be written as

$$(Eaz')'(t) + \varepsilon E(t)f(t)z^{\beta}(\sigma(t)) \le 0, \quad t \ge t_3 \ge t_2. \tag{3.2}$$

From (3.2), we have

$$z(t) > 0, \quad z'(t) < 0, \quad \text{and} \quad (Eaz')'(t) \le 0$$

for $t \ge t_3$. Since Eaz' is nonincreasing, we have

$$z(t) \ge -\int_{t}^{\infty} \frac{E(s)a(s)z'(s)}{a(s)E(s)}\mathrm{d}s \ge -F(t)E(t)a(t)z'(t)$$

and so

$$\left(\frac{z}{F}\right)'(t) \ge 0 \quad \text{for} \quad t \ge t_3.$$

Hence, z/F is nondecreasing, so there exist a constant $\overline{M} > 0$ and $T \geq t_3$ such that

$$z(t) \geq \overline{M}F(t) \quad \text{for} \quad t \geq T.$$

Using this in (3.2), we obtain

$$(Eaz')'(t) + M_4 E(t) f(t) F^\beta(\sigma(t)) \leq 0, \tag{3.3}$$

where $M_4 := \varepsilon\overline{M}^\beta$ for $t \geq T$. Integrating (3.3) from T to t, we obtain

$$\begin{aligned} E(t)a(t)z'(t) &\leq E(T)a(T)z'(T) - M_4 \int_T^t E(s)f(s)F^\beta(\sigma(s))\mathrm{d}s \\ &\leq -M_4 \int_T^t E(s)f(s)F^\beta(\sigma(s))\mathrm{d}s. \end{aligned}$$

Integrating the latter inequality again and taking (3.1) into account, we get

$$\begin{aligned} 0 < z(t) &\leq z(T) - M_4 \int_T^t \frac{1}{a(s)E(s)} \left(\int_T^s E(u)f(u)F^\beta(\sigma(u))\mathrm{d}u \right) \mathrm{d}s \\ &\to -\infty \end{aligned}$$

as $t \to \infty$, a contradiction. The proof is complete. □

Theorem 3.3. *Let conditions* (D_1)–(D_5) *and* (2.2) *hold. If* $\beta = 1$ *and*

$$\limsup_{t\to\infty} \left\{ F(t) \int_{t_0}^t E(s)f(s)\mathrm{d}s \right\} > 1, \tag{3.4}$$

then (E) *is oscillatory.*

Proof. Let x be a positive solution of (E), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$. In view of (2.2), Lemma 2.3 implies that z satisfies (P_2) of Lemma 2.1 for all $t \geq t_2 \geq t_1$. Also, by Lemma 2.4, z satisfies the inequality (2.5), which is valid for any $\varepsilon \in (0, 1)$ and $t \geq t_3$ for some $t_3 \in [t_2, \infty)$. Further, (2.5) can be written as (3.2) for $t \geq t_3$. Integrating (3.2) from t_3 to t yields

$$\begin{aligned} -E(t)a(t)z'(t) &\geq \varepsilon \int_{t_3}^t E(s)f(s)z^\beta(\sigma(s))\mathrm{d}s \\ &\geq \varepsilon z^\beta(\sigma(t)) \int_{t_3}^t E(s)f(s)\mathrm{d}s. \end{aligned} \tag{3.5}$$

Using (3.2) in the latter inequality, we have

$$\begin{aligned} -E(t)a(t)z'(t) &\geq \varepsilon z^\beta(t) \int_{t_3}^t E(s)f(s)\mathrm{d}s \\ &\geq \varepsilon F^\beta(t)(-E(t)a(t)z'(t))^\beta \int_{t_3}^t E(s)f(s)\mathrm{d}s, \end{aligned}$$

that is,

$$(-E(t)a(t)z'(t))^{1-\beta} \geq \varepsilon F^\beta(t) \int_{t_3}^t E(s)f(s)\mathrm{d}s \tag{3.6}$$

for any $\varepsilon \in (0, 1)$ and $t \geq t_3$. If we take $\beta = 1$, then (3.6) becomes

$$1 \geq \varepsilon F(t) \int_{t_3}^t E(s)f(s)\mathrm{d}s$$

for any $\varepsilon \in (0, 1)$, which clearly contradicts (3.4). The proof is complete. □

Theorem 3.4. *Let conditions* (D_1)–(D_5) *and* (2.2) *hold. Assume that* σ *is nondecreasing and* $\sigma(t) < t$. *If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} \frac{1}{a(s)E(s)} \left(\int_{t_0}^{s} E(u)f(u)\mathrm{d}u \right) \mathrm{d}s > \frac{1}{e} \quad \textit{when} \quad \beta = 1 \tag{3.7}$$

or

$$\int_{t_0}^{\infty} \frac{1}{a(t)E(t)} \left(\int_{t_0}^{t} E(s)f(s)\mathrm{d}s \right) \mathrm{d}t = \infty \quad \textit{when} \quad \beta < 1, \tag{3.8}$$

then (E) *is oscillatory.*

Proof. Let x be a positive solution of (E), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$. In view of (2.2), Lemma (2.3) implies that z satisfies (P_2) of Lemma 2.1 for all $t \geq t_2 \geq t_1$. From the proof of Theorem 3.3, we obtain (3.5), that is, z is a positive solution of the first-order delay differential inequality

$$z'(t) + \left(\frac{\varepsilon}{a(t)E(t)} \int_{t_3}^{t} E(s)f(s)\mathrm{d}s \right) z^{\beta}(\sigma(t)) \leq 0,$$

which holds for all $\varepsilon \in (0,1)$ for $t \geq t_3$. It follows from [8, Theorem 5.1.1] that the associated delay differential equation

$$z'(t) + \left(\frac{\varepsilon}{a(t)E(t)} \int_{t_3}^{t} E(s)f(s)\mathrm{d}s \right) z^{\beta}(\sigma(t)) = 0 \tag{3.9}$$

also has a positive solution. However, [13, Theorem 2.1.1] and [12, Theorem 2], respectively, imply that (3.7) and (3.8) ensure oscillation of (3.9) in the case $\beta = 1$ and $\beta < 1$, respectively. This in turn means that (E) cannot have an eventually positive solution, a contradiction. The proof is complete. □

4. Examples

In this section, we present two examples to illustrate the importance of our main results.

Example 4.1. Consider the differential equation with a superlinear neutral term and a damping term

$$(t^2 z'(t))' + t z'(t) + t^6 x^3(t/3) = 0, \quad t \geq 1 \tag{4.1}$$

with

$$z(t) = x(t) + \frac{x^3(2t)}{t^2}.$$

Here,

$$a(t) = t^2, \quad b(t) = \frac{1}{t^2}, \quad d(t) = t, \quad f(t) = t^6,$$
$$\tau(t) = 2t, \quad \sigma(t) = \frac{t}{3}, \quad \alpha = \beta = 3.$$

Then it is easy to see that (D_1)–(D_4) and (2.2) hold. Also,

$$A(t) = 1 - \frac{1}{t} \quad \text{and} \quad \lim_{t\to\infty} \frac{b(t)A^{\alpha}(\tau(t))}{A(t)} = \lim_{t\to\infty} \frac{\left(1 - \frac{1}{2t}\right)^3}{t(t-1)} = 0,$$

and so (D_5) holds. Moreover,

$$E(t) = t \quad \text{and} \quad F(t) = \frac{1}{2t^2}.$$

Finally,

$$\int_1^\infty \frac{1}{t^3}\int_1^t \frac{729}{8} s\mathrm{d}s = \frac{729}{16}\int_1^\infty \left(\frac{1}{t}-\frac{1}{t^3}\right)\mathrm{d}t = \infty,$$

that is, (3.1) holds. Thus, all conditions of Theorem 3.2 hold. Therefore, by Theorem 3.2, (4.1) is oscillatory.

Example 4.2. Consider the differential equation with a superlinear neutral term and a damping term

$$(t^{3/2}z'(t))' + t^{1/2}z'(t) + \frac{c}{\sqrt{t}}x(t/2) = 0, \quad t \geq 1 \tag{4.2}$$

with

$$z(t) = x(t) + \frac{x^3(2t)}{t} \quad \text{and} \quad c > 0.$$

Here,

$$a(t) = t^{3/2}, \quad b(t) = \frac{1}{t}, \quad d(t) = t^{1/2}, \quad f(t) = \frac{c}{\sqrt{t}},$$
$$\tau(t) = 2t, \quad \sigma(t) = \frac{t}{2}, \quad \alpha = 3, \quad \beta = 1.$$

Then it is easy to see that (D_1)–(D_4) and (2.2) hold. Also,

$$A(t) = 2 - \frac{2}{\sqrt{t}} \quad \text{and} \quad \lim_{t\to\infty}\frac{b(t)A^\alpha(\tau(t))}{A(t)} = \lim_{t\to\infty}\frac{4\left(1-\frac{1}{\sqrt{2t}}\right)^3}{t\left(1-\frac{1}{\sqrt{t}}\right)} = 0,$$

and so (D_5) holds. Moreover,

$$E(t) = t \quad \text{and} \quad F(t) = \frac{2}{3t^{3/2}}.$$

Finally,

$$\begin{aligned}\limsup_{t\to\infty}\left\{F(t)\int_1^t E(s)f(s)\mathrm{d}s\right\} &= \lim_{t\to\infty}\frac{2}{3t^{3/2}}\int_1^t cs^{1/2}\mathrm{d}s \\ &= \lim_{t\to\infty}\left(\frac{4c}{9}-\frac{4c}{9t^{3/2}}\right) \\ &= \frac{4c}{9},\end{aligned}$$

that is, (3.4) holds if $c > 2.25$. Thus, by Theorem 3.3, (4.2) is oscillatory if $c > 2.25$.

5. Conclusion

In this paper, we have established some new criteria for the oscillatory behavior of solutions to a second-order differential equation with a damping term and a superlinear neutral term. The results obtained in [10,11,20,23–26] cannot be applied to our examples since $b(t) \to 0$ as $t \to \infty$ and $\alpha > 1$. Thus, the results in this paper are new and complement existing results reported in the literature.

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