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NEW OPTIMAL BOUNDS FOR LOGARITHMIC AND EXPONENTIAL FUNCTIONS

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Abstract. We present new lower and upper bounds for the logarithmic function $\ln(1+x)$. These bounds involve a parameter m and they become optimal as $m \to 0$. Moreover, by using these bounds we also present new optimal bounds for the exponential function e^x. We compare our bounds with other known bounds from the literature and we support their optimum performance.

1. Introduction

Upper and lower bounds for the logarithmic function $\ln(1 + x)$ have been extensively presented in the literature. The most famous and simplest bounds for $\ln(1 + x)$ are given by (see e.g. [7]),

$$\frac{x}{1+x} \le \ln(1+x) \le x, \qquad x > -1.$$

If we restrict our interest to positive real numbers, then the above bounds become strict, i.e.

$$\frac{x}{1+x} < \ln(1+x) < x, \qquad x > 0. \tag{1.1}$$

In [7], a better lower bound which is a refinement of (1.1) was proved:

$$\frac{x}{1+\frac{1}{2}x} < \ln(1+x), \qquad x > 0.$$

Later, (see [8] and the references therein) another refinement of (1.1) was also presented:

$$\frac{x(1+\frac{5}{6}x)}{(1+x)(1+\frac{1}{3}x)} < \ln(1+x) < \frac{x(1+\frac{1}{6}x)}{1+\frac{2}{3}x}, \qquad x > 0.$$

More upper bounds (already known or new ones) for $\ln(1 + x)$ were also proved in [10], specifically:

$$\ln(1+x) \le \frac{x(2+x)}{2+2x}, \qquad x > -1,$$

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$$\ln(1+x) \leq \frac{x}{\sqrt{1+x}}, \qquad x > -1, \tag{1.2}$$

Additionally in [10], a methodology is described on how to construct new upper and lower bounds for $\ln(1+x)$, which are fractions of polynomials of increasing order (see also Table 1 in [10]).

Another interesting and different upper bound, which involves the *arctan* function, is given in [4]:

$$\ln(1+x) \leq \frac{\pi + \frac{1}{2}(4+\pi)x - 2(x+2)\arctan(\sqrt{x+1})}{\sqrt{x+1}}, \qquad x \geq 0.$$

A proof given in [6] (Theorem 1, page 292), supports the fact that the above upper bound is always better than any other bound which is a polynomial fraction.

Another lower bound for $\ln(1+x)$, which involves a parameter k, was presented in [5]:

$$\frac{1-k+k(1+x)-(1+x)^k}{k(1-k)x} \leq \ln(1+x), \qquad x > 0,$$

where $0 < k < 1$. Interestingly, this lower bound becomes better as $k \to 1$.

Closely related to the bounds for the logarithmic function, are the bounds for the exponential function e^x. The following bounds are best possible in a certain sense (see [1])

$$\left(1+\frac{x}{a}\right)^{\sqrt{a(a+x)}} \leq e^x \leq \left(1+\frac{x}{a}\right)^{\sqrt{a(a+x)+\frac{1}{12}x^2}},$$

where $a > 0$ and $x > 0$. The above lower bound was firstly proved in [3] and the upper bound in this form was proved in [1].

Lastly, the book [9] devotes a whole section for inequalities involving exponential, logarithmic and gamma functions (section 3.6, pages 266-289).

In this article we will present new lower and upper bounds for the logarithmic function $\ln(1+x)$ which involve a parameter $m \in (0,1)$, and we will prove that they become optimal as $m \to 0$. We will compare our bounds with other already known bounds and we will support their optimum performance. As an application, we will also present optimal bounds for the logarithmic mean. Moreover by using the bounds for $\ln(1+x)$, we will find optimal bounds for the exponential function e^x and we will also compare them with other known bounds for e^x from the literature.

2. New optimal bounds for the logarithmic function $\ln(1+x)$

In this section we will present and prove new bounds for the function $\ln(1+x)$ and we will also prove results which indicate their optimum performance. Then, we will compare our bounds with other known bounds from the literature. Let us first present the well-known Chebyshev's inequality for integrals (see [9]). Let $f, g : [a,b] \to R$ be two integrable functions, which are both increasing or decreasing, then

$$\int_a^b f(t)g(t)dt \geq \frac{1}{b-a}\int_a^b f(t)dt \int_a^b g(t)dt. \tag{2.1}$$

Moreover, if one of the functions in (2.1) is increasing and the other one is decreasing the inequality is reversed. We also note that the equality holds if and only if one of the functions f, g is a constant function (see [9], Theorem 10, page 40).

Firstly, we will find a new upper bound for $\ln(1+x)$. Let us use (2.1) with $a = 0$, $b = 1$ and $f(t) = (1+xt)^m$, $g(t) = \frac{1}{(1+xt)^{m+1}}$, where $x > 0$ and $0 < m < 1$ are fixed real numbers. We observe that f, g are of opposite monotonicity and so we have

$$\int_0^1 \frac{1}{(1+xt)}dt < \int_0^1 (1+xt)^m dt \int_0^1 \frac{1}{(1+xt)^{m+1}}dt,$$

where the strict inequality holds since f and g are not constant functions. By using the substitution $y = 1 + xt$ and after the calculations we easily get

$$\ln(1+x) < \frac{1}{m(m+1)}\frac{1}{x}\left[(1+x)^{m+1} + (1+x)^{-m} - (x+2)\right], \tag{2.2}$$

which is a new upper bound for the logarithmic function $\ln(1 + x)$ for $x > 0$. Let us denote this upper bound as

$$\psi_m(x) = \frac{1}{m(m+1)}\frac{1}{x}\left[(1+x)^{m+1} + (1+x)^{-m} - (x+2)\right], \tag{2.3}$$

where $x > 0$ and $0 < m < 1$.

Next, as we have promised, we will prove a result which indicates the fact that the upper bound $\psi_m(x)$ is optimal as we let $m \to 0$. Thus, for the upper bound $\psi_m(x)$ we can also prove that for a fixed $x > 0$ it holds

$$\lim_{m\to 0} \psi_m(x) = \ln(1+x). \tag{2.4}$$

The above can be easily proved by using L' Hospital's rule. To see this let us rewrite $\psi_m(x)$ as

$$\psi_m(x) = \frac{(1+x)^{m+1} + (1+x)^{-m} - (x+2)}{m(m+1)x}.$$

Let a fixed $x > 0$, then for $m \to 0$ the above is of the form $\frac{0}{0}$. Thus, calculating the derivatives with respect to m of the numerator and of the denominator we have

$$\lim_{m\to 0} \psi_m(x) = \lim_{m\to 0} \frac{(1+x)^{m+1}\ln(1+x) - (1+x)^{-m}\ln(1+x)}{(2m+1)x} = \ln(1+x).$$

So, the upper bound in (2.2) is optimal in the sense that as m becomes smaller, $\psi_m(x)$ is getting closer to the best possible approximation from above of $\ln(1+x)$. The approximation is characterized as the best possible since (2.4) holds, i.e. the limit of $\psi_m(x)$ as $m \to 0$ for a given $x > 0$, is the logarithmic function $\ln(1+x)$ itself.

Let us now compare our upper bound (2.3) with the upper bound obtained in [4]:

$$\ln(1+x) \le h(x) = \frac{\pi + \frac{1}{2}(4+\pi)x - 2(x+2)\arctan\left(\sqrt{x+1}\right)}{\sqrt{x+1}}, \qquad x \ge 0.$$

Let us remember from the introduction that the above bound $h(x)$ is better than any other bound which is a polynomial fraction ([6]). By numerical experimentation we discovered that even if the upper bound $h(x)$ may be better than $\psi_m(x)$ for small values of $x > 0$, our upper bound $\psi_m(x)$ for $0 < m < 1/2$ becomes better as x increases. The fact that our upper bound is better for large values of $x > 0$, will be rigorously proved.

So, we can prove that by choosing an $m < 1/2$, our upper bound $\psi_m(x)$ is better than the upper bound $h(x)$ for large enough $x > 0$. In fact, since $\lim_{x\to\infty} \arctan\left(\sqrt{x+1}\right) = \pi/2$, it can be seen that

$$\lim_{x\to\infty} \frac{h(x)}{\sqrt{x}} = \frac{4-\pi}{2}.$$

Moreover for a fixed $m \in \left(0, \frac{1}{2}\right)$, we also have

$$\lim_{x\to\infty} \frac{\psi_m(x)}{x^m} = \frac{1}{m(m+1)}.$$

From the above two limits we can get

$$\lim_{x\to\infty} \frac{\psi_m(x)}{h(x)} x^{\frac{1}{2}-m} = \frac{2}{m(m+1)(4-\pi)} > 0,$$

and since $x^{\frac{1}{2}-m} \to \infty$ as $x \to \infty$ (remember that we have chosen an $m < 1/2$), it holds that

$$\lim_{x\to\infty} \frac{\psi_m(x)}{h(x)} = 0.$$

Thus, there is an $M > 0$ such that for $x \geq M$ it holds $\psi_m(x) < h(x)$ and so our upper bound is sharper for large values of x.

Let us next investigate and find a new sharp lower bound for $\ln(1+x)$. Let us use again (2.1) with $a = 0$, $b = 1$ and $f(t) = \frac{1}{1+xt}$, $g(t) = (1+xt)^m$ for a fixed $x > 0$ and a real number $m \in (0, 1)$. We observe that f and g are of opposite monotonicity and they are not constant functions. Thus (2.1) becomes

$$\int_0^1 (1+xt)^{m-1} dt < \int_0^1 \frac{1}{(1+xt)} dt \int_0^1 (1+xt)^m dt.$$

After the calculations (using the substitution $y = 1+xt$) we get the following lower bound

$$\frac{m+1}{m}\left[x\frac{(1+x)^m - 1}{(1+x)^{m+1}-1}\right] < \ln(1+x).$$

We denote this lower bound as

$$\varphi_m(x) = \frac{m+1}{m}\left[x\frac{(1+x)^m - 1}{(1+x)^{m+1}-1}\right]. \tag{2.5}$$

For the lower bound $\varphi_m(x)$ we can also prove that for a fixed $x > 0$ it holds

$$\lim_{m\to 0} \varphi_m(x) = \ln(1+x).$$

The above limit can be easily proved by using again L' Hospital's rule. So, the lower bound $\varphi_m(x)$ is also optimal in the sense that for a given $x > 0$, as $m \to 0$ the bound $\varphi_m(x)$ is getting closer to the best possible approximation from below of $\ln(1+x)$, i.e. is getting closer to $\ln(1+x)$ itself.

Another interesting lower bound from the literature for the logarithmic function $\ln(1+x)$, which also involves a parameter k, is given in [5]:

$$\gamma_k(x) = \frac{1-k+k(1+x)-(1+x)^k}{k(1-k)x} \leq \ln(1+x), \qquad x > 0,$$

where $0 < k < 1$. This bound will be used to check the performance of our bound. We can observe that $\gamma_k(x)$ becomes better as $k \to 1$ (remember that our bound

(2.5) becomes better as $m \to 0$). It can be easily proved by using L'Hospital's rule that for a fixed $x > 0$,

$$\lim_{k \to 1} \gamma_k(x) = -1 + \ln(1+x) + \frac{\ln(1+x)}{x}.$$

We can now compare the two bounds and prove that the bound $\gamma_k(x)$ at the limit is not as good as our bound $\varphi_m(x)$ at the limit. So, we want to prove that for a given $x > 0$ it holds

$$\ln(1+x) > -1 + \ln(1+x) + \frac{\ln(1+x)}{x},$$

or

$$x > \ln(1+x),$$

which is true (see (1.1)) for $x > 0$. So, our bound at the limit is better. This means that as $m \to 0$ our bound $\varphi_m(x)$ becomes sharper than the bound $\gamma_k(x)$ when $k \to 1$.

To sum up, we have found new optimal bounds for the logarithmic function $\ln(1+x)$ for $x > 0$. These bounds are given in (2.3) and (2.5) and so

$$\varphi_m(x) < \ln(1+x) < \psi_m(x),$$

for a real number $m \in (0,1)$. We also proved that our bounds as $m \to 0$ become optimal in the sense that for a fixed $x > 0$ we have

$$\lim_{m \to 0} \varphi_m(x) = \lim_{m \to 0} \psi_m(x) = \ln(1+x).$$

From the above it is clear that as $m \to 0$ the logarithmic function $\ln(1+x)$ is squeezed from below and above by the bounds $\varphi_m(x)$ and $\psi_m(x)$ respectively. So, if we consider $\ln(1+x)$ to be defined in a closed interval $[a,b] \subset (0,\infty)$ we can approximate it as good as we want by letting $m \to 0$, i.e. by choosing a small enough $m > 0$.

The following Theorem summarizes our results of this section.

Theorem 2.1. *Let $x > 0$ and let a parameter $m \in (0,1)$, then the following bounds for the logarithmic function $\ln(1+x)$ hold:*

$$\varphi_m(x) < \ln(1+x) < \psi_m(x),$$

where

$$\varphi_m(x) = \frac{m+1}{m}\left[x\frac{(1+x)^m - 1}{(1+x)^{m+1} - 1}\right],$$

and

$$\psi_m(x) = \frac{(1+x)^{m+1} + (1+x)^{-m} - (x+2)}{m(m+1)x}.$$

Moreover, these bounds are optimal in the following sense:

$$\lim_{m \to 0} \varphi_m(x) = \lim_{m \to 0} \psi_m(x) = \ln(1+x),$$

for a given $x > 0$.

We end up this section by presenting an application of Theorem 2.1. We will use it to find bounds for the logarithmic mean. The logarithmic mean of two positive real numbers, which has applications in physics (see [2]), is defined as

$$L(a,b) = \frac{b-a}{\ln b - \ln a},$$

where $a, b > 0$ and $a \neq b$. For the trivial case $a = b$, the logarithmic mean becomes $L(a, a) = a$. Our interest here is only in the non-trivial case and so from now on we assume that $a \neq b$. We also note that the logarithmic mean is symmetric, i.e. $L(a, b) = L(b, a)$. Let us first assume that $b > a$ (as we will see later this condition is not necessary) and let us define $x = \frac{b-a}{a} > 0$. By applying Theorem 2.1 for $x = \frac{b-a}{a} > 0$ we have

$$\ln b - \ln a > \varphi_m\left(\frac{b-a}{a}\right) = \frac{m+1}{m}\left[(b-a)\frac{b^m - a^m}{b^{m+1} - a^{m+1}}\right],$$

and

$$\ln b - \ln a < \psi_m\left(\frac{b-a}{a}\right) = \frac{1}{m(m+1)}\frac{a}{b-a}\left[\left(\frac{b}{a}\right)^{m+1} + \left(\frac{b}{a}\right)^{-m} - \frac{b+a}{a}\right].$$

Thus, the above two inequalities become

$$L(a, b) = \frac{b-a}{\ln b - \ln a} < \frac{m}{m+1}\frac{b^{m+1} - a^{m+1}}{b^m - a^m}$$

and

$$L(a, b) = \frac{b-a}{\ln b - \ln a} > \frac{m(m+1)(b-a)^2}{a\left[(\frac{b}{a})^{m+1} + (\frac{b}{a})^{-m} - \frac{b+a}{a}\right]} = \frac{m(m+1)a(\frac{b}{a}-1)^2}{(\frac{b}{a})^{m+1} + (\frac{b}{a})^{-m} - \frac{b}{a} - 1}.$$

Now, we can easily observe that the above upper bound is always positive for $a \neq b$. Moreover, the lower bound is also always positive for $a \neq b$. This is true since the numerator is positive and the denominator is also positive (the function $f(\lambda) = \lambda^{m+1} + \lambda^{-m} - \lambda - 1 > 0$ for $\lambda > 0$ and $\lambda \neq 1$). This was expected since, as we have noted, $L(a, b)$ is symmetric and so the bounds should be also true for $L(b, a)$. Thus we have found the following bounds for the logarithmic mean

$$\frac{m(m+1)a(\frac{b}{a}-1)^2}{(\frac{b}{a})^{m+1} + (\frac{b}{a})^{-m} - \frac{b}{a} - 1} < L(a, b) < \frac{m}{m+1}\frac{b^{m+1} - a^{m+1}}{b^m - a^m}, \tag{2.6}$$

for $a, b > 0$, $a \neq b$ and $m \in (0, 1)$. We can also see that the bounds (2.6) are optimal as $m \to 0$, since it holds

$$\lim_{m\to 0}\frac{m(m+1)a(\frac{b}{a}-1)^2}{(\frac{b}{a})^{m+1} + (\frac{b}{a})^{-m} - \frac{b}{a} - 1} = \lim_{m\to 0}\frac{m}{m+1}\frac{b^{m+1} - a^{m+1}}{b^m - a^m} = L(a, b).$$

3. New optimal bounds for the exponential function e^x

In this section, we will use Theorem 2.1 to find optimal bounds for the exponential function e^x for $x > 0$. Moreover, we will support the fact that our bounds are, in a sense, sharper than other known bounds for the exponential function. As we saw in the introduction the following bounds are in a way best possible (see [1]):

$$\left(1 + \frac{x}{a}\right)^{\sqrt{a(a+x)}} \leq e^x \leq \left(1 + \frac{x}{a}\right)^{\sqrt{a(a+x) + \frac{1}{12}x^2}}, \tag{3.1}$$

for $x > 0$ and $a > 0$. These bounds (3.1) will be used to check the performance of our new bounds.

First, we will find a new optimal upper bound by using the lower bound that we have found for the logarithmic function $\ln(1 + x)$. Thus, from Theorem 2.1 we have

$$x < \frac{m}{m+1}\left[\frac{(1+x)^{m+1}-1}{(1+x)^m-1}\right]\ln(1+x).$$

By replacing $x > 0$ with $\frac{x}{a} > 0$ $(a > 0)$ in the above inequality, we get

$$x < \frac{m}{m+1}\left[\frac{(a+x)^{m+1}-a^{m+1}}{(a+x)^m-a^m}\right]\ln\left(1+\frac{x}{a}\right).$$

Hence

$$e^x < \left(e^{\ln\left(1+\frac{x}{a}\right)}\right)^{\frac{m}{m+1}\frac{(a+x)^{m+1}-a^{m+1}}{(a+x)^m-a^m}},$$

and so we have proved the following new upper bound for e^x:

$$e^x < \left(1+\frac{x}{a}\right)^{\frac{m}{m+1}\frac{(a+x)^{m+1}-a^{m+1}}{(a+x)^m-a^m}}, \tag{3.2}$$

where $x > 0$ and $a > 0$. We will next see that our upper bound (3.2) is better than the upper bound (3.1). So, we want to prove

$$\left(1+\frac{x}{a}\right)^{\frac{m}{m+1}\frac{(a+x)^{m+1}-a^{m+1}}{(a+x)^m-a^m}} < \left(1+\frac{x}{a}\right)^{\sqrt{a(a+x)+\frac{1}{12}x^2}},$$

for small values of m. What we really want to prove is that the above inequality is true as $m \to 0$. Equivalently, it is enough to prove that for the exponents it holds

$$\frac{m}{m+1}\frac{(a+x)^{m+1}-a^{m+1}}{(a+x)^m-a^m} < \sqrt{a(a+x)+\frac{1}{12}x^2},$$

for $x, a > 0$ and $m \in (0,1)$. Equivalently, we have

$$\frac{m}{m+1}\frac{(1+\frac{x}{a})^{m+1}-1}{(1+\frac{x}{a})^m-1} < \sqrt{1+\frac{x}{a}+\frac{1}{12}\left(\frac{x}{a}\right)^2}.$$

For simplicity we define $y = x/a > 0$ and so

$$\frac{m}{m+1}\frac{(1+y)^{m+1}-1}{(1+y)^m-1} < \sqrt{1+y+\frac{1}{12}y^2}.$$

Now, by letting $m \to 0$ and using L' Hospital's rule we find (for a fixed $y > 0$) that

$$\lim_{m\to 0}\frac{m}{m+1}\frac{(1+y)^{m+1}-1}{(1+y)^m-1} = \frac{y}{\ln(1+y)}. \tag{3.3}$$

So, we have to prove that for $y > 0$ the following inequality is true

$$\ln(1+y) > \frac{y}{\sqrt{1+y+\frac{1}{12}y^2}}.$$

Interestingly, this inequality holds and it has already been proved in [1]. So, at the limit $m \to 0$ our upper bound becomes better. This was expected since we have seen in Theorem 2.1 that our bounds for $\ln(1+x)$ become optimal as $m \to 0$.

Moreover, we can also use the sharp upper bound for $\ln(1+x)$ from Theorem 2.1 in order to find an optimal lower bound for e^x. Thus

$$\ln(1+x) < \frac{1}{m(m+1)}\frac{1}{x}\left[(1+x)^{m+1}+(1+x)^{-m}-(x+2)\right].$$

The inequality holds for $x > 0$ and so we have

$$\ln(1+x) < \frac{1}{m(m+1)}\frac{x}{x^2}\left[(1+x)^{m+1} + (1+x)^{-m} - (x+2)\right],$$

or

$$\frac{m(m+1)x^2}{(1+x)^{m+1} + (1+x)^{-m} - (x+2)}\ln(1+x) < x.$$

By replacing $x > 0$ with $\frac{x}{a} > 0$ ($a > 0$) in the above inequality and after some calculations we get

$$\left(1+\frac{x}{a}\right)^{\frac{m(m+1)x^2}{a\left[\left(1+\frac{x}{a}\right)^{m+1}+\left(1+\frac{x}{a}\right)^{-m}-\left(2+\frac{x}{a}\right)\right]}} < e^x, \tag{3.4}$$

for $x, a > 0$ and $m \in (0,1)$. This is a new lower bound for the exponential function.

Furthermore, we can prove that the lower bound (3.4) is better (as $m \to 0$) than the lower bound given in (3.1). For the exponent of $\left(1+\frac{x}{a}\right)$ we have the following limit

$$\lim_{m\to 0}\frac{m(m+1)x^2}{a\left[\left(1+\frac{x}{a}\right)^{m+1}+\left(1+\frac{x}{a}\right)^{-m}-\left(2+\frac{x}{a}\right)\right]} = \frac{x}{\ln\left(1+\frac{x}{a}\right)}. \tag{3.5}$$

So, by comparing the exponents, our bound (3.4) is better at the limit ($m \to 0$) provided that the next inequality holds

$$\frac{x}{\ln\left(1+\frac{x}{a}\right)} > \sqrt{a(a+x)}$$

or

$$\ln\left(1+\frac{x}{a}\right) < \frac{\frac{x}{a}}{\sqrt{1+\frac{x}{a}}},$$

for $\frac{x}{a} > 0$. Note that the above is the known inequality (1.2) which holds strictly for positive real numbers. So again our lower bound for e^x is better as $m \to 0$.

To sum up, we have found optimal bounds for the exponential function e^x which are given by

$$\left(1+\frac{x}{a}\right)^{\frac{m(m+1)x^2}{a\left[\left(1+\frac{x}{a}\right)^{m+1}+\left(1+\frac{x}{a}\right)^{-m}-\left(2+\frac{x}{a}\right)\right]}} < e^x < \left(1+\frac{x}{a}\right)^{\frac{m}{m+1}\frac{(a+x)^{m+1}-a^{m+1}}{(a+x)^m-a^m}}, \tag{3.6}$$

for $x, a > 0$ and $m \in (0,1)$. We can also support their optimum performance as $m \to 0$. Indeed, we have already proved the limits (3.5) and (3.3) and so we can easily deduce the following:

$$\lim_{m\to 0}\left(1+\frac{x}{a}\right)^{\frac{m(m+1)x^2}{a\left[\left(1+\frac{x}{a}\right)^{m+1}+\left(1+\frac{x}{a}\right)^{-m}-\left(2+\frac{x}{a}\right)\right]}} = \left(1+\frac{x}{a}\right)^{\frac{x}{\ln\left(1+\frac{x}{a}\right)}}$$

and

$$\lim_{m\to 0}\left(1+\frac{x}{a}\right)^{\frac{m}{m+1}\frac{(a+x)^{m+1}-a^{m+1}}{(a+x)^m-a^m}} = \left(1+\frac{x}{a}\right)^{\frac{x}{\ln\left(1+\frac{x}{a}\right)}}.$$

So the limits, as it was expected, are identical and they are optimal because the following is true

$$\left(1+\frac{x}{a}\right)^{\frac{x}{\ln\left(1+\frac{x}{a}\right)}} = e^x.$$

We have the same situation as in Theorem 2.1, and so the exponential function e^x can be approximated as good as we want by using (3.6) and by letting $m \to 0$. For convenience, our results of this section are summarized in the next theorem:

Theorem 3.1. *Let $x > 0$, $a > 0$ and let a parameter $m \in (0, 1)$, then the following bounds for the exponential function e^x hold:*

$$\left(1 + \frac{x}{a}\right)^{\frac{m(m+1)x^2}{a\left[\left(1+\frac{x}{a}\right)^{m+1} + \left(1+\frac{x}{a}\right)^{-m} - \left(2+\frac{x}{a}\right)\right]}} < e^x < \left(1 + \frac{x}{a}\right)^{\frac{m}{m+1} \frac{(a+x)^{m+1} - a^{m+1}}{(a+x)^m - a^m}}.$$

Moreover, these bounds are optimal in the following sense:

$$\lim_{m\to 0} \left(1 + \frac{x}{a}\right)^{\frac{m(m+1)x^2}{a\left[\left(1+\frac{x}{a}\right)^{m+1} + \left(1+\frac{x}{a}\right)^{-m} - \left(2+\frac{x}{a}\right)\right]}} = \lim_{m\to 0} \left(1 + \frac{x}{a}\right)^{\frac{m}{m+1} \frac{(a+x)^{m+1} - a^{m+1}}{(a+x)^m - a^m}} = e^x,$$

for a given $x > 0$ and $a > 0$.

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