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LYAPUNOV INEQUALITIES FOR A CLASS OF ψ-LAPLACE EQUATIONS

XU GUO, HAOFAN WANG, JUN ZHENG*

Abstract. In this work, we establish several Lyapunov inequalities for a class of nonlinear higher-order ψ-Laplace equations, where ψ satisfies the Tolksdorf-type structural conditions without any restriction on the convexity of $t\psi(t)$ or $\frac{1}{\psi(t)}$.

1. Introduction

In 1907, Lyapunov considered the Hill's equation

$$\begin{aligned} u''(x) + r(x)u(x) =& 0, x \in (a, b),\\ u(a) = u(b) =& 0, \end{aligned}$$

and proved that if there exists a nontrivial solution u, then the inequality

$$\int_a^b r(x)\mathrm{d}x \geq \frac{4}{b-a} \tag{1.1}$$

holds true, where r is a non-negative continuous function on $[a, b]$; see [1]. The Lyapunov inequality (1.1) and many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications in the theories of differential and difference equations, and also in time scales; see, e.g., [2]-[8] for comprehensive surveys. The Lyapunov-type inequalities for second order differential equations, and higher-order differential equations, have been well addressed in e.g., [9]-[17], and [18]-[29], respectively.

It is worth noting that most of the existing literature focused on establishing the Lyapunov inequalities for the p-Laplace equations, while few considered equations having a general form, except, e.g., [30]-[32], where the Lyapunov inequalities were established for ψ-Laplace equations. More precisely, in [30], the Lyapunov inequality

$$2\left(\frac{k_1}{2}\right)^{[1-\log_2(b-a)]} \leq \int_a^b r(x)\mathrm{d}x \tag{1.2}$$

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was obtained for the following ψ-Laplace equation

$$\begin{aligned}(\psi(u'(x)))' + r(x)\psi(u(x)) =& 0, x \in (a,b),\\ u(a) = u(b) =& 0,\end{aligned}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing function satisfying $2\psi(2t) \leq k_1\psi(t)$ for $t \geq 0$ with a certain constant $k_1 > 0$, and such that $t\psi(t)$ is convex in t, r is a positive integrable function, and $[v]$ denotes the largest integer less than or equal to v.

In [31], the Lyapunov-type inequality

$$\frac{2}{\psi\left(\frac{b-a}{2}\right)} \leq \lambda \int_a^b r(x)\frac{f(u(x))}{\psi(u(x))}\mathrm{d}x \tag{1.3}$$

was established for the ψ-Laplace equation with a nonlinear term having a general form

$$(\psi(u'(x)))' + \lambda r(x) f(u(x)) = 0, x \in (a,b), \tag{1.4a}$$

$$u(a) = u(b) = 0, \tag{1.4b}$$

under the assumptions that ψ is odd, increasing, and sub-multiplicative on $[0,+\infty)$, and $\frac{1}{\psi(t)}$ is convex on $(0,+\infty)$, $f \in C(\mathbb{R};\mathbb{R})$ is odd and satisfies $tf(t) > 0$ for $t \neq 0$, and $r \in C([a,b];(0,+\infty))$, where $\lambda > 0$ is a constant.

It is worth mentioning that the convexity condition of $t\psi(t)$ (and $\frac{1}{\psi(t)}$) proposed in [30] (and [31]) plays an essential role in establishing the Lyapunov-type inequalities for ψ-Laplace equations. In order to avoid employing the convexity conditions (and the sub-multiplicative condition of ψ in [31]), the authors of [32] proposed the Tolksdorf-type structural conditions to obtain the Lyapunov inequalities for a class of ψ-Laplace equations. Specifically, under the assumptions that

- **(A.1)** $\psi, f \in C(-\infty,+\infty) \cap C^1(0,+\infty)$ with $f \not\equiv 0$ on $(-\infty,\infty)$, $r \in L^1(a,b)$ with $r \not\equiv 0$ on (a,b),
- **(A.2)** ψ is odd on $(-\infty,+\infty)$,
- **(A.3)** $f \geq 0$ on $[0,+\infty)$,
- **(A.4)** there exists $k_0 > 0$ such that $|f(t)| \leq k_0\psi(|t|)$, $\forall t \in (-\infty,+\infty)$,
- **(A.5)** there exist constants $\delta_0, \delta_1 \geq 0$ such that $\delta_0\psi(t) \leq t\psi'(t) \leq \delta_1\psi(t), \forall t > 0$, or
- **(A.5')** there exist constants $\theta_0, \theta_1 \geq 0$ such that $\theta_0 f(t) \leq tf'(t) \leq \theta_1 f(t), \forall t > 0$,

several Lyapunov inequalities were established for the ψ-Laplace equation (1.4a) under the boundary condition (1.4b) with $\lambda = 1$ in [32]. In particular, for ψ, f, r satisfying **(A.1)-(A.5)**, if u is a nontrivial solution of (1.4a), (1.4b) , then it holds that

$$\int_a^b |r(x)|\mathrm{d}x \geq \frac{2}{k_0}\cdot\frac{1+\delta_0}{1+\delta_1}\cdot\min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\},$$

and for ψ, f, r satisfying **(A.1)-(A.4)** and **(A.5')**, if u is a nontrivial solution of (1.4a), (1.4b) , then it holds that

$$\int_a^b |r(x)|\mathrm{d}x \geq \frac{2}{k_0}\cdot\frac{1+\theta_0}{1+\theta_1}\cdot\min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}.$$

It should be noticed that the structural condition **(A.5)** (or **(A.5')**), whose slight version originally introduced by Tolksdorf in [33] has important applications in the regularity theory of partial differential equations (see, e.g., [34, 35]), allows for not only the p-Laplacian case by setting $\psi(t) = |t|^{p-2}t$ with $\delta_0 = \delta_1 = p - 1$ for $p > 1$, but also more nonlinear cases, e.g., $\psi(t) = f(t) = |t|^{a-1}t \log_c(b|t| + d)$ with $\delta_0 = a$, $\delta_1 = a + \frac{1}{\ln d}$ for $a, b > 0, c, d > 1$, and $\psi(t) = f(t) = \frac{|t|^{a-1}t}{\log_c(b|t|+d)}$ with $\delta_0 = a - \frac{1}{\ln d}$, $\delta_1 = a$ for $b > 0$, $c, d > 1$, $a > \frac{1}{\ln d}$. More examples can be found in [32].

The purpose of this paper is to continue the work presented in [32] by establishing Lyapunov-type inequalities for a class of nonlinear ψ-Laplace equations of higher-order. The main tool used in this paper is the Taylor's formula guaranteeing that the proofs are easy-to-follow. The results obtained in this paper can be seen as generalizations and complements of the ones presented in [30]-[32].

2. Problem setting and main results

In this paper, we consider the following higher-order ψ-Laplace equation having a nonlinear term

$$(\psi(u^{(m)}(x)))^{(n)} + r(x)f(u^{(q)}(x)) = 0, x \in (a, b), \tag{2.1a}$$

$$u^{(m-i)}(a) = u^{(j)}(b) = 0, i = 1, 2, ..., n, j = 0, 1, ..., m - 1, \tag{2.1b}$$

where $a, b \in \mathbb{R}$ with $a < b$, $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$, $q \in \mathbb{N}$ with $0 \le m - n \le q \le m - 1$, and ψ, f, r satisfy

(A.1') $\psi \in C^{n-1}(-\infty, +\infty) \cap C^n(0, +\infty)$, $f \in C(-\infty, +\infty) \cap C^1(0, +\infty)$, $f \not\equiv 0$ on $(0, +\infty)$, and $r \in L^1(a, b)$ with $r \not\equiv 0$ on (a, b),

and **(A.2) - (A.5)** (or **(A.2) - (A.4)**, **(A.5')**).

We say that u is a nontrivial solution of the ψ-Laplace equation (2.1a) under the boundary condition (2.1b) if $u \not\equiv 0$ on (a, b), $u \in C^m(a, b) \cap C^{m-1}([a, b])$, $\left(\psi\left(u^{(m)}(x)\right)\right)^{(n-1)}$ is absolutely continuous in x, u satisfies (2.1a) almost everywhere in (a, b), and u satisfies (2.1b). The main results are as follows.

Theorem 2.1. *Assume that ψ, f, r satisfy **(A.1')**, **(A.2) - (A.5)**, and u is a nontrivial solution of the ψ-Laplace equation* (2.1a) *under the boundary condition* (2.1b). *Then the following inequality holds:*

$$\int_a^b |r(x)|dx \ge \frac{2}{k_0} \cdot \frac{(n-1)!}{(b-a)^{n-1}} \cdot \frac{1+\delta_0}{1+\delta_1} \cdot \min\left\{\left(2 \cdot \frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\delta_0}, \left(2 \cdot \frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\delta_1}\right\}. \tag{2.2}$$

Furthermore, if $\psi(t)t$ is a convex function on $[0, +\infty)$, *then it holds that*

$$\int_a^b |r(x)|\,dx \ge \frac{2}{k_0} \cdot \frac{(n-1)!}{(b-a)^{n-1}} \cdot \min\left\{\left(2 \cdot \frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\delta_0}, \left(2 \cdot \frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\delta_1}\right\}. \tag{2.3}$$

Theorem 2.2. *Assume that* ψ, f, r *satisfy* ***(A.1')***, ***(A.2) - (A.4)***, ***(A.5')***, *and* u *is a nontrivial solution of the* ψ*-Laplace equation* (2.1a) *under the boundary condition* (2.1b), *then the following inequality holds:*

$$\int_a^b |r(x)|\,dx \geq \frac{2}{k_0}\cdot\frac{(n-1)!}{(b-a)^{n-1}}\cdot\frac{1+\theta_0}{1+\theta_1} \cdot \min\left\{\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\theta_0}, \left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\theta_1}\right\}. \tag{2.4}$$

Furthermore, if $\psi(t)t$ *is a convex function on* $[0,+\infty)$, *then it holds that*

$$\int_a^b |r(x)|\,dx \geq \frac{2}{k_0}\cdot\frac{(n-1)!}{(b-a)^{n-1}}\cdot\min\left\{\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\theta_0}, \left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\theta_1}\right\}. \tag{2.5}$$

Corollary 2.3. *Let* $r \in L^1(a,b)$ *with* $r \not\equiv 0$ *on* (a,b).

(i) If $\psi(t) = f(t) = |t|^{p-2}t$ *in* (2.1a) *with* $p > 1$, *and* u *is a nontrivial solution of the* ψ*-Laplace equation* (2.1a) *under the boundary condition* (2.1b), *then the following inequality holds:*

$$\int_a^b |r(x)|\,dx \geq 2\cdot\frac{(n-1)!}{(b-a)^{n-1}}\cdot\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{p-1}.$$

In particular, for $m = n = 1, q = 0$, *it holds that* $\int_a^b |r(x)|dx \geq \frac{2^p}{(b-a)^{p-1}}$, *which is the same as one of the results obtained in* [15, 16, 30].

(ii) If $\psi(t) = f(t) = |t|^{a-1}t\log_c(b|t|+d)$ *in* (2.1a) *with* $a, b > 0$ *and* $c, d > 1$, *and* u *is a nontrivial solution of the* ψ*-Laplace equation* (2.1a) *under the boundary condition* (2.1b), *then the following inequality holds:*

$$\int_a^b |r(x)|dx \geq 2\cdot\frac{(n-1)!}{(b-a)^{n-1}}\cdot\frac{(1+a)\ln d}{(1+a)\ln d+1} \cdot \min\left\{\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{a}, \left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{a+\frac{1}{\ln d}}\right\}.$$

(iii) If $\psi(t) = f(t) = \frac{|t|^{a-1}t}{\log_c(b|t|+d)}$ *in* (2.1a) *with* $b > 0$, $c, d > 1$, *and* $a > \frac{1}{\ln d}$, *and* u *is a nontrivial solution of the* ψ*-Laplace equation* (2.1a) *under the boundary condition* (2.1b), *then the following inequality holds:*

$$\int_a^b |r(x)|dx \geq 2\cdot\frac{(n-1)!}{(b-a)^{n-1}}\cdot\frac{(1+a)\ln d-1}{(1+a)\ln d} \cdot \min\left\{\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{a-\frac{1}{\ln d}}, \left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{a}\right\}.$$

Remark 2.1. *Compared with the results obtained in* [31] *(see* (1.3)*), the estimates presented in Theorem 2.1 and Theorem 2.2 provide explicit forms of the Lyapunov inequalities.*

3. Proof of main results

Lemma 3.1 ([32]). *Let $\psi(t)$ satisfy **(A.1')**, **(A.2)**, **(A.5)**, and $\Psi(t) := \int_0^t \psi(s)ds$ for $t \geq 0$. Then the following statements hold true:*

(i) $\psi(st) \leq \max\{s^{\delta_0}, s^{\delta_1}\}\psi(t), \forall s, t \geq 0$;
(ii) $\Psi(t)$ *is* C^{n+1}*-continuous on* $(0, +\infty)$ *and convex on* $[0, +\infty)$;
(iii) $\frac{t\psi(t)}{1+\delta_1} \leq \Psi(t) \leq \frac{t\psi(t)}{1+\delta_0}, \forall t \geq 0.$

Proof of Theorem 2.1. We prove first (2.2). Note that the Taylor's Theorem (see, e.g., [36, pp. 470-471]) gives

$$\begin{aligned} v(x) =& v(a) + v'(a)(x-a) + \frac{v''(a)}{2!}(x-a)^2 + ... + \frac{v^{(k-1)}(a)}{(k-1)!}(x-a)^{k-1} \\ &+ \frac{(-1)^{k-1}}{(k-1)!}\int_a^x (t-x)^{k-1}v^{(k)}(t)\mathrm{dt}, \forall v \in C^k(a,b), \forall k \in \mathbb{Z}^+, \forall x \in (a,b), \\ v(x) =& v(b) + v'(b)(x-b) + \frac{v''(b)}{2!}(x-b)^2 + ... + \frac{v^{(k-1)}(b)}{(k-1)!}(x-b)^{k-1} \\ &+ \frac{(-1)^{k-1}}{(k-1)!}\int_b^x (t-x)^{k-1}v^{(k)}(t)\mathrm{dt}, \forall v \in C^k(a,b), \forall k \in \mathbb{Z}^+, \forall x \in (a,b). \end{aligned}$$

Letting $v := u^{(q)}$ and $k := m - q$, we get

$$\begin{aligned} u^{(q)}(x) =& u^{(q)}(a) + u^{(q+1)}(a)(x-a) + ... + \frac{u^{(m-q-1)}(a)}{(m-q-1)!}(x-a)^{m-q-1} \\ &+ \frac{(-1)^{m-q-1}}{(m-q-1)!}\int_a^x (t-x)^{m-q-1}u^{(m)}(t)\mathrm{dt}, \forall x \in (a,b), \\ u^{(q)}(x) =& u^{(q)}(b) + u^{(q+1)}(b)(x-b) + ... + \frac{u^{(m-q-1)}(b)}{(m-q-1)!}(x-b)^{m-1} \\ &+ \frac{(-1)^{m-q-1}}{(m-q-1)!}\int_b^x (t-x)^{m-q-1}u^{(m)}(t)\mathrm{dt}, \forall x \in (a,b). \end{aligned}$$

Note that $u^{(m-i)}(a) = u^{(j)}(b) = 0, i = 1, 2, ..., n, j = 0, 1, ..., m-1$, where $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$, $q \in \mathbb{N}$ satisfying $0 \leq m - n \leq q \leq m - 1$. It follows that

$$u^{(q)}(x) = \frac{(-1)^{m-q-1}}{(m-q-1)!}\int_a^x (t-x)^{m-q-1}u^{(m)}(t)\mathrm{dt}, \forall x \in (a,b), \tag{3.1}$$

$$u^{(q)}(x) = \frac{(-1)^{m-q-1}}{(m-q-1)!}\int_b^x (t-x)^{m-q-1}u^{(m)}(t)\mathrm{dt}, \forall x \in (a,b). \tag{3.2}$$

Then, by (3.1) and (3.2), we get

$$\begin{aligned} &\left|u^{(q)}(x)\right| \\ =&\left|\frac{1}{2}\cdot\frac{(-1)^{m-q-1}}{(m-q-1)!}\left(\int_a^x (t-x)^{m-q-1}u^{(m)}(t)\mathrm{dt} + \int_b^x (t-x)^{m-q-1}u^{(m)}(t)\mathrm{dt}\right)\right| \\ \leq&\frac{1}{2}\cdot\frac{(b-a)^{m-q-1}}{(m-q-1)!}\cdot\int_a^b \left|u^{(m)}(t)\right|\mathrm{dt}, \forall x \in (a,b). \end{aligned} \tag{3.3}$$

Similarly, we have

$$\begin{aligned}\left|u^{(m-n)}(x)\right| =&\left|\frac{1}{2}\cdot\frac{(-1)^{n-1}}{(n-1)!}\left(\int_a^x (t-x)^{n-1}u^{(m)}(t)\mathrm{d}t+\int_b^x (t-x)^{n-1}u^{(m)}(t)\mathrm{d}t\right)\right|\\ \le&\frac{1}{2}\cdot\frac{(b-a)^{n-1}}{(n-1)!}\cdot\int_a^b\left|u^{(m)}(t)\right|\mathrm{d}t, \forall x\in(a,b). \qquad (3.4)\end{aligned}$$

Then we infer from (3.3), (3.4) and Lemma 3.1 that

$$\begin{aligned}&\psi\left(\left|u^{(q)}(x)\right|\right)\left|u^{(m-n)}(x)\right|\\ \le&\psi\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q-1}}{(m-q-1)!}\cdot\int_a^b\left|u^{(m)}(x)\right|\mathrm{d}x\right)\cdot\frac{1}{2}\cdot\frac{(b-a)^{n-1}}{(n-1)!}\int_a^b\left|u^{(m)}(x)\right|\mathrm{d}x\\ \le&\frac{1}{2}\cdot\frac{(b-a)^n}{(n-1)!}\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\}\\ &\cdot\psi\left(\frac{1}{b-a}\int_a^b\left|u^{(m)}(x)\right|\mathrm{d}x\right)\cdot\frac{1}{b-a}\cdot\int_a^b\left|u^{(m)}(x)\right|\mathrm{d}x\\ \le&\frac{1}{2}\cdot\frac{(b-a)^n}{(n-1)!}\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\}\\ &\cdot(1+\delta_1)\cdot\Psi\left(\frac{1}{b-a}\int_a^b\left|u^{(m)}(x)\right|\mathrm{d}x\right)\\ \le&\frac{1+\delta_1}{2}\cdot\frac{(b-a)^{n-1}}{(n-1)!}\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\}\\ &\cdot\int_a^b\Psi\left(\left|u^{(m)}(x)\right|\right)\mathrm{d}x, \forall x\in(a,b). \qquad (3.5)\end{aligned}$$

Using Lemma 3.1 (iii) and **(A.2)**, we get

$$\begin{aligned}\int_a^b\Psi\left(\left|u^{(m)}(x)\right|\right)\mathrm{d}x \le&\frac{1}{1+\delta_0}\int_a^b\psi\left(\left|u^{(m)}(x)\right|\right)\left|u^{(m)}(x)\right|\mathrm{d}x\\ =&\frac{1}{1+\delta_0}\int_a^b\psi\left(u^{(m)}(x)\right)u^{(m)}(x)\mathrm{d}x. \qquad (3.6)\end{aligned}$$

By integrating by parts, we have

$$\begin{aligned}\frac{1}{1+\delta_0}\int_a^b\psi\left(u^{(m)}(x)\right)u^{(m)}(x)\mathrm{d}x =&\frac{(-1)^n}{1+\delta_0}\int_a^b u^{(m-n)}(x)\left(\psi\left(u^{(m)}(x)\right)\right)^{(n)}\mathrm{d}x\\ =&\frac{(-1)^{n+1}}{1+\delta_0}\int_a^b r(x)f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\mathrm{d}x\\ \le&\frac{1}{1+\delta_0}\int_a^b\left|r(x)f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\right|\mathrm{d}x. \qquad (3.7)\end{aligned}$$

We deduce from **(A.4)** and (3.5) that

$$\begin{aligned}
&\frac{1}{1+\delta_0}\int_a^b \left|r(x)f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\right|\mathrm{d}x\\
\leq&\frac{1}{1+\delta_0}\max_{x\in[a,b]}\left(\left|f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\right|\right)\int_a^b |r(x)|\,\mathrm{d}x\\
\leq&\frac{k_0}{1+\delta_0}\max_{x\in[a,b]}\left(\psi\left(\left|\left(u^{(q)}(x)\right)\right|\right)\left|u^{(m-n)}(x)\right|\right)\int_a^b |r(x)|\,\mathrm{d}x\\
\leq&\frac{k_0(b-a)^{n-1}(1+\delta_1)}{2(n-1)!(1+\delta_0)}\int_a^b \Psi\left(\left|u^{(m)}(x)\right|\right)\mathrm{d}x\int_a^b |r(x)|\mathrm{d}x\\
&\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\}. \qquad (3.8)
\end{aligned}$$

If $\int_a^b\Psi(|u^{(m)}(x)|)\mathrm{d}x=0$, then $\Psi(|u^{(m)}(x)|)\equiv 0$, which is guaranteed by Lemma 3.1, **(A.2)** and **(A.5)**. It follows that $u^{(m)}(x)\equiv 0$, and hence $u(x)\equiv A_m x^{m-1}+A_{m-1}x^{m-2}+...+A_2x+A_1$ with some constants $A_m, A_{m-1}, ..., A_1$. By the boundary conditions $u^{(j)}(b)=0, j=0,1,...,m-1$, it follows that $A_m=A_{m-1}=...=A_1=0$, and hence $u(x)\equiv 0$, which is a contradiction with the assumption that u is a nontrivial solution. Therefore, $\int_a^b\Psi(|u^{(m)}(x)|)\mathrm{d}x>0$, which along with (3.6), (3.7) and (3.8) implies that

$$\begin{aligned}
\int_a^b |r(x)|\mathrm{d}x\geq&\frac{2}{k_0}\cdot\frac{(n-1)!}{(b-a)^{n-1}}\cdot\frac{1+\delta_0}{1+\delta_1}\\
&\cdot\min\left\{\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\delta_0},\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\delta_1}\right\}.
\end{aligned}$$

Thus (2.2) has been proved.

In order to prove (2.3), let $\Phi(t):=\psi(t)t$ for $t\geq 0$. By (3.5), for all $x\in(a,b)$, we get

$$\begin{aligned}
&\psi\left(\left|u^{(q)}(x)\right|\right)\left|u^{(m-n)}(x)\right|\\
\leq&\frac{1}{2}\cdot\frac{(b-a)^n}{(n-1)!}\cdot\psi\left(\frac{1}{b-a}\int_a^b\left|u^{(m)}(x)\right|dx\right)\cdot\frac{1}{b-a}\cdot\int_a^b\left|u^{(m)}(x)\right|dx\\
&\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\}\\
=&\frac{1}{2}\cdot\frac{(b-a)^n}{(n-1)!}\cdot\Phi\left(\frac{1}{b-a}\int_a^b\left|u^{(m)}(x)\right|dx\right)\\
&\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\}\\
\leq&\frac{1}{2}\cdot\frac{(b-a)^n}{(n-1)!}\cdot\frac{1}{b-a}\int_a^b\Phi\left(\left|u^{(m)}(x)\right|\right)dx\\
&\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}\cdot\frac{(b-a)^{n-1}}{(n-1)!}\cdot\int_a^b\Phi\left(\left|u^{(m)}(x)\right|\right)dx\\
&\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\}.
\end{aligned}\tag{3.9}
$$

We infer from (2.1a), (2.1b), **(A.4)** and (3.9) that

$$
\begin{aligned}
\int_a^b\Phi\left(\left|u^{(m)}(x)\right|\right)\mathrm{d}x=&\int_a^b\psi\left(\left|u^{(m)}(x)\right|\right)\left|u^{(m)}(x)\right|\mathrm{d}x\\
=&\int_a^b\psi\left(u^{(m)}(x)\right)u^{(m)}(x)\mathrm{d}x\\
=&(-1)^n\int_a^b u^{(m-n)}(x)\left(\psi\left(u^{(m)}(x)\right)\right)^{(n)}\mathrm{d}x\\
=&(-1)^{n-1}\int_a^b r(x)f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\mathrm{d}x\\
\le&\int_a^b\left|r(x)f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\right|\mathrm{d}x\\
\le&k_0\int_a^b\left|r(x)\psi\left(u^{(q)}(x)\right)u^{(m-n)}(x)\right|\mathrm{d}x\\
\le&k_0\max_{x\in[a,b]}\left(\psi\left(u^{(q)}(x)\right)u^{(m-n)}(x)\right)\int_a^b|r(x)|\mathrm{d}x\\
\le&\frac{k_0}{2}\cdot\frac{(b-a)^{n-1}}{(n-1)!}\cdot\int_a^b\Phi\left(\left|u^{(m)}(x)\right|\right)\cdot\int_a^b|r(x)|\mathrm{d}x\\
&\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\delta_1}\right\},
\end{aligned}
$$

which implies (2.3). □

Proof of Theorem 2.2. We only prove (2.4) due to the fact that (2.5) can be proved in a similar way. Let $F(t):=\int_0^t f(s)\mathrm{d}s$ for $t\ge 0$. Note that $f(t)$ satisfies **(A.1')**, **(A.2) - (A.4)** and **(A.5')**. Applying Lemma 3.1 to $F(t)$, for all $x\in(a,b)$, we have

$$
\begin{aligned}
f\left(\left|u^{(q)}(x)\right|\right)\left|u^{(m-n)}(x)\right|\le&\frac{1+\theta_1}{2}\cdot\frac{(b-a)^{n-1}}{(n-1)!}\cdot\int_a^b F\left(\left|u^{(m)}(x)\right|\right)\mathrm{d}x\\
&\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\theta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\theta_1}\right\}.
\end{aligned}\tag{3.10}
$$

Then using integration by parts, Lemma 3.1, **(A.2)**, **(A.4)**, (2.1a), (2.1b) and (3.10), we obtain

$$
\begin{aligned}
\int_a^b F\left(\left|u^{(m)}(x)\right|\right)\mathrm{d}x\le&\frac{1}{1+\theta_0}\int_a^b f\left(\left|u^{(m)}(x)\right|\right)\left|u^{(m)}(x)\right|\mathrm{d}x\\
\le&\frac{k_0}{1+\theta_0}\int_a^b\psi\left(\left|u^{(m)}(x)\right|\right)\left|u^{(m)}(x)\right|\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{k_0}{1+\theta_0}\int_a^b \psi\left(u^{(m)}(x)\right)u^{(m)}(x)\mathrm{d}x\\
&=\frac{(-1)^n k_0}{1+\theta_0}\int_a^b u^{(m-n)}(x)\left(\psi(u^{(m)}(x))\right)^{(n)}\mathrm{d}x\\
&=\frac{(-1)^{n+1}k_0}{1+\theta_0}\int_a^b r(x)f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\mathrm{d}x\\
&\le\frac{k_0}{1+\theta_0}\int_a^b \left|r(x)f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\right|\mathrm{d}x\\
&\le\frac{k_0}{1+\theta_0}\max_{x\in[a,b]}\left(\left|f\left(u^{(q)}(x)\right)u^{(m-n)}(x)\right|\right)\int_a^b |r(x)|\,\mathrm{d}x\\
&\le\frac{k_0}{1+\theta_0}\cdot\frac{1+\theta_1}{2}\cdot\frac{(b-a)^{n-1}}{(n-1)!}\int_a^b F\left(\left|u^{(m)}(x)\right|\right)\mathrm{d}x\int_a^b |r(x)|\mathrm{d}x\\
&\quad\cdot\max\left\{\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\theta_0},\left(\frac{1}{2}\cdot\frac{(b-a)^{m-q}}{(m-q-1)!}\right)^{\theta_1}\right\}.
\end{aligned}
$$

Note that $\int_a^b F\left(\left|u^{(m)}(x)\right|\right)\mathrm{d}x > 0$, which can be proceeded in the same way as in the proof of Theorem 2.1. Then we conclude that

$$
\begin{aligned}
\int_a^b |r(x)|\mathrm{d}x \ge&\frac{2}{k_0}\cdot\frac{(n-1)!}{(b-a)^{n-1}}\cdot\frac{1+\theta_0}{1+\theta_1}\\
&\cdot\min\left\{\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\theta_0},\left(2\cdot\frac{(m-q-1)!}{(b-a)^{m-q}}\right)^{\theta_1}\right\}.
\end{aligned}
$$

Finally, (2.4) has been proved. □

Proof of Corollary 2.3. For (i), it should be noticed that $\delta_0=\theta_0=\delta_1=\theta_1=p-1$ in **(A.5)** (or **(A.5')**).

For (ii) and (iii), it should be noticed that $\delta_0=\theta_0=a>0, \delta_1=\theta_1=a+\frac{1}{\ln d}>0$ in**(A.5)** (or **(A.5')**), and $\delta_0=\theta_0=a-\frac{1}{\ln d}>0, \delta_1=\theta_1=a>0$ in **(A.5)** (or **(A.5')**), respectively, which have been verified in [32]. □

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Xu Guo
School of Information Science & Technology, Southwest Jiaotong University, Chengdu, 611756, China
E-mail address: guoxu@my.swjtu.edu.cn

Haofan Wang
School of Information Science & Technology, Southwest Jiaotong University, Chengdu, 611756, China
E-mail address: wanghaofan@my.swjtu.edu.cn

Jun Zheng[1,2]
[1]School of Mathematics, Southwest Jiaotong University, Chengdu, 611756, China
[2]Department of Electrical Engineering, Polytechnique Montréal, P.O. Box 6079, Station Centre-Ville, Montreal, QC, Canada H3T 1J4
E-mail address: zhengjun@swjtu.edu.cn