Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 12 Issue 3(2021), Pages 1-12.

NEW UPPER BOUNDS RELATED TO THE BEREZIN NUMBER INEQUALITIES

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Abstract. The Berezin symbol $\widetilde{A}$ of an operator A on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω with the reproducing kernel $\mathcal{K}_\lambda$ is defined by

$$\tilde{A}(\lambda) = \left\langle A\frac{\mathcal{K}_\lambda}{\|\mathcal{K}_\lambda\|}, \frac{\mathcal{K}_\lambda}{\|\mathcal{K}_\lambda\|}\right\rangle, \ \lambda \in \Omega.$$

The Berezin number of an operator A is defined by

$$\operatorname{ber}(A) := \sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|.$$

We study some problems of operator theory by using this bounded function $\widetilde{A}$, including estimates for upper bounds for the Berezin numbers of some operators. We also establish some inequalities involving of the Berezin number inequalities.

1. Introduction

Recall that a reproducing kernel Hilbert space (shortly, RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that the evaluation functionals $\varphi_\lambda(f) = f(\lambda),\ \lambda \in \Omega$, are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem, for each $\lambda \in \Omega$ there exists a unique function $\mathcal{K}_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, \mathcal{K}_\lambda\rangle$ for all $f \in \mathcal{H}$. The family $\{\mathcal{K}_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. The prototypical RKHSs are the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in C : |z| < 1\}$ is the unit disc, the Bergman space $L^2_a(\mathbb{D})$, the Dirichlet space $D^2(\mathbb{D})$ and the Fock space $F(\mathbb{C})$. A detailed presentation of the theory of RKHSs and reproducing kernels is given, for instance, in Aronzajn [1], Bergman [7] and Saitoh and Sawano [29]. It is well known that (see Aronzajn [1])

$$\mathcal{K}_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n(z)\}_{n\geq 0}$ of the space $\mathcal{H}(\Omega)$.

For A a bounded linear operator on $\mathcal{H}$ (i.e., for $A \in \mathcal{B}(\mathcal{H})$, the Banach algebra of all bounded linear operators on $\mathcal{H}$), its Berezin symbol (also called Berezin

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2000 *Mathematics Subject Classification.* 47A30, 47A63.
Key words and phrases. Berezin symbol; Berezin number; numerical radius; reproducing kernel Hilbert space; bounded linear operator; inequality.

Submitted June 16, 2021. Published July 7, 2021.
Communicated by F. Kittaneh.

transform) $\widetilde{A}$ is defined on Ω by (see Berezin [5, 6], and also Engliš [11])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{\mathcal{K}}_{\lambda}, \widehat{\mathcal{K}}_{\lambda} \right\rangle \ (\lambda \in \Omega),$$

where $\widehat{\mathcal{K}}_{\lambda} := \frac{\mathcal{K}_{\lambda}}{\|\mathcal{K}_{\lambda}\|}$ is the normalized reproducing kernel of the space $\mathcal{H}$ and the inner product $< , >$ is taken in the space $\mathcal{H}$. It is obvious that the Berezin symbol $\widetilde{A}$ is a bounded function on Ω and $\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|$, which is called the Berezin number of operator A (see Karaev [20, 21]), does not exceed $\|A\|$, i.e.,

$$\operatorname{ber}(A) := \sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right| \leq \|A\|.$$

The Berezin number of an operator A satisfies the following properties:
(a) $\operatorname{ber}(\alpha A) = |\alpha| \operatorname{ber}(A)$, for all $\alpha \in \mathbb{C}$;
(b) $\operatorname{ber}(A+B) \leq \operatorname{ber}(A) + \operatorname{ber}(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.

It is also clear from the definition of Berezin symbol that the range of the Berezin symbol $\widetilde{A}$, which is called the Berezin set of operator A (see Karaev [20, 21]), lies in the numerical range $W(A)$ of operator A, i.e.,

$$\operatorname{Ber}(A) := \operatorname{Range}\left(\widetilde{A}\right) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\} \subset W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

which implies that $\operatorname{ber}(A) \leq w(A) := \sup_{\|x\|=1} |\langle Ax, x\rangle|$ (numerical radius of operator A) (for more information, see [8, 10, 18, 24, 25, 26, 27]). So, many questions, which are well studied for the numerical radius $w(A)$ of operator A, can be naturally asked for the Berezin number $\operatorname{ber}(A)$ of operator A. For example, is it true, or under which additional conditions the following are true:
(i) $\operatorname{ber}(A) \geq \frac{1}{2}\|A\|$;
(ii) $\operatorname{ber}(A^n) \leq \operatorname{ber}(A)^n$ for any integer $n \geq 1$; more generally, if A is not nilpotent, then

$$C_1 \operatorname{ber}(A)^n \leq \operatorname{ber}(A^n) \leq C_2 \operatorname{ber}(A)^n$$

for some constants $C_1, C_2 > 0$;
(iii) $\operatorname{ber}(AB) \leq \operatorname{ber}(A)\operatorname{ber}(B)$, where $A, B \in \mathcal{B}(\mathcal{H})$.

If $A = cI$ with $c \neq 0$, then obviously $\operatorname{ber}(A) = |c| > \dfrac{|c|}{2} = \dfrac{\|A\|}{2}$. However, it is known that in general the above inequality (i) is not satisfied (see Karaev [22]).

Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [20]. For the basic properties and facts on these new concepts, see [2, 3, 4, 22, 28, 30].

It is well-known that

$$\operatorname{ber}(A) \leq w(A) \leq \|A\| \tag{1.1}$$

and

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\| \tag{1.2}$$

for any $A \in B(\mathcal{H})$ .Also, Berezin number inequalities were given by using the other inequalities in [12, 13, 14, 15, 16, 17, 19, 31, 32, 33].

We also define the following so-called Berezin norm of operators $A \in \mathcal{B}(\mathcal{H})$:

$$\|A\|_{\text{Ber}} := \sup_{\lambda\in\Omega}\left\|A\widehat{\mathcal{K}}_{\lambda}\right\|.$$

It is easy to see that actually $\|A\|_{\text{Ber}}$ determines a new operator norm in $\mathcal{B}(\mathcal{H}(\Omega))$ (since the set of reproducing kernels $\{\mathcal{K}_\lambda : \lambda \in \Omega\}$ span the space $\mathcal{H}(\Omega)$). It is also trivial that $\text{ber}(A) \leq \|A\|_{\text{Ber}} \leq \|A\|$.

Kittaneh [24, 25] improved on the inequality (1.1), to prove that

$$\frac{1}{4}\left\||A|^2+|A^*|^2\right\| \leq w^2(A) \leq \frac{1}{2}\left\||A|^2+|A^*|^2\right\|, \tag{1.3}$$

$$w(A) \leq \frac{1}{2}\left\||A|^2+|A^*|^2\right\|^{\frac{1}{2}}. \tag{1.4}$$

In the present paper, we study some problems of operator theory by using this bounded function $\tilde{A}$, including estimates for upper bounds for the Berezin numbers of some operators. We also establish some inequalities involving of the Berezin inequalities.

2. Known Lemmas

Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. In this case we will write $A \geq 0$. The classical operator Jensen inequality for the positive operators $S \in \mathcal{B}(\mathcal{H})$ is

$$\langle Sx, x\rangle^r \leq (\geq) \langle S^r x, x\rangle, \ r \geq 1 \ \ (0 \leq r \leq 1). \tag{1.5}$$

Kittaneh [23] obtained the following result which is a generalized mixed Cauchy-Schwartz.

Lemma 2.1. *Let $S \in \mathcal{B}(\mathcal{H})$. Let f and g be a nonnegative functions on $[0,\infty]$ which are continous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty]$.*

$$|\langle Sx, y\rangle| \leq \|f(|S|)x\| \, \|g(|S^*|)y\|, \tag{1.6}$$

for all $x, y \in \mathcal{H}$.

The following result [34] is a consequence of the convexity the function $f(t) = t^r$, $r \geq 1$.

Lemma 2.2. *If a_i, $i = 1, 2, \ldots, n$, are positive real numbers, then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r \tag{1.7}$$

for $r \geq 1$.

3. The main results

Our first result is an improvement of the inequalities (1.2).

Theorem 3.1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS. If $A \in B(\mathcal{H})$, then*

$$\frac{1}{4}\|A^*A + AA^*\| \leq (\text{ber}(A))^2 \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{2.1}$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of A. Then B and C are self-adjoint, and $A^*A + AA^* = 2\left(B^2 + C^2\right)$. Let x be any vector in $\mathcal{H}$. From ([25], Theorem 1), then, by the convexity of the function $f(t) = t^2$, we get

$$\sup|\langle Ax, x\rangle|^2 \geq \frac{1}{2}\left\|(B \pm C)^2\right\|$$

for $x \in \mathcal{H}$ and $\|x\| = 1$. Let $\widehat{\mathcal{K}}_\lambda$ be a normalized reproducing kernel. Putting $x = \widehat{\mathcal{K}}_\lambda$ in the above inequality, we have

$$(\mathrm{ber}\,(A))^2 = \sup_{\lambda\in\Omega} \left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right|^2 \geq \frac{1}{2}\left\|(B \pm C)^2\right\|.$$

Thus,

$$\begin{aligned} 2\,(\mathrm{ber}\,(A))^2 &\geq \frac{1}{2}\left(\left\|(B+C)^2\right\| + \left\|(B-C)^2\right\|\right) \\ &\geq \frac{1}{2}\left\|(B+C)^2 + (B-C)^2\right\| = \left\|B^2 + C^2\right\| \\ &= \frac{1}{2}\left\|A^*A + AA^*\right\| \end{aligned}$$

and hence

$$(\mathrm{ber}\,(A))^2 \geq \frac{1}{4}\left\|A^*A + AA^*\right\|.$$

On the other hand, for every normalized reproducing kernel $\widehat{\mathcal{K}}_\lambda$, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right|^2 &= \left\langle B\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle^2 + \left\langle C\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle^2 \\ &\leq \left\|B\widehat{\mathcal{K}}_\lambda\right\|^2 + \left\|C\widehat{\mathcal{K}}_\lambda\right\|^2 \\ &= \left\langle B^2\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle + \left\langle C^2\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle = \left\langle \left(B^2 + C^2\right)\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle \end{aligned}$$

and

$$\sup_{\lambda\in\Omega}\left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right|^2 \leq \sup_{\lambda\in\Omega}\left\langle \left(B^2 + C^2\right)\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle$$

which is equivalent to

$$(\mathrm{ber}\,(A))^2 \leq \left\|B^2 + C^2\right\| = \frac{1}{2}\left\|A^*A + AA^*\right\|,$$

and completes the proof of the theorem. □

We now present the second inequality.

Theorem 3.2. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS. If $A \in B(\mathcal{H})$, then for every $r \geq 1$*

$$(\mathrm{ber}\,(A))^{2r} \leq \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}\mathrm{ber}\left(|A^*|^r\,|A|^r\right). \tag{2.2}$$

Proof. Let $\widehat{\mathcal{K}}_\lambda$ be a reproducing kernel of space $\mathcal{H}(\Omega)$. Considering $f(t))g(t) = \sqrt{t}$ in inequality (1.6) we have that

$$\left|\widetilde{A}(\lambda)\right|^2 = \left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right|^2 \leq \left\langle |A|\,\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle \left\langle |A^*|\,\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle.$$

By inequality (1.5), we get

$$\left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right|^{2r} \leq \left\langle |A|^r\,\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle \left\langle |A^*|^r\,\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle = \left\langle |A^*|^r\,\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle \left\langle \widehat{\mathcal{K}}_\lambda, |A|^r\,\widehat{\mathcal{K}}_\lambda \right\rangle. \tag{2.3}$$

The following refinement of the Cauchy-Schwarz inequality proved by Buzano [9]:

$$\|x\|\,\|y\| \geq |\langle x, y\rangle - \langle x, e\rangle\,\langle e, y\rangle| + |\langle x, e\rangle\,\langle e, y\rangle| \geq |\langle x, y\rangle|, \tag{2.4}$$

for all $x, y, e \in \mathcal{H}$ and $\|e\| = 1$. From inequality (2.4), we conclude that

$$\frac{1}{2}\left(\|x\|\,\|y\| + |\langle x, y\rangle|\right) \geq |\langle x, e\rangle \langle e, y\rangle| . \tag{2.5}$$

Putting $e = \widehat{\mathcal{K}}_\lambda$, $x = |A^*|^r \widehat{\mathcal{K}}_\lambda$ and $y = |A|^r \widehat{\mathcal{K}}_\lambda$ in the above inequality and using (2.3), we have

$$\begin{aligned}
\left|\widetilde{A}(\lambda)\right|^{2r} &\leq \left\langle |A^*|^r \widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle \left\langle \widehat{\mathcal{K}}_\lambda, |A|^r \widehat{\mathcal{K}}_\lambda \right\rangle \\
&\leq \frac{1}{4}\left(\left\||A^*|^r \widehat{\mathcal{K}}_\lambda\right\|^2 + \left\||A|^r \widehat{\mathcal{K}}_\lambda\right\|^2\right) + \frac{1}{2}\left|\left\langle |A|^r |A^*|^r \widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right| \\
&= \frac{1}{4}\left(\left\langle |A^*|^{2r} \widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle + \left\langle |A|^{2r} \widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right) + \frac{1}{2}\left|\left\langle |A|^r |A^*|^r \widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right| \\
&= \frac{1}{4}\left\langle \left(|A^*|^{2r} + |A|^{2r}\right) \widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle + \frac{1}{2}\left|\left\langle |A|^r |A^*|^r \widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle\right|
\end{aligned}$$

and

$$\sup_{\lambda \in \Omega} \left|\widetilde{A}(\lambda)\right|^{2r} \leq \frac{1}{4} \sup_{\lambda \in \Omega} \left\langle \left(|A^*|^{2r} + |A|^{2r}\right) \widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle + \frac{1}{2} \sup_{\lambda \in \Omega} \left|\widetilde{|A|^r |A^*|^r}(\lambda)\right|$$

which is equivalent to

$$\operatorname{ber}^{2r}(A) \leq \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}\operatorname{ber}\left(|A^*|^r |A|^r\right).$$

□

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS. If $A \in B(\mathcal{H})$, then*

$$(\operatorname{ber}(A))^2 \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\operatorname{ber}(|A^*|\,|A|).$$

Remark. *(i) If $|A^*|\,|A| = 0$, then $(\operatorname{ber}(A))^2 \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|$.*

(ii) The inequality in Corollary 3.3 developes into the right hand inequality in (2.1). Clearly, $\operatorname{ber}(|A|\,|A^*|) \leq \||A|\,|A^*|\| = \left\|A^2\right\|$. Therefore,

$$\begin{aligned}
\operatorname{ber}^2(A) &\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\operatorname{ber}(|A^*|\,|A|) \\
&\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\left\|A^2\right\| \\
&\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \\
&\leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
\operatorname{ber}^2(A) &\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\operatorname{ber}(|A|\,|A^*|) \\
&\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\left\|A^2\right\| \\
&\leq \left(\frac{1}{2}\|A\| + \frac{1}{2}\left\|A^2\right\|^{1/2}\right)^2.
\end{aligned}$$

Next we obtain the following inequality for Berezin number of the sum of n operators which generalizes Theorem 3.2.

Theorem 3.4. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $A_i \in B(\mathcal{H})$, $i=1,2,...,n$, then*

$$\left(\operatorname{ber}\left(\sum_{i=1}^{n} A_i\right)\right)^{2r} \leq \frac{n^{2r-1}}{4}\left\|\sum_{i=1}^{n}|A_i|^2+|A_i^*|^2\right\|+\frac{n^{2r-1}}{2}\left(\sum_{i=1}^{n}\operatorname{ber}\left(|A_i|^r|A_i^*|^r\right)\right),$$

for all $r \geq 1$.

Proof. Let $\widehat{\mathcal{K}}_\lambda$ be a normalized reproducing kernel. Then we get

$$\begin{aligned}
&\left|\widetilde{\left(\sum_{i=1}^{n} A_i\right)}(\lambda)\right|^{2r} \\
&=\left|\left\langle\left(\sum_{i=1}^{n} A_i\right)\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|^{2r} \\
&=\left|\sum_{i=1}^{n}\left\langle A_i\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|^{2r} \leq \left(\sum_{i=1}^{n}\left|\left\langle A_i\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|\right)^{2r} \\
&\leq n^{2r-1}\left(\sum_{i=1}^{n}\left|\left\langle A_i\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|^{2r}\right) \\
&\leq n^{2r-1}\left(\sum_{i=1}^{n}\frac{1}{4}\left(\left\langle\left(|A_i|^{2r}+|A_i^*|^{2r}\right)\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right)+\frac{1}{2}\left|\left\langle|A_i^*|^r|A_i|^r\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|\right) \\
&\text{(by the inequalities (2.3) and (2.4))} \\
&\leq \frac{n^{2r-1}}{4}\sum_{i=1}^{n}\left(\left\langle\left(|A_i|^{2r}+|A_i^*|^{2r}\right)\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right)+\frac{n^{2r-1}}{2}\sum_{i=1}^{n}\left|\left\langle|A_i^*|^r|A_i|^r\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|.
\end{aligned}$$

By taking the supremum over $\lambda \in \Omega$ above inequality, we have

$$\begin{aligned}
\sup_{\lambda\in\Omega}\left|\widetilde{\left(\sum_{i=1}^{n} A_i\right)}(\lambda)\right|^{2r} &\leq \frac{n^{2r-1}}{4}\sup_{\lambda\in\Omega}\sum_{i=1}^{n}\left(\left\langle\left(|A_i|^{2r}+|A_i^*|^{2r}\right)\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right) \\
&\quad+\frac{n^{2r-1}}{2}\sup_{\lambda\in\Omega}\sum_{i=1}^{n}\left|\widetilde{|A_i^*|^r|A_i|^r}(\lambda)\right|
\end{aligned}$$

which clearly implies that

$$\left(\operatorname{ber}\left(\sum_{i=1}^{n} A_i\right)\right)^{2r} \leq \frac{n^{2r-1}}{4}\left\|\sum_{i=1}^{n}|A_i|^2+|A_i^*|^2\right\|+\frac{n^{2r-1}}{2}\left(\sum_{i=1}^{n}\operatorname{ber}\left(|A_i|^r|A_i^*|^r\right)\right).$$

□

The next result gives as follows.

Theorem 3.5. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS and $A,B \in B(\mathcal{H})$ selfadjoint. Then*

$$\|A+B\| \leq \sqrt{\operatorname{ber}^2(A+iB)+\|A\|\|B\|+\operatorname{ber}(BA)} \leq \|A\|+\|B\|.$$

Proof. $\widehat{\mathcal{K}}_\lambda$ be a normalized reproducing kernel. Then, we have,

$$\begin{aligned}
\|A+B\|^2 &= \operatorname{ber}^2(A+B) \\
&= \sup_{\lambda\in\Omega}\left|\left\langle (A+B)\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|^2 \\
&\le \sup_{\lambda\in\Omega}\left(\left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right| + \left|\left\langle B\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|\right)^2 \\
&= \sup_{\lambda\in\Omega}\left(\left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|^2 + \left|\left\langle B\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|^2 + 2\left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|\left|\left\langle B\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|\right) \\
&= \sup_{\lambda\in\Omega}\left(\left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|^2 + i\left|\left\langle B\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|^2 + 2\left|\left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|\left|\left\langle \widehat{\mathcal{K}}_\lambda, B\widehat{\mathcal{K}}_\lambda\right\rangle\right|\right) \\
&\le \sup_{\lambda\in\Omega}\left(\left|\left\langle (A+iB)\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|^2 + \left\|A\widehat{\mathcal{K}}_\lambda\right\|\left\|B\widehat{\mathcal{K}}_\lambda\right\| + \left\langle A\widehat{\mathcal{K}}_\lambda, B\widehat{\mathcal{K}}_\lambda\right\rangle\right) \\
&\quad \text{(by the inequality (2.5))} \\
&= \sup_{\lambda\in\Omega}\left(\left|\left\langle (A+iB)\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|^2 + \left\|A\widehat{\mathcal{K}}_\lambda\right\|\left\|B\widehat{\mathcal{K}}_\lambda\right\| + \left\langle BA\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right) \\
&\le \operatorname{ber}^2(A+iB) + \|A\|_{\mathrm{Ber}}\|B\|_{\mathrm{Ber}} + \operatorname{ber}(BA). \\
&\le \operatorname{ber}^2(A+iB) + \|A\|\|B\| + \operatorname{ber}(BA).
\end{aligned}$$

Hence,

$$\|A+B\| \le \sqrt{\operatorname{ber}^2(A+iB) + \|A\|\|B\| + \operatorname{ber}(BA)}.$$

On the other hand,

$$\begin{aligned}
\operatorname{ber}^2(A+iB) &= \sup_{\lambda\in\Omega}\left(\left|\left\langle (A+iB)\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right|^2\right) \\
&\le \sup_{\lambda\in\Omega}\left(\left\langle |A+iB|\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle^2\right) \\
&\le \sup_{\lambda\in\Omega}\left(\left\langle \left(A^2+B^2\right)\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right) \\
&\le \sup_{\lambda\in\Omega}\left|\left\langle \left(A^2+B^2\right)\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda\right\rangle\right| \le \left\|A^2+B^2\right\|.
\end{aligned}$$

Therefore, we have

$$\operatorname{ber}^2(A+iB) + \|A\|\|B\| + \operatorname{ber}(BA) \le (\|A\|+\|B\|)^2.$$

This completes the proof. □

Next we prove the following inequality.

Theorem 3.6. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS and $A_i, B_i, X_i \in B(\mathcal{H})$, $i=1,2,...,n$. Let f and g be two nonnegative functions on $[0,\infty)$ which are continouns and satisfy the relation $f(t)\,g(t)=t$ for all $t\in[0,\infty)$. Then*

$$\left(\operatorname{ber}\left(\sum_{i=1}^n A_i^* X_i B_i\right)\right)^r \le \frac{n^{2r-1}}{\sqrt{2}}\operatorname{ber}\left(\sum_{i=1}^n \left[B_i^* f^2(|X_i|)\,B_i\right]^r + i\left[A_i^* g^2(X_i)\,A_i\right]^r\right),$$

for all $r\ge 1$.

Proof. $\widehat{\mathcal{K}}_\lambda$ be a normalized reproducing kernel. Then, we have,

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^n A_i^* X_i B_i\right)\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|^r\\
&\le\left(\sum_{i=1}^n\left|\left\langle A_i^* X_i B_i\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|\right)^r\\
&\le n^{r-1}\left(\sum_{i=1}^n\left|\left\langle A_i^* X_i B_i\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|^r\right)\\
&\text{(by the inequality (1.7))}\\
&= n^{r-1}\left(\sum_{i=1}^n\left|\left\langle X_i B_i\widehat{\mathcal{K}}_\lambda,A_i\widehat{\mathcal{K}}_\lambda\right\rangle\right|^r\right)\\
&\le n^{r-1}\left(\sum_{i=1}^n\left\|f\left(|X_i|\right)B_i\widehat{\mathcal{K}}_\lambda\right\|^r\left\|g\left(|X_i^*|\right)A_i\widehat{\mathcal{K}}_\lambda\right\|^r\right)\\
&\text{(by the inequality (1.6))}\\
&= n^{r-1}\left(\sum_{i=1}^n\left\langle f^2\left(|X_i|\right)B_i\widehat{\mathcal{K}}_\lambda,B_i\widehat{\mathcal{K}}_\lambda\right\rangle^{\frac{r}{2}}\left\langle g^2\left(|X_i^*|\right)A_i\widehat{\mathcal{K}}_\lambda,A_i\widehat{\mathcal{K}}_\lambda\right\rangle^{\frac{r}{2}}\right)\\
&= n^{r-1}\left(\sum_{i=1}^n\left\langle B_i^* f^2\left(|X_i|\right)B_i\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle^{\frac{r}{2}}\left\langle A_i^* g^2\left(|X_i^*|\right)A_i\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle^{\frac{r}{2}}\right)\\
&\le n^{r-1}\left(\sum_{i=1}^n\left\langle\left[B_i^* f^2\left(|X_i|\right)B_i\right]^r\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle^{\frac{1}{2}}\left\langle\left[A_i^* g^2\left(|X_i^*|\right)A_i\right]^r\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle^{\frac{1}{2}}\right)\\
&\text{(by the inequality (1.5))}\\
&\le\frac{n^{r-1}}{2}\left(\sum_{i=1}^n\left\langle\left[B_i^* f^2\left(|X_i|\right)B_i\right]^r\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle+\left\langle\left[A_i^* g^2\left(|X_i^*|\right)A_i\right]^r\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right)\\
&\le\frac{n^{r-1}}{\sqrt{2}}\left(\left|\sum_{i=1}^n\left\langle\left[B_i^* f^2\left(|X_i|\right)B_i\right]^r\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle+i\sum_{i=1}^n\left\langle\left[A_i^* g^2\left(|X_i^*|\right)A_i\right]^r\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|\right)
\end{aligned}$$

(as $|a+b|\le\sqrt{2}\,|a+ib|$, $\forall a,b\in\mathbb{R}$)

$$\le\frac{n^{r-1}}{\sqrt{2}}\left(\left|\sum_{i=1}^n\left\langle\left(\left[B_i^* f^2\left(|X_i|\right)B_i\right]^r+i\left[A_i^* g^2\left(|X_i^*|\right)A_i\right]^r\right)\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|\right).$$

So,

$$\left|\left\langle\widetilde{\sum_{i=1}^n A_i^* X_i B_i}\left(\lambda\right)\right\rangle\right|^r=\frac{n^{r-1}}{\sqrt{2}}\left(\left|\sum_{i=1}^n\left\langle\left(\left[B_i^* f^2\left(|X_i|\right)B_i\right]^r+i\left[A_i^* g^2\left(|X_i^*|\right)A_i\right]^r\right)\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|\right)$$

and by taking supremum over $\lambda \in \Omega$,

$$\sup_{\lambda\in\Omega}\left|\left(\widetilde{\sum_{i=1}^{n}A_i^*X_iB_i}\right)(\lambda)\right|^r$$
$$\leq \sup_{\lambda\in\Omega}\frac{n^{r-1}}{\sqrt{2}}\left(\left|\sum_{i=1}^{n}\left\langle\left(\left[B_i^*f^2\left(|X_i|\right)B_i\right]^r+i\left[A_i^*g^2\left(|X_i^*|\right)A_i\right]^r\right)\widehat{\mathcal{K}}_\lambda,\widehat{\mathcal{K}}_\lambda\right\rangle\right|\right)$$

which is equivalent to

$$\left(\mathrm{ber}\left(\sum_{i=1}^{n}A_i^*X_iB_i\right)\right)^r\leq\frac{n^{r-1}}{\sqrt{2}}\mathrm{ber}\left(\sum_{i=1}^{n}\left[B_i^*f^2\left(|X_i|\right)B_i\right]^r+i\left[A_i^*g^2\left(|X_i^*|\right)A_i\right]^r\right)$$

as required. □

Considering $f(t)=t^{\alpha}$ and $g(t)=t^{1-\alpha}$, $0\leq\alpha\leq 1$ in Theorem 3.6, we get the following corollary.

Corollary 3.7. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS and $A_i,B_i,X_i\in B(\mathcal{H})$, $i=1,2,...,n$. Then every $r\geq 1$.*

$$\left(\mathrm{ber}\left(\sum_{i=1}^{n}A_i^*X_iB_i\right)\right)^r\leq\frac{n^{r-1}}{\sqrt{2}}\mathrm{ber}\left(\sum_{i=1}^{n}\left[B_i^*f^{2\alpha}\left(|X_i|\right)B_i\right]^r+i\left[A_i^*g^{2(1-\alpha)}\left(|X_i^*|\right)A_i\right]^r\right).$$

From Theorem 3.6, we have the following corollary.

Corollary 3.8. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS and $X_i\in B(\mathcal{H})$, $i=1,2,...,n$. Let f and g be nonnegative functions on $[0,\infty)$ which are continous and satisfy the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then*

$$\left(\mathrm{ber}\left(\sum_{i=1}^{n}X_i\right)\right)^r\leq\frac{n^{r-1}}{\sqrt{2}}\mathrm{ber}\left(\sum_{i=1}^{n}\left(f^{2r}\left(|X_i|\right)+ig^{2r}\left(|X_i^*|\right)\right)\right),$$

for all $r\geq 1$.

In particular, taking $n=1$, $r=1$ and $f(t)=g(t)=t^{\frac{1}{2}}$ in Corollary 3.8 we get the following inequality.

Corollary 3.9. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $A\in B(\mathcal{H})$, then*

$$\mathrm{ber}(A)\leq\frac{1}{\sqrt{2}}\mathrm{ber}\left(|A|+i\,|A^*|\right).$$

It is easy to observe that $(ber\,(A))^2\leq\left\|\,|A|^2+|A^*|^2\right\|$. Therefore

$$(\mathrm{ber}(A))^2\leq\frac{1}{2}\left(\mathrm{ber}\left(|A|+i\,|A^*|\right)\right)^2\leq\frac{1}{2}\left\|\left(|A|^2+i\,|A^*|^2\right)\right\|.$$

Hence, Corollary 3.9 is sharper than that in (2.1).

Next, we obtain an inequality which follows from Corollary 3.8.

Corollary 3.10. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $A,B\in B(\mathcal{H})$, then*

$$\left(\mathrm{ber}\left(A^*B\right)\right)^r\leq\frac{1}{2}\left(\mathrm{ber}\left(|B|^r+i\,|A|^r\right)\right)^2,$$

for all $r\geq 2$.

Theorem 3.11. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. For $A,B\in B(\mathcal{H})$ and $r\geq 1$, we have inequality:*

$$(\operatorname{ber}(A^{*}B))^{r}\leq\frac{1}{2}\left\||B|^{2r}+|A|^{2r}\right\|. \tag{2.6}$$

Proof. $\widehat{\mathcal{K}}_{\lambda}$ be a normalized reproducing kernel. By the Schwarz inequality, we have

$$\begin{aligned}\left|\left\langle A^{*}B\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle\right| &= \left|\left\langle B\widehat{\mathcal{K}}_{\lambda},A\widehat{\mathcal{K}}_{\lambda}\right\rangle\right|\leq\left\|B\widehat{\mathcal{K}}_{\lambda}\right\|\left\|A\widehat{\mathcal{K}}_{\lambda}\right\|\\ &= \left\langle B\widehat{\mathcal{K}}_{\lambda},B\widehat{\mathcal{K}}_{\lambda}\right\rangle^{1/2}\left\langle A\widehat{\mathcal{K}}_{\lambda},A\widehat{\mathcal{K}}_{\lambda}\right\rangle^{1/2}\\ &= \left\langle B^{*}B\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle^{1/2}\left\langle A^{*}A\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle^{1/2}. \end{aligned}\tag{2.7}$$

Applying again the AM-GM inequality and then the convexity of function $f(t)=t^{r}$, $r\geq 1$, we get

$$\begin{aligned}\left\langle B^{*}B\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle^{1/2}\left\langle A^{*}A\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle^{1/2} &\leq \frac{\left\langle B^{*}B\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle+\left\langle A^{*}A\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle}{2}\\ &\leq \left(\frac{\left\langle B^{*}B\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle^{r}+\left\langle A^{*}A\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle^{r}}{2}\right)^{1/r}\\ &\leq \left(\frac{\left\langle |B|^{2}\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle^{r}+\left\langle |A|^{2}\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle^{r}}{2}\right)^{1/r}\\ &\leq \left(\frac{\left\langle |B|^{2r}\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle+\left\langle |A|^{2r}\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle}{2}\right)^{1/r}\\ &\quad\text{(by the inequality (1.5))}\\ &\leq \left(\frac{\left\langle\left(|B|^{2r}+|A|^{2r}\right)\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle}{2}\right)^{1/r} \end{aligned}\tag{2.8}$$

for all $\widehat{\mathcal{K}}_{\lambda}\in\mathcal{H}$. Now, on making use of the inequalities (2.7) and (2.8), we get the inequality

$$\left|\widetilde{(A^{*}B)^{r}}(\lambda)\right|^{r}\leq\frac{1}{2}\left\langle\left(|B|^{2r}+|A|^{2r}\right)\widehat{\mathcal{K}}_{\lambda},\widehat{\mathcal{K}}_{\lambda}\right\rangle.$$

So, by taking supremum over $\lambda\in\Omega$, we get the desired result. □

The following is an immediate corollary of the inequality (2.6).

Corollary 3.12. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. For $A,B\in B(\mathcal{H})$ and $r\geq 2$, we get*

$$(\operatorname{ber}(A^{*}B))^{r}\leq\frac{1}{2}\left(\operatorname{ber}\left(|B|^{r}+i|A|^{r}\right)\right)^{2}\leq\frac{1}{2}\left\||B|^{2r}+|A|^{2r}\right\|.$$

Acknowledgments. The abstract of this work was presented in the "1-st Western Balkan Conference in Mathematics and Applications (WBCMA2021)" which was held between 10–12 June 2021 in Prishtina/Kosovo.

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